On the Capacity of 2-Dimensional Channels
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Abstract—For a 2-dimensional (2D) Gaussian inter-symbol interference (ISI) channel with discrete input and a 2D discrete memoryless channel with a special class of irreducible constraints, we show that the information capacity is equal to the stationary capacity. As a byproduct, these capacities are shown to be equal to the operational capacity.

I. INTRODUCTION

In storage systems such as holographic memory [18] and non-volatile memory (e.g., NAND flash memory [4], 3D NAND flash memory [15] and 3D XPoint memory [10]), data are stored on 2D or 3D devices. For holographic memory, the storage channel is modeled as a 2D Gaussian ISI channel due to interference from neighboring bits. In [4], the authors propose a 2D communication channel to model multilevel NAND flash memory. To mitigate interference in holographic memory and NAND flash memory, high dimensional constrained coding schemes have been proposed [1], [2], [20]. In [22], [23], [16], [12], the authors derived some bounds on the noiseless capacity of some 2D constraints. In this paper, we study various concepts of capacity of 2D Gaussian ISI channels and input-constrained 2D discrete memoryless channels.

In Section II, we consider the 2D Gaussian ISI channel model:

\[ Y_{k,l} = X_{k,l} + \sum_{(i,j) \in U} h_{i,j} X_{i+k,j+l} + W_{k,l}, \]

where \( U = \{(i,j) : |i| \leq k_0, |j| \leq l_0\} \), \( h_{i,j} \) are real numbers, \( W_{k,l} \) are independent Gaussian random variables with mean 0 and variance 1, and \( \{W_{k,l}\} \) is independent of \( \{X_{k,l}\} \). The input random field \( \{X_{k,l}\} \) takes real values in a finite set \( X \).

For a 2D channel, we can define several notions of capacity. The operational capacity \( C \), roughly speaking, is the largest rate under which reliable communication is possible. The information capacity or Shannon capacity \( C_{\text{Shannon}} \) is defined in [9, p. 256] as

\[ C_{\text{Shannon}} = \lim_{m,n \to \infty} C_{m,n}, \]  

(2)

where

\[ C_{m,n} = \frac{1}{(m+1)(n+1)} \sup_{p(x_{0,0}^{m,n})} I(X_{0,0}^{m,n} ; Y_{0,0}^{m,n}), \]

and \( X \) and \( Y \) are input and output random fields, respectively. Finally, the stationary capacity \( C_S \) is defined as

\[ C_S = \sup_X I(X; Y), \]

where the supremum is taken over all the stationary random fields and

\[ I(X; Y) = \lim_{m,n \to \infty} \frac{1}{(m+1)(n+1)} I(X_{0,0}^{m,n} ; Y_{0,0}^{m,n}). \]

For such 2D channels, many tools available in 1D information theory are not applicable. First, a Shannon-McMillan-Breiman (SMB) theorem for continuous-valued 2D stationary ergodic random fields is not established yet, although a SMB theorem for discrete-valued stationary ergodic random fields was proved in [19]. Second, Ordentlich and Roth in [17] showed that the 2D maximum-likelihood detection problem is NP hard. However, some results relating to 2D capacity are known. In [6], Chen and Siegel derived bounds on the symmetric information rate of a 2D Gaussian ISI channel. In [21], the authors mapped the 2D Gaussian ISI channels to a graphical model and then applied the generalized belief propagation method to estimate the a posteriori probability and information rate. However, in contrast to the 1D case, for 2D channels with memory and states it is not known whether a stationary ergodic random field will achieve the information capacity, as was pointed out as an open problem in [7]. In this work, for a special class of 2D channels, we will show that the operational capacity \( C \) and information capacity \( C_{\text{Shannon}} \) are equal and both can be achieved by a stationary ergodic random field, i.e., \( C = C_S = C_{\text{Shannon}} \).

