Performance Bound for Parity-Check Coded Partial-Response Channels
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Abstract—We consider maximum-likelihood decoder performance in additive white Gaussian noise for the serial concatenation of an outer code comprising multiple, independent odd-parity-check codes and an inner precoded dicode partial-response channel through a random interleaver. Using a technique proposed in [1],[2], we derive an approximation of the average weight enumerator, assuming that code bit values within error events are independent, identically distributed, and equiprobable. We show that the union bound on word-error-rate based upon the approximate weight enumerator is actually an upper bound to that based upon the true average weight enumerator, for suitably large signal-to-noise ratios.

I. INTRODUCTION

Recently, several authors have investigated turbo coding and decoding architectures for digital recording channels [3],[4],[5],[6],[7]. The system model they consider is represented in Fig. 1, where the outer code is a high-rate systematic code (e.g., truncated convolutional code, turbo code, or repeated short block code) and the channel is a binary-input partial-response model of a digital recording channel (e.g., dicode, PR4, EPR4, E7PR4, and variants).

![Trellis coded partial response system](image)

Fig. 1. Trellis coded partial response system.

Simulation results demonstrate that gains as large as 5 dB over uncoded partial-response channels can be achieved using high-rate punctured convolutional codes of modest complexity and iterative decoding (turbo-equalization), even in the presence of equalization loss, channel nonlinearities, and data-dependent media noise. In [7], it was shown that even a repeated single-parity-check outer code can provide gains on the order of 3.5 - 4 dB.

In view of the potential benefits that these techniques appear to offer, there is a need for analytical tools that can be used to accurately predict their performance. For memoryless channels, the introduction of the uniform random interleaver [8] permitted the analysis of the average Hamming weight enumerator for turbo-like codes, requiring only the Hamming weight enumerators of the constituent codes. From this, one can derive an average union bound on maximum-likelihood decoder performance.

For partial-response channels, however, the calculation of the average Euclidean weight enumerator corresponding to the uniform random interleaver is more complex, even for a simple channel such as the dicode channel. In addition to the Hamming weight enumerator of the outer code and the error event structure of the channel, one must further determine the distribution of interleaved code bits within error events. This appears to be prohibitively complex in most situations.

To overcome this obstacle, a technique for approximating the average Euclidean weight enumerator was proposed in [1],[2]. The calculation treats the code bits within error events as independent, equiprobable bits. This "i.i.d. assumption" was considered to be reasonable for high-rate, systematic outer codes and a random interleaver structure. As shown in [1],[2], certain qualitative features of simulated error-rate comparisons using iterative decoders were indeed reflected in the union bounds derived from the approximate weight enumerator. The application of the "i.i.d. assumption" to turbo-coded high-order partial-response channels has recently been investigated in [9].

In this paper, we demonstrate that, for an outer code corresponding to multiple, independent odd-parity-check codes, and a precoded dicode partial-response channel, the analysis based upon the "i.i.d. assumption" actually yields a true upper bound on the union bound for the uniform random interleaver for sufficiently high signal-to-noise ratios.

II. UPPER BOUND FOR PARITY-CHECK CODE

We consider the serial concatenation of an outer code comprising multiple single-parity-check codes and a precoded dicode partial-response channel through a uniform interleaver. More specifically, the outer code consists of codewords of length $N$, each containing $N/n$ single-parity-check codewords of length $n$. Each single-parity-check codeword contains $k = n - 1$ information bits and a single parity bit. The overall parity of each $n$-bit parity-check codeword is chosen to be odd. The

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input sequence to the encoder is a sequence of \( kN/n \) independent, identically distributed (i.i.d.) information bits. Since a parity-check bit is generated by \( k \) information symbols, any set of \( k \) code bits are i.i.d. bits. Within a parity-check codeword, the bits are, of course, linearly dependent, since any symbol is the complement of the modulo-2 sum of the other symbols. On the other hand, a code symbol in one parity-check codeword is linearly independent of any set of code symbols taken from other parity-check codewords.

