Ladder Codes: A Class of Error-Correcting Codes with Multi-Level Shared Redundancy

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***Abstract*—Error-correcting codes play an important role in storage systems to maintain data integrity. In this work, we propose a new class of linear error-correcting codes, called ladder codes, whose codeword structure consists of multiple codewords of certain component codes and also their shared redundancy. First, we give a general construction for an m-level ladder code, determine the code length and dimension, and also derive a lower bound d_L^* on the minimum distance. Some examples of ladder codes are presented. Then, we study correctable error-erasure patterns of ladder codes and give a corresponding decoding algorithm. Finally, we compare a two-level ladder code with a concatenated code, and show that the former can outperform the latter in many cases. Ladder codes have potential to be used for data protection in flash memories where only a few pages may suffer from severe errors in a block.**

I. Introduction

As the volume of data continues to explode, error-correcting codes (ECCs) with multi-level redundancy become increasingly important in data storage, since they can balance the reliability and the total redundancy cost. The idea of using multi-level redundancy dates back to Patel [6]. In [6], a two-level coding scheme was used for a data *block*, which consists of several *sub-blocks* and also extra parity-check symbols shared by all these sub-blocks. The scheme in [6] was later extended in [1]. In [3], integrated-interleaving codes with two-level protection were proposed for data storage. A codeword (block) of a two-level integrated-interleaving code is comprised of several component codewords (sub-blocks) of a Reed-Solomon (RS) code $\mathcal{C}$, satisfying the constraints that some linear combinations of these component codewords are codewords of a subcode of $\mathcal{C}$. More recently, generalized integrated-interleaving codes were studied in [9], [12].

In this paper, we present a new class of ECCs with multi-level shared redundancy. We call them ladder codes, since the decoding procedure, which uses multi-level redundancy successively from the lowest level to the highest level, mimics climbing up a ladder.

Our construction is motivated by the construction of tensor product codes, first proposed by Wolf in [10], and later generalized in [5]. A tensor product code is defined by a parity-check matrix that is the tensor product of the parity-check matrices of component codes. Tensor product codes and integrated-interleaving codes are similar. In fact, integrated-interleaving codes can be treated as a subclass of tensor product codes [4]. A codeword of a tensor product code consists of multiple component codewords (sub-blocks) of equal length. As shown by Example 1 in [11], the encoding steps of a tensor product code involve using *phantom* syndrome symbols, which only appear in the encoding procedure but are not stored in the encoded codeword. By imposing constraints on these phantom syndrome symbols (this step is done over an extension field [5], [10]), some of the information symbols of some sub-blocks are turned into parity-check symbols commonly shared by all the sub-blocks. However, in our ladder codes, these shared parity-check symbols do not reside in sub-blocks; instead, they are protected by other levels of coding. Thus, in some sense, ladder codes can be considered as an *external* version of tensor product codes. As a result, to provide extra protection for sub-blocks, unlike tensor product codes, the encoder for each sub-block in ladder codes can be kept intact and we only need to generate the extra shared redundancy part, which seems an attractive feature for some data storage applications.

Aiming at the specific code structure consisting of multiple sub-blocks and their exterior shared redundancy, ladder codes provide a systematic way to generate multi-level shared redundancy successively. However, due to this particular structure embedded in the code, the performance of a three-level (or higher) ladder code might be worse than that of a corresponding generalized tensor product code, if one directly compares their rates and minimum distances.

One possible application of ladder codes could be for flash memories [2]. However, in this paper, we only focus on the *theoretical* aspects of ladder codes, leaving their applications as a future work. Our contributions are as follows: **1)** We propose a new class of ECCs with multi-level shared redundancy. Specifically, we present a general construction of an m-level ladder code, and determine the code length and dimension; in addition, we derive a lower bound d_L^* on the minimum distance. We also provide explicit examples of ladder codes, some of which turn out to be optimal with respect to the minimum distance. **2)** We present a general result on the correctable error-erasure patterns for ladder codes and give a corresponding decoding algorithm. With respect to erasure correction, it is shown that ladder codes can correct at least $d_L^* - 1$ erasures; as for error correction, ladder codes can correct at least $\lfloor \frac{d_L^*-1}{2} \rfloor$ errors. **3)** We compare two-level ladder codes with concatenated codes [7]. Our first code design results in a ladder code possessing the same code parameters as those of a corresponding concatenated code. The second design shows that a ladder code can even outperform a concatenated code in some cases.

The remainder of the paper is organized as follows. In Section II, we present a general construction of ladder codes, and

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determine the corresponding code parameters. In Section III, we study the correctable error-erasure patterns of ladder codes and give a corresponding decoding algorithm. In Section IV, we compare two-level ladder codes with concatenated codes.

Notation: Throughout the paper, we use the following notation. The transpose of a matrix H is written as H^T. The cardinality of a set A is denoted by $|A|$. For a vector $\boldsymbol{v}$ over $\mathbb{F}_q$, we use $w_q(\boldsymbol{v})$ to represent its Hamming weight. A linear code over $\mathbb{F}_q$ of length n, dimension k, and minimum distance d is denoted by $[n,k,d]_q$, where q may be omitted if the field is clear from the context. A linear code which consists of all length-n vectors over $\mathbb{F}_q$ is denoted by $[n,n,1]_q$, and its dual code only has the all-zero codeword, so it is denoted by $[n,0,\infty]_q$.

II. Ladder Codes: Construction and Minimum Distance

In this section, we present a general construction for ladder codes that have multi-level shared redundancy. We then give the code parameters of a ladder code; in particular, we derive a lower bound on its minimum distance.