In Section III, we consider a 2D memoryless channel \( W \) with input alphabet \( A \) and output alphabet \( Y \), where \( A \) and \( Y \) are both finite sets. For a finite subset \( S \subset \mathbb{Z}^2 \), an element \( u \in A^S \) is called a configuration. For \( x \in A^\mathbb{Z}^2 \), \( x|_S \) is the restriction of \( x \) to \( S \). Given a set of finite configurations \( F \), we define a 2D constraint \( X(F) = \{x \in A^\mathbb{Z}^2 : \text{for any finite subset } S \subset \mathbb{Z}^2, x|_S \notin F \} \) that forbids \( F \). The language of \( X(F) \) is \( \mathcal{L}(X(F)) = \cup_{U \in \mathcal{U}} X_U(F) \), where \( X_U(F) = \{x \in A^U : \text{for any finite subset } S \subset U, x|_S \notin F \} \) and the union is taken over all finite \( U \subset \mathbb{Z}^2 \). For \( U \subset \mathbb{Z}^2 \), let \( \partial_U = \{(i,j) : (i,j) \in U \text{ and } |i-k| + |j-l| \leq d+1 \text{ for some } (k,l) \notin U\} \). A constraint \( X(F) \) is said to have a safe symbol if there is some \( a \in A \) and some positive integer \( d \) such that for any configurations \( u \in X_U(F) \), if \( u_{i,j} = a \) for \( (i,j) \in \partial_U \), then \( u_{i,j} \notin \mathcal{L}(X(F)) \) for any \( v \in \mathcal{L}(X(F)) \). We call such an \( a \) and \( d \) a safe symbol and safe distance, respectively. For example for the 2D-(1,∞)-run-length-limited constraint \( X(F) \) with

\[ F = \{(11,1), (1,1)\}, \]

\( a = 0 \) is the safe symbol with safe distance \( d = 1 \). The constraint \( X(F) \) with

\[ F = \{(101,1), (1,0)\}, \]

\( a = 0 \) is relevant to mitigation of inter-cell interference in NAND flash memory [20]. For this constraint, \( a = 0 \) is the safe symbol.
with safe distance $d = 2$.

In [11] and [13], the authors characterized asymptotics of
the capacity of a 1D binary symmetric channel with the input
supported on an irreducible finite-type constraint. In [14], Li
and Han derived the asymptotics of the capacity of a 1D
erasure channel with input supported on an irreducible finite-
type constraint. For a 2D memoryless channel with input
constraint $X(F)$ with a safe symbol $a$ and safe distance $d$,
we consider the corresponding notions of capacity and
prove a result analogous to the result above for 2D ISI
channels. Namely, we show that the operational capacity $C_{Shannon}$
and information capacity $C_{Shannon}$ are equal and both can be
achieved by a stationary ergodic random field supported on
$X(F)$.

Throughout the paper, $X \sim \mu$ means that $\mu$ is the probability
measure induced by the random field $X$ and we use the
logarithm with base $e$. Also both entropy and differential
entropy of some random variable $X$ are denoted by $H(X)$.
For $U \subset \mathbb{Z}^2$, $X_U$ means $\{X_{i,j} : (i,j) \in U\}$ and is denoted by $X_{m,n}^M,N$ when $U = \{(i,j) : m \leq i \leq M, n \leq j \leq N\}$.

II. 2D GAUSIAN ISI CHANNELS

In this section we consider the 2D Gaussian ISI channel (1)
and show that its operational capacity $C_{Shannon}$ and information capacity $C_{Shannon}$ are equal and both can be achieved by
a stationary and ergodic random field.

Remark II.1. In the definition of $C_{Shannon}$, $X_{i,j} = x_{i,j}$ is
fixed for $(i,j) \in \{0 \leq i \leq m + k, 0 \leq j \leq n + l\} - \{(i,j) : 0 \leq i \leq m, 0 \leq j \leq n\}$ and it can be verified that $C_{Shannon}$ is
independent of the choice of $x_{i,j}$.