We will show that for uniformly interleaved, multiple, independent odd-parity-check codes, the i.i.d. assumption will give an upper bound on the maximum likelihood (ML) union bound. First we will show that if we have the correct distance spectrum for all but \( n \) linearly dependent symbols, then the i.i.d. assumption applied to these symbols will give us an upper bound. The desired result then follows directly.

We denote by \( B(\gamma|n) \) the number of arrangements of \( n \) linearly dependent code symbols with \( \gamma \) ones and \( n-\gamma \) zeros. Since the code has odd parity, there are no codewords with an even number of ones. Therefore,

\[
B(\gamma|n) = \begin{cases} 0 & \text{if } \gamma \text{ is even}, \\ \binom{n}{\gamma} & \text{if } \gamma \text{ is odd}. \end{cases}
\]  

(1)

Similarly we denote by \( B^{i.i.d.}(\gamma|n) \) the number of arrangements of \( n \) i.i.d. code symbols with \( \gamma \) ones; that is,

\[
B^{i.i.d.}(\gamma|n) = \binom{n}{\gamma}.
\]  

(2)

We write the ML union bound for the probability of word error, \( P_w \), as

\[
P_w \leq \sum_{d_E} \bar{T}(d_E)Q\left(\frac{d_E}{2\sigma}\right),
\]  

(3)

where \( \bar{T}(d_E) \) denotes the average number of codewords with noiseless channel outputs at Euclidean distance \( d_E \) from that of a given codeword.

Suppose \( i \) of the \( n \) linearly dependent symbols are located within sub-error events, where \( 0 \leq i \leq n \). (See [1], [2] for a detailed analysis of the sub-error event structure for the precoded diode channel.) Denote by \( \bar{t}(d_E|i) \) the average number of codewords with noiseless channel output sequences at Euclidean distance \( d_E \) from that of a given codeword, excluding the contribution from the \( n \) linearly dependent symbols. For \( i < n \) the \( i \) symbols are i.i.d., and for \( i = n \), the distribution of \( \delta \) ones within the \( n \) symbols is given by \( B(\delta|n) \).

Recall that a one that lies within a sub-error event contributes 4 to \( d_E^2 \). Therefore, if \( \delta \) of the \( i \) symbols located in sub-error events are ones, we have \( d_E^2 = d_E^2 + 4\delta \). This implies that

\[
\bar{T}(d_E) = \sum_{i=0}^{n-1} \sum_{\delta \in \mathbb{Z}} \bar{t}(d_E|i) \binom{i}{\delta} \left(0.5^i \sum_{\delta \in \mathbb{Z}} \bar{t}(d_E|\delta) \binom{i}{\delta} B(\delta|n)\right) Q\left(\frac{d_E}{2\sigma}\right).
\]  

(4)

Now we can write the ML union bound as

\[
P_w \leq \sum_{d_E} \left(\sum_{i=0}^{n-1} \sum_{\delta \in \mathbb{Z}} \bar{t}(d_E|i) \binom{i}{\delta} 0.5^i \sum_{\delta \in \mathbb{Z}} \bar{t}(d_E|\delta) \binom{i}{\delta} B(\delta|n)\right) Q\left(\frac{d_E}{2\sigma}\right).
\]  

(5)

Using an outer summation over \( \hat{d}_E \), we obtain

\[
P_w \leq \sum_{d_E} \left(\sum_{i=0}^{n-1} \sum_{\delta \in \mathbb{Z}} \bar{t}(d_E|i) 0.5^i \sum_{\delta \in \mathbb{Z}} \bar{t}(d_E|\delta) B(\delta|n)\right) Q\left(\frac{d_E}{2\sigma}\right).
\]  

(6)

By substituting \( B^{i.i.d.}(\delta|n) \) for \( B(\delta|n) \) in (6) we get

\[
P_w^{i.i.d.} \leq \sum_{d_E} \left(\sum_{i=0}^{n-1} \sum_{\delta \in \mathbb{Z}} \bar{t}(d_E|i) 0.5^i \sum_{\delta \in \mathbb{Z}} \bar{t}(d_E|\delta) B^{i.i.d.}(\delta|n)\right) Q\left(\frac{d_E}{2\sigma}\right).
\]  