Theorem II.2. For the 2D Gaussian ISI channel (1),

$$C = C_S = C_{Shannon}.$$  

Proof. Proof of $C_S \leq C$. This follows from a standard
“achievability part” proof. For any rate $R < C_S$ and $\varepsilon > 0$, choose a stationary ergodic input $\{X_{m,n}\}$ such that $R < I(X;Y) - \varepsilon$. First choose $N$ such that

$$\lim_{m,n \rightarrow \infty} I_{m,n}^{X_{0,n};Y_{0,n},0} < I(X;Y) - \varepsilon/2.$$  

Then it follows from the generalized Shannon-McMillan-
Breiman Theorem in [5] that $\{(X_{m,n}, Y_{m,n}) : m \in \mathbb{Z}, 0 \leq n \leq N\}$ satisfies the asymptotic equipartition property. Then
the proof of the achievability can be completed by going
through the usual random coding argument [3, p. 200].

Proof of $C \leq C_{Shannon}$. This follows from a standard
“converse part” proof [3, p. 207].

Proof of $C_{Shannon} \leq C_S$. First we find a capacity-
achieving distribution for $C_{Shannon}$ on a large 2D grid, then
we construct a block independent process with mutual information close to $C_{Shannon}$. Then using a classical argument in
probability [8], we construct a stationary and ergodic random field from this block independent random field that achieves
$C_{Shannon}$. The proof is similar to the one in [8], so we just
outline the main steps.

Step 0. Let $\alpha = \sup_{x \in X} |x|$. First, for any $\varepsilon > 0$, choose
sufficiently large $M$ and $N$ such that

$$(Ml_0 + Nk_0) \log 2 \pi e \left( 2 + \sum_{i,j \in \mathbb{Z}} (H^2(x_{i,j})) ((|U| + 1)\alpha^2 + 1) 
+ 2(Ml_0 + Nk_0) \log |I| 
(\frac{M+1}{M+1} |N+1|) \right) \leq \varepsilon$$

and

$$\sup_{x_{0,0} \in X_{0,0}} I_{X_{0,0}}^{Y_{0,0}^{M,N}} > C_{Shannon} - \varepsilon/2.$$  

Step 1. Now, let $\tilde{X} = \{\tilde{X}_{i,j}\}$ be the “independent block
random field defined as follows:

(i) $\tilde{X}_{k_1}(M+1)+k_2(N+1)+N$ are i.i.d. for $k_1, k_2 \in \mathbb{Z}$;

(ii) $\tilde{X}_{0,0}$ has the same distribution as $X_{0,0}^M,N$.

Let $\tilde{Y}$ be the output obtained by passing $\tilde{X}$ through the
channel (1). Let $\nu$ be independent of $\{\tilde{X}_{i,j}, W_{i,j}\}$ and
uniformly distributed over $\{(i,j) : 0 \leq i \leq M, 0 \leq j \leq N\}$, and let $\tilde{X}_{i,j} = \tilde{X}_{i+1,j}$. It can be verified that $\{\tilde{X}_{i,j}\}$ is a stationary
and ergodic random field.

Step 2. Let $\tilde{Y}_{i,j}$ be the output obtained by passing
the stationary random field $\{\tilde{X}_{i,j}\}$ through the channel (1). Letting

$$I(\tilde{X};\tilde{Y}) = \lim_{k_1, k_2 \rightarrow \infty} I_{k_1}^{X_{k_1}(M+1)+k_2(N+1)+N} \tilde{X}_{k_1}(M+1)+k_2(N+1)+N(I/(M+1)(N+1)),$$

we will show that

$$I(\tilde{X};\tilde{Y}) - I_{X_{0,0}}^{Y_{0,0}^{M,N}} \geq -\varepsilon/2,$$

which, together with (5) and the arbitrariness of $\varepsilon$, will imply
$C_S \geq C_{Shannon}$.