(7)

We now proceed to show that \( P_w^{i.i.d.} - P_w \geq 0 \), for sufficiently high signal-to-noise ratios. We note that the first term in (6) and (7) is the same and will cancel. We also note that

\[
\sum_{\delta \in \mathbb{Z}} B(\delta|n) = \sum_{\delta \in \mathbb{Z}, \delta \text{ odd}} \binom{n}{\delta} = 2^{n-1},
\]  

(8)
and similarly
\[ \sum_{\delta=0}^{n} B^{\text{i.i.d.}}(\delta|n) = \sum_{\delta=0}^{n} \binom{n}{\delta} = 2^n. \tag{9} \]

Thus, we find
\[
P_w^{\text{i.i.d.}} - P_w = \sum_{d_E} \sum_{\delta=0}^{n} B^{\text{i.i.d.}}(\delta|n) Q\left( \frac{\sqrt{d_E^2 + 4\delta}}{4\sigma^2} \right) - \sum_{d_E} \sum_{\delta=0}^{n} B(\delta|n) Q\left( \frac{\sqrt{d_E^2 + 4\delta}}{4\sigma^2} \right) = \frac{1}{2^n} \sum_{d_E} \sum_{\delta=0}^{n} (-1)^{\delta} \binom{n}{\delta} Q\left( \frac{\sqrt{d_E^2 + 4\delta}}{4\sigma^2} \right). \tag{10} \]

Since \( l(d_E|n) \geq 0 \) for all \( d_E \), it suffices to show that
\[ \sum_{\delta=0}^{n} (-1)^{\delta} \binom{n}{\delta} Q\left( \frac{\sqrt{d_E^2 + 4\delta}}{4\sigma^2} \right) \geq 0. \tag{11} \]

Unfortunately this is not true for all values of \( n \) and \( \sigma \). However, we will now show that the inequality holds for rates below the computational cutoff-rate \( R_0 \), which is the region where union bounds generally apply.

From [10, p. 369] we have
\[ R_0 = \log_2 \frac{2}{1 + e^{-E_s/N_0}}, \tag{12} \]

where \( E_s = 0.5 \) is the energy per symbol, and \( N_0 = 2\sigma^2 \) is the double-sided spectral noise density. The single-parity-check code of length \( n \) has rate \( R = (n-1)/n \). The noise variance \( \sigma_n^2 \) corresponding to a cutoff rate \( R \) is
\[ \sigma_n^2 = \frac{E_s}{2 \ln(21-R-1)} = -\frac{1}{4 \ln(21/n-1)}. \tag{13} \]

To prove that the inequality in (11) holds for \( \sigma^2 \leq \sigma_n^2 \), we will make use of the following upper and lower bounds on the \( Q \)-function from [11, p. 380]:
\[ \left(1 - \frac{1}{x^2}\right) \frac{e^{-x^2/2}}{x \sqrt{2\pi}} < Q(x) < \frac{e^{-x^2/2}}{x \sqrt{2\pi}}. \tag{14} \]

We use the lower bound for even terms (positive) and the upper bound for the odd terms (negative) to get a lower bound for the sum in (11). Specifically, we find
\[
\sum_{\delta=0}^{n} (-1)^{\delta} \binom{n}{\delta} Q\left( \frac{\sqrt{d_E^2 + 4\delta}}{4\sigma^2} \right) > h_0 \sum_{\delta=0, \delta \text{ even}}^{n} \binom{n}{\delta} e^{-\delta/2\sigma^2} \left( \frac{1}{\sqrt{d_E^2 + 4\delta}} \right) - \frac{n - \delta}{\delta + 1} \frac{e^{-1/2\sigma^2}}{\sqrt{d_E^2 + 4\delta + 4}} - \frac{4\sigma^2}{(d_E^2 + 4\delta)^{3/2}}, \tag{15} \]