For $0 \leq i \leq M, 0 \leq j \leq N$, let $\tilde{X}_{i,j,m,n} \triangleq \tilde{X}_{i+m,n+j}$
and $Y_{i,j} = \{Y_{i,j,m,n}\}$ be the output random field obtained
by passing the random field $X_{i,j} = \{X_{i,j,m,n}\}$ through
the channel (1). Now one checks that

$$P_{X_{0,0}^{m,n},Y_{0,0}^{m,n}}(x_{0,0}^{m,n},y_{0,0}^{m,n}) = \sum_{i=0}^{M} \sum_{j=0}^{N} \frac{P(X_{0,0}^{m,n}, x_{0,0}^{m,n} | (i,j)) f_{X_{0,0}^{m,n}, Y_{0,0}^{m,n}}(y_{0,0}^{m,n} | x_{0,0}^{m,n})}{(M+1)(N+1)}.$$  

Then it follows from Lemma 2 in [8] that

$I(\tilde{X};\tilde{Y}) = \frac{1}{(M+1)(N+1)} \sum_{i=0}^{M} \sum_{j=0}^{N} I(\tilde{X}_{i,j};\tilde{Y}_{i,j})$, 

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where
\[ I(\hat{X}_{(i,j)}; \hat{Y}_{(i,j)}) = \lim_{k_1,k_2 \to \infty} \frac{I(X_{0,0}^{\hat{X}_{(i,j)},(M+1)+M,k_1(N+1)+N}, \hat{Y}_{(i,j)}^{\hat{X}_{(i,j)},(M+1)+M,k_2(N+1)+N})}{(k_1+1)(k_2+1)(M+1)(N+1)}. \]

To prove (6), it suffices to establish that for any \((i,j)\),
\[ I(\hat{X}_{(i,j)}; \hat{Y}_{(i,j)}) \geq \frac{I(X_{0,0}^{M,N}; Y_{0,0}^{M,N})}{(M+1)(N+1)} - \frac{\varepsilon}{2}, \tag{7} \]

The proof of (7) is similar for all values of \((i,j)\), so in the following we only show it holds true for \((i,j) = (0,0)\). Here, we note that when \((i,j) = (0,0)\),
\[ I(\hat{X}_{(0,0)}; \hat{Y}_{(0,0)}) = I(\hat{X}; \hat{Y}) \]

Let
\[ G_{s,t} = \left\{ (u,v) : s(M+1) \leq u \leq (s+1)(M+1) - 1, \frac{t(N+1)}{(M+1)(N+1)} \leq v \leq \frac{(t+1)(N+1)}{(M+1)(N+1)} - 1 \right\} \]

and
\[ \Gamma_{s,t} = \bigcup_{s_1 < s} G_{s_1,t}. \]

Intuitively, \(\Gamma_{s,t}\) is the “block history” of \(G_{s,t}\) (see \(\Gamma_{4,3}\) in Fig. 1).

Using the chain rule for mutual information, we have,
\[
\begin{align*}
\sum_{k_1,k_2} \sum_{s=0}^{k_1} \sum_{t=0}^{k_2} I(X_{s,t}; Y_{s,t}) &\leq I(X_{0,0}^{(M+1)+M,k_1(N+1)+N}, \hat{Y}_{0,0}^{k_1(M+1)+M,k_2(N+1)+N}) \\
&\leq \frac{1}{(M+1)(N+1)} I(X_{0,0}^{M,N}; Y_{0,0}^{M,N}) - \frac{\varepsilon}{2},
\end{align*}
\]

which means that, to prove (7), it suffices to show that
\[
\frac{I(X_{s,t}^{(M+1)+M,k_1(N+1)+N}, \hat{Y}_{s,t}^{k_1(M+1)+M,k_2(N+1)+N})}{(M+1)(N+1)} \geq \frac{1}{(M+1)(N+1)} I(X_{0,0}^{M,N}; Y_{0,0}^{M,N}) - \frac{\varepsilon}{2}.
\]

Without loss of generality, we prove this holds true for \((s,t) = (1,1)\). Let
\[ U_{M,N} = \left\{ (u,v) : M + 1 + k_0 \leq u \leq 2M + 1 - k_0, \quad N + 1 + k_0 \leq v \leq 2N + 1 - k_0 \right\} \]

and
\[ U_0 = G_{1,1} - U_{M,N}. \]

Geometrically, \(U_0\) is the difference of the rectangle \(G_{1,1}\) and \(U_{M,N}\) (see Fig. 2). In the following, we will first apply the chain rule to decompose \(I(X_{G_{1,1}}; Y_{G_{1,1}}|X_{G_{1,1}}, Y_{G_{1,1}})\) into three parts. Then we will show that each part is close to the counterpart of \(I(X_{0,0}^{M,N}; Y_{0,0}^{M,N})\).

Note that
\[
\begin{align*}
I(X_{G_{1,1}}; Y_{G_{1,1}}|X_{G_{1,1}}, Y_{G_{1,1}}) &= I(X_{G_{1,1}}; Y_U|X_{G_{1,1}}, Y_{G_{1,1}}) + I(X_{G_{1,1}}; Y_{U_{M,N}}|X_{G_{1,1}}, Y_{G_{1,1}}, U_0) \\
&= I(X_{G_{1,1}}; Y_U|X_{G_{1,1}}, Y_{G_{1,1}}) + I(X_{G_{1,1}}; Y_{U_{M,N}}|X_{G_{1,1}}, Y_{G_{1,1}}, U_0) \\
&\quad + I(X_{U_{M,N}}; Y_{U_{M,N}}|X_{G_{1,1}}, Y_{G_{1,1}}, U_0) \\
&= I(X_{G_{1,1}}; Y_U|X_{G_{1,1}}, Y_{G_{1,1}}) + I(X_{G_{1,1}}; Y_{U_{M,N}}|X_{G_{1,1}}, Y_{G_{1,1}}, U_0) \\
&\quad + I(X_{U_{M,N}}; Y_{U_{M,N}}|X_{G_{1,1}}, Y_{G_{1,1}}, U_0), \tag{9}
\end{align*}
\]

where in (9) we use the fact that given \(X_U\), \((\hat{X}_{G_{1,1}}, \hat{Y}_{G_{1,1}})\) is independent of \((\hat{X}_{G_{1,1}}, \hat{Y}_{G_{1,1}}, U_0)\).

Let
\[ V_0 = \{(u,v) : 0 \leq u \leq M, 0 \leq v \leq N\} \]

and
\[ V_0 = \{(u,v) : k_0 \leq u \leq M - k_0, \quad l_0 \leq v \leq N - l_0\}. \]

Since
\[
\begin{align*}
&I(X_{0,0}^{M,N}; Y_{0,0}^{M,N}) = I(X_{G_{1,1}}, Y_{G_{1,1}}|X_{G_{1,1}}, Y_{G_{1,1}}) - I(X_{0,0}^{M,N}; Y_{0,0}^{M,N}) \\
&\leq I(X_{G_{1,1}}; Y_U|X_{G_{1,1}}, Y_{G_{1,1}}) + I(X_{G_{1,1}}; Y_{U_{M,N}}|X_{G_{1,1}}, Y_{G_{1,1}}, U_0) \\
&\quad + I(X_{G_{1,1}}; Y_{U_{M,N}}|X_{G_{1,1}}, Y_{G_{1,1}}, U_0) + I(\hat{X}_{U_{0}}, Y_{U_{0}}|X_{U_{0}}) \\
&\quad + I(\hat{X}_{k_0,N_{0}}; Y_{k_0,N_{0}}|X_{U_{0}}),
\end{align*}
\]