where \( h_0 = \frac{2e^{-d_E/2\sigma^2}}{\sqrt{2\pi}} \geq 0 \). The inequality in (15) follows from
\[
\sum_{\delta=0}^{n} (-1)^{\delta} \binom{n}{\delta} \frac{e^{-\delta/2\sigma^2}}{\sqrt{d_E^2 + 4\delta}} > \sum_{\delta=0, \delta \text{ even}}^{n} \binom{n}{\delta} e^{-\delta/2\sigma^2} \left( \frac{1}{\sqrt{d_E^2 + 4\delta}} \right) - \sum_{\delta=1, \delta \text{ odd}}^{n+1} \binom{n}{\delta} e^{-\delta/2\sigma^2} \left( \frac{1}{\sqrt{d_E^2 + 4\delta}} \right) - \frac{n - \delta}{\delta + 1} \frac{e^{-1/2\sigma^2}}{\sqrt{d_E^2 + 4\delta + 4}}, \tag{16} \]

where we have used the identity
\[ \binom{n}{\delta} e^{-\delta/2\sigma^2} \left( \frac{1}{\sqrt{d_E^2 + 4\delta}} \right) = \binom{n}{\delta + 1}. \tag{17} \]

It now suffices to show that
\[
\frac{1}{\sqrt{d_E^2 + 4\delta}} - \frac{n - \delta}{\delta + 1} \frac{e^{-1/2\sigma^2}}{\sqrt{d_E^2 + 4\delta + 4}} - \frac{4\sigma^2}{(d_E^2 + 4\delta)^{3/2}} \geq 0 \text{ for even } \delta \leq n. \tag{18} \]

Clearly this is true if
\[
\frac{1}{\sqrt{d_E^2 + 4\delta}} \left(1 - \frac{n - \delta}{\delta + 1} e^{-1/2\sigma^2} \right) - \frac{4\sigma^2}{(d_E^2 + 4\delta)^{3/2}} \geq 0 \text{ for even } \delta \leq n. \tag{19} \]
Since the last two terms on the left-hand-side of (19) are maximized for $\delta = 0$ it is sufficient to show that
\[
1 - n e^{-1/2\sigma^2} - \frac{4\sigma^2}{\delta_E} \geq 1 - n e^{-1/2\sigma^2} - 4\sigma^2 \geq 0. \tag{20}
\]
We note that if there is a $\sigma^2$ such that, for some $n$, $1 - n e^{-1/2\sigma^2} - 4\sigma^2 \geq 0$ then $1 - n e^{-1/2\sigma^2} - 4\delta^2 \geq 0$ if $\delta \leq \sigma$. We substitute into (20) the largest $\sigma^2$ of interest for each $n$ as given in (13), and we define
\[
f(n) := 1 - n(2^{1/n} - 1)^2 + \frac{1}{\ln(2^{1/n} - 1)} \geq 0. \tag{21}
\]
The function $f(n)$ is shown graphically for $n \geq 2$ in Fig. 2. We find that $f(n) > 0$ for $n > 2$, and, therefore,
\[
\sum_{\delta = 0}^{n} (-1)^{\delta} \binom{n}{\delta} Q \left( \sqrt{\frac{d_E^2 + 4\delta}{4\sigma^2}} \right) \geq 0
\]
for $n > 2$ and $\sigma^2 \leq \sigma^2_n$.

For the case $n = 2$, we directly evaluate the sum in (11) to show
\[
\sum_{\delta = 0}^{2} (-1)^{\delta} \binom{2}{\delta} Q \left( \sqrt{\frac{d_E^2 + 4\delta}{4\sigma^2}} \right)
\]
\[
= Q \left( \sqrt{\frac{d_E^2}{4\sigma^2}} \right) - 2Q \left( \sqrt{\frac{d_E^2 + 4}{4\sigma^2}} \right) + Q \left( \sqrt{\frac{d_E^2 + 8}{4\sigma^2}} \right)
\]
\[
\geq \frac{1}{2\sigma\sqrt{2\pi}} \left( \sqrt{d_E^2 + 4} - \sqrt{d_E^2} \right) \geq 0, \text{ for } d_E^2 \geq 1. \tag{22}
\]
thus, for the case of $n = 2$, the i.i.d. assumption gives an upper bound for all values of $\sigma^2$.