we have that
\[
\begin{align*}
&I(\hat{X}_{G_{1,1}}; Y_{G_{1,1}}|X_{G_{1,1}}, Y_{G_{1,1}}) - I(X_{0,0}^{M,N}; Y_{0,0}^{M,N}) \\
&\leq I(X_{G_{1,1}}; Y_U|X_{G_{1,1}}, Y_{G_{1,1}}) + I(X_{G_{1,1}}; Y_{U_{M,N}}|X_{G_{1,1}}, Y_{G_{1,1}}, U_0) \\
&\quad + I(X_{G_{1,1}}; Y_{U_{M,N}}|X_{G_{1,1}}, Y_{G_{1,1}}, U_0) + I(X_{G_{1,1}}; Y_{U_{M,N}}|X_{G_{1,1}}, Y_{G_{1,1}}, U_0) \\
&\quad + I(X_{G_{1,1}}; Y_{U_{M,N}}|X_{G_{1,1}}, Y_{G_{1,1}}, U_0) + I(\hat{X}_{U_{0}}, Y_{U_{0}}|X_{U_{0}}),
\end{align*}
\]

in (a) we use the fact that
\[ p_{X_{0,0}^{M,N}}(\cdot) = p_{X_{M+1,N+1}}(\cdot) \]

and in (b) we use the inequalities
\[
\begin{align*}
&I(\hat{X}_{U_{0}}; Y_{U_{M+1+k_0,N+1+l_0}}|X_{G_{1,1}}, Y_{G_{1,1}}, U_0) \leq H(\hat{X}_{U_{0}}) \\
&\quad \leq |U_0| \log |X| \quad \text{and} \\
&I(\hat{X}_{U_{0}}; Y_{U_{M+k_0,N-1+l_0}}|X_{G_{1,1}}, Y_{G_{1,1}}, U_0) \leq H(\hat{X}_{U_{0}}) \leq |U_0| \log |X|.
\end{align*}
\]

Recall that \(\alpha = \sup_{x \in X} \log |x|\). Then it follows from Hölder’s inequality that
\[ E[|Y_{1,i,j}|^2] \leq (2 + \sum_{(i,j) \in \mathcal{U}} h_{i,j}^2)(|U| + 1)\alpha^2 + E[W_{i,j}^2]). \]

Then we have
\[
\begin{align*}
H(\hat{Y}_{U_0}|X_{G_{1,1}}, Y_{G_{1,1}}) &\leq H(\hat{Y}_{U_0}) = \sum_{(i,j) \in \mathcal{U}} H(\hat{Y}_{i,j}) \\
&\leq \frac{|U_0|}{2} \log 2pe((2 + \sum_{(i,j) \in \mathcal{U}} h_{i,j}^2)(|U| + 1)\alpha^2 + 1) \tag{11}
\end{align*}
\]

where (11) follows from the fact that Gaussian distribution maximizes the entropy given the second moment.

It then follows from (11) and
\[ H(\hat{Y}_{U_0}|X_{G_{1,1}}, Y_{G_{1,1}}) = H(W_{U_0}) > 0, \]
that
\[ I(\hat{X}^{2M+1,2N+1}_{M+1,N+1} ; \hat{Y}_{U0} | \hat{X}^{1,1}_{\Gamma1,1}, \hat{Y}^{1,1}_{\Gamma1,1}) \leq H(\hat{Y}_{U0} | \hat{X}^{1,1}_{\Gamma1,1}, \hat{Y}^{1,1}_{\Gamma1,1}) \leq |U_0|^2 \log 2\pi e \left( 2 + \sum_{(i,j) \in U} h^2_{i,j} \right) (|U| + 1) + 1). \] (12)

Now, with (5) and the fact that
\[ |U_0| = |V_0| \leq 2(MI_0 + NK_0), \]
we conclude that
\[ I(\hat{X}^{2M+1,2N+1}_{M+1,N+1} ; \hat{Y}^{2M+1,2N+1}_{M+1,N+1} | \hat{Y}_{U0}^{1,1}) \]
\[ \leq \frac{|U_0|^2}{2} \log 2\pi e \left( 2 + \sum_{(i,j) \in U} h^2_{i,j} \right) (|U| + 1) + 1). \] (12)

Theorem III.1. Let \( X(F) \) be a constraint with a safe symbol. For a 2D memoryless channel with input constraint \( X(F) \),
\[ C = C_{\text{Shannon}} = C_S, \] (14)
where
\[ C_{\text{Shannon}} = \lim_{m,n \to \infty} \frac{1}{(m+1)(n+1)} \sup_{\mu \in M_{m,n}} I(X^{m,n} ; Y^{m,n}) \]
and
\[ C_S = \sup_{X \sim \mu} I(X;Y). \]

Proof. We will apply similar arguments as in the proof of Theorem II.2 to prove (14). The main difference is the construction of the capacity-achieving random field. Due to the limitation of the constraint \( X(F) \), we cannot put independent blocks together in a straightforward way. However, for two adjacent blocks with sufficiently large distance, we can, using the property of safe symbol of the constraint \( X(F) \), put extra symbols between independent blocks to preserve the constraint \( X(F) \). Then using the same idea as in Theorem II.2, we construct a stationary and ergodic random field from this random field that achieves \( C_{\text{Shannon}} \). In the following, we outline the main steps.

Step 0. First of all, for any \( \varepsilon > 0 \), choose \( M \) and \( N \) such that
\[ \left( 1 - \frac{(M + 1)(N + 1)}{(M + d + 1)(N + d + 1)} \right) \log |Y| \]
\[ + \frac{(\log |A| + \log |Y|)(M + N)d}{(M + d + 1)(N + d + 1)} \leq \frac{\varepsilon}{2} \]
and then choose \( X^{M,N}_{0,0} \sim p(x^{M,N}_{0,0}) \) such that
\[ I(X^{M,N}_{0,0} ; X^{M,N}_{0,0}) \geq C^{M,N}_{\text{Shannon}} - \frac{\varepsilon}{2}. \] (17)

Step 1. Let \( R(k_1, k_2) \) be the rectangle consisting of \((i, j)\) such that
\[ k_1(M + d + 1) \leq i \leq (k_1 + 1)(M + d + 1) - 1, \]
\[ k_2(N + d + 1) \leq j \leq (k_2 + 1)(N + d + 1) - 1. \]
Let \( R_1(k_1, k_2) \) be the complement of \((i, j) : k_1(M + d + 1) \leq i \leq k_1(M + d + 1) + M, k_2(N + d + 1) \leq j \leq k_2(N + d + 1) + N \) in \( R(k_1, k_2) \). More precisely, \( R_1(k_1, k_2) \) is the set of \((i, j)\) such that either
\[ k_1(M + d + 1) \leq i \leq (k_1 + 1)(M + d + 1) - 1 \]
\[ k_2(N + d + 1) + N + 1 \leq j \leq k_2(N + d + 1) + N + d, \]
or
\[ k_1(M + d + 1) + M + 1 \leq i \leq k_1(M + d + 1) + M + d, \]
\[ k_2(N + d + 1) \leq j \leq (k_2 + 1)(N + d + 1) - 1, \]
Now, let \( \bar{X} = \{ \bar{X}_{i,j} \} \) be the “block” random field defined as:
(i) \( \bar{X}_{k_1(M + d + 1) + M, k_2(N + d + 1)+N} \) and \( \bar{X}_{k_1(M + d + 1), k_2(N + d + 1)} \) are i.i.d. for \( k_1, k_2 \in \mathbb{Z} \);
(ii) \( \bar{X}_{i,j} = a \) for \((i, j) \in R_1(k_1, k_2) \).
(iii) \( \bar{X}^{M,N}_{0,0} \) has the same distribution as \( X^{M,N}_{0,0} \).
From the definition of \( a \) and \( d \), it follows that \( \{ \bar{X}_{i,j} \} \) is a random field supported on the constraint \( X(F) \). One can easily checks that
\[ \bar{X}^{k_1(M + d + 1) + M, k_2(N + d + 1)+N} \]
\[ \bar{X}^{k_1(M + d + 1), k_2(N + d + 1)} \]
are also i.i.d. Let \( \nu \) be independent of \( \{ \bar{X}_{i,j} \} \) and uniformly distributed over \( \{ (i, j) : 0 \leq i \leq M + d, 0 \leq j \leq N + d \} \), and let \( \bar{X}_{i,j} = \bar{X}^{\nu} \).
It can be verified that \( \{ \bar{X}_{i,j} \} \) is a stationary and ergodic random field.