We have shown that applying the i.i.d. assumption to a set of $n$ linearly dependent symbols, using the correct distance spectrum for the remaining symbols, gives an upper bound on the ML union bound for the uniform interleaver. If we instead use an approximate spectrum that overestimates the contributions of the remaining symbols, the same argument can be applied to show that the i.i.d. assumption still yields an upper bound. Thus, we can conclude that the i.i.d. assumption yields an upper bound to the ML union bound for the uniformly interleaved, odd-parity check code on the precoded dicode channel. We remark that this result can be extended to the precoded PR4 channel, as well, along the lines of the discussion in [2].

III. Example

We apply the preceding results to a specific outer code, consisting of 128 rate $n - 1/n = 8/9$ odd-parity-check codewords, with overall codeword length $N = 128 \times 9 = 1152$.

Since the odd parity-check code is a coset of the even parity-check code, the weight enumerating function for the even parity-check code can be used to enumerate the weights of error words for the odd parity-check code. For the concatenation of $N/n$ $(n,n-1)$ even parity-check codes, the Hamming input-output weight enumerating function, denoted by $IOWEF(D,I)$, is the product of $N$ weight enumerating functions for a single $(n,n-1)$ even parity-check code. Thus,
\[
IOWEF(D,I) = \sum_{i \geq 0, d \geq 0} A(d,i)D^d I^i
\]
\[
= \left[ \sum_{j=0}^{n-1} \binom{n-1}{j} D^{2[j/2]} I^j \right]^{N/n}, \tag{23}
\]
where the exponent of the indeterminate $D$ reflects output Hamming weight and the exponent of the indeterminate $I$ reflects the input Hamming weight.

To compute the approximate Euclidean weight enumerator as in [1],[2] we need the output weight enumerators for the outer code, as well as the average input weight for codewords of a given output weight. The output weight enumerator is given by
\[
A(k) = \sum_i A(k,i). \tag{24}
\]
The average input weight of codewords with output
weight $k$ is given by
\[ W(k) = \frac{1}{A(k)} \sum_i i A(k, i). \] (25)

Both of these can be obtained from (23).

In [7], we used the i.i.d. assumption to compute an estimate for the word-error-rate (WER) of this rate-8/9 outer code on the precoded dicode channel with a uniform interleaver of length $N = 1152$, shown in Fig. 3 as the dashed curve. The analysis in this paper implies that this estimate actually represents an upper bound on the average ML union bound corresponding to the uniform interleaver.

The figure also shows simulation results for a specific pseudorandom interleaver, Interleaver 1, with an iterative decoder incorporating a posteriori probability (APP) decoders for the outer code and precoded channel. We have also plotted simulation results at $E_b/N_0 = 8.0$ dB for five additional interleavers, denoted as Interleavers 2, 3, 4, 5, 6.

IV. CONCLUSIONS

In this paper, we have investigated a union bound on the word-error-rate performance of a maximum-likelihood decoder in AWGN for the serial concatenation of an outer code comprising multiple, independent odd-parity-check codes and an inner precoded dicode partial-response channel through a random interleaver. For this system, we analyzed a performance estimate derived from an approximate Euclidean weight enumerator reflecting an “i.i.d. assumption” on the distribution of code bit values within error events, as proposed in [1],[2]. We proved that, for sufficiently high signal-to-noise ratios, the union bound based upon this approximate weight enumerator is a true upper bound on that derived from the exact average weight enumerator.

We compared the analytical bound to computer simulation results for a specific rate-8/9 outer code with codeword length $N = 1152$.

REFERENCES