Step 2. Let \( \{ \tilde{Y}_{i,j} \} \) be the output obtained by passing the stationary random field \{ \bar{X}_{i,j} \} through the channel \( W \). We will show that
\[ I(\bar{X};\tilde{Y}) \leq I(X^{M+d+N+d} ; Y^{M+d+N+d} | \bar{X}^{M+d+N+d} , \bar{Y}^{M+d+N+d}) \geq \varepsilon - \frac{\varepsilon}{2} \] (18)
which, together with (17) and the arbitrariness of \( \varepsilon \), will imply
\[ C_S \geq C_{\text{Shannon}}. \]
For \( 0 \leq i \leq M + d \) and \( 0 \leq j \leq N + d \), let \( \bar{X}_{(i,j),m,n} = \bar{X}_{m+i,n+j} \) and \( \bar{Y}_{(i,j),m,n} = \bar{Y}_{m+i,n+j} \) denote the output random field obtained by passing the random field \( \{ \bar{X}_{i,j} \} \) through the channel \( W \).
\[
I(\hat{X}_{(0,0)};\hat{Y}_{(0,0)}) = I(\hat{X};\hat{Y}) = \lim_{k_1,k_2 \to \infty} I(\hat{X}_{(k_1+1)(M+d+1)-1,(k_2+1)(N+d+1)-1};\hat{Y}_{(0,0)}) = \sum_{i=1}^{k_1} \sum_{j=1}^{k_2} I(\hat{X}_{R(i,k_1)},\hat{Y}_{R(j,k_2)}).
\]

Theorem II.2, to prove (18), it suffices to show that

\[
I(\hat{X}_{(i,j)};\hat{Y}_{(i,j)}) \geq \frac{I(X_{M+d,N+d};Y_{M+d,N+d})}{(M+d+1)(N+d+1)} - \frac{\varepsilon}{2}. \tag{19}
\]

The proof of (19) is similar for all values of \((i,j)\), so in the following we only show it holds true for \((i,j) = (0,0)\). Note that when \((i,j) = (0,0)\), (15) holds.

Using the chain rule for mutual information and the fact that \((X_{R(k_1,k_2)},Y_{R(k_1,k_2)})\) are independent, we have (16), which means that, to prove (19), it suffices to show that

\[
I(\hat{X}_{R(1,1)};\hat{Y}_{R(1,1)}) \geq \frac{I(X_{M,N};Y_{M,N})}{(M+N)(M+N)} - \frac{\log|A| + \log|Y|}{R(1,1)}, \tag{20}
\]

where in (20) we use the fact that

\[
P_{X_{M,N}}(1) = p_{X_{R(1,1)},R(1,1)}(1)
\]

and the two inequalities

\[
I(\hat{X}_{R(1,1)};\hat{Y}_{R(1,1)}|X_{R(1,1)},R(1,1)) \leq H(\hat{X}_{R(1,1)})
\]

\[
\leq |R(1,1)| \log|A|
\]

and

\[
I(\hat{X}_{R(1,1)};R(1,1)|Y_{R(1,1)},R(1,1)) \leq H(\hat{Y}_{R(1,1)})
\]

\[
\leq |R(1,1)| \log|Y|
\]

Since \(|R(1,1)| \leq (M+N)d\), it then follows that

\[
I(\hat{X}_{R(1,1)};\hat{Y}_{R(1,1)}) \geq \frac{(M+d+1)(N+d+1)}{(M+1)(N+1)} - \frac{\log|A| + \log|Y|}{R(1,1)} \geq \frac{(M+N)(N+1)d}{(M+1)(N+1)} - \frac{(M+1)(N+1)}{(M+1)(N+1)} \frac{\log|Y|}{R(1,1)} \geq \frac{(M+N)(N+1)}{2}\]

as desired.

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