A. Construction of m-Level Ladder Codes

An m-level ladder code $\mathcal{C}_L$ over $\mathbb{F}_q$ is based on the following component codes.

1) A collection of m nested $[n,k_i,d_i]_q$ codes $\mathcal{C}_i$, $1 \leqslant i \leqslant m$, over $\mathbb{F}_q$, such that

$$\mathcal{C}_m \subset \mathcal{C}_{m-1} \subset \cdots \subset \mathcal{C}_1.$$

The corresponding dimensions satisfy $k_m < k_{m-1} < \cdots < k_1$ and the minimum distances satisfy $d_m \geqslant d_{m-1} \geqslant \cdots \geqslant d_1$. We denote the parity-check matrix of $\mathcal{C}_i$ by

$$H_{\mathcal{C}_i} = \begin{bmatrix} H_1 \\ H_2 \\ \vdots \\ H_i \end{bmatrix},$$

where H_i, $1 \leqslant i \leqslant m$, is a matrix of size $(k_{i-1}-k_i)\times n$, by defining $k_0 = n$. The encoder of $\mathcal{C}_1$ is denoted by $\mathcal{E}_{\mathcal{C}_1} : \mathbb{F}_q^{k_1} \to \mathcal{C}_1$. We also use $\mathcal{E}_{\mathcal{C}_1}^{-1}$ as the inverse of the encoding mapping.

2) A collection of $m-1$ $[n'_i, k' = \ell, \delta_i]_{q^{v_i}}$ codes $\mathcal{C}'_i$, $2 \leqslant i \leqslant m$, over $\mathbb{F}_{q^{v_i}}$, where $v_i = k_{i-1} - k_i$. Without loss of generality, we assume that $n'_2 > n'_3 > \cdots > n'_m$ and $\delta_2 > \delta_3 > \cdots > \delta_m$. The encoder of $\mathcal{C}'_i$ is systematic and is denoted by $\mathcal{E}_{\mathcal{C}'_i} : \mathbb{F}^{\ell}_{q^{v_i}} \to \mathcal{C}'_i$. We also use $\mathcal{E}_{\mathcal{C}'_i}^{-1}$ as the inverse of the encoding mapping.

3) A collection of $m-1$ $[n''_i, k''_i, d''_i]_q$ codes $\mathcal{C}''_i$, $2 \leqslant i \leqslant m$, over $\mathbb{F}_q$. The encoder of $\mathcal{C}''_i$ is denoted by $\mathcal{E}_{\mathcal{C}''_i} : \mathbb{F}_q^{v_i} \to \mathcal{C}''_i$. We also use $\mathcal{E}_{\mathcal{C}''_i}^{-1}$ as the inverse of the encoding mapping.

With the component codes introduced above, the construction of an m-level ladder code $\mathcal{C}_L$ is outlined in the following procedure.

Construction 1: Encoding Procedure for Ladder Codes

Input: ℓ information vectors $\boldsymbol{u}_i \in \mathbb{F}_q^{k_1}$, $1 \leqslant i \leqslant \ell$.
Output: a codeword $\boldsymbol{c}_L = (\boldsymbol{c}_1, \ldots, \boldsymbol{c}_\ell, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_m)$ of the ladder code $\mathcal{C}_L$ over $\mathbb{F}_q$, where

- $\boldsymbol{c}_i \in \mathcal{C}_1$, $1 \leqslant i \leqslant \ell$, from step 1.
- $\boldsymbol{r}_i = (\boldsymbol{g}^i_1, \ldots, \boldsymbol{g}^i_{n'_i-\ell})$, $2 \leqslant i \leqslant m$, from step 5.

1: For $1 \leqslant i \leqslant \ell$, encode $\boldsymbol{u}_i$ according to the code $\mathcal{C}_1$ to obtain a codeword $\boldsymbol{c}_i = (c_{i,1}, c_{i,2}, \ldots, c_{i,n})$, i.e.,

$$\boldsymbol{c}_i = \mathcal{E}_{\mathcal{C}_1}(\boldsymbol{u}_i),\ \forall 1 \leqslant i \leqslant \ell.$$

2: For $2 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant \ell$, compute the ith level syndromes by

$$\boldsymbol{s}^i_j = (s^i_{j,1}, s^i_{j,2}, \ldots, s^i_{j,v_i}) = \boldsymbol{c}_j H_i^T.$$

Since $\boldsymbol{s}^i_j$ is a v_i-dimensional vector over $\mathbb{F}_q$, we treat it as a symbol in $\mathbb{F}_{q^{v_i}}$. See Fig. 1.

3: For $2 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n'_i - \ell$, calculate the ith level parity-check symbols $\boldsymbol{p}^i_j$ in $\mathbb{F}_{q^{v_i}}$, by encoding syndromes $(\boldsymbol{s}^i_1, \ldots, \boldsymbol{s}^i_\ell)$ with the code $\mathcal{C}'_i$, i.e.,

$$(\boldsymbol{s}^i_1, \ldots, \boldsymbol{s}^i_\ell, \boldsymbol{p}^i_1, \ldots, \boldsymbol{p}^i_{n'_i-\ell}) = \mathcal{E}_{\mathcal{C}'_i}(\boldsymbol{s}^i_1, \ldots, \boldsymbol{s}^i_\ell).$$

See Fig. 2.

4: Encode the parity-check symbols obtained in step 3. For $2 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n'_i - \ell$, apply the code $\mathcal{C}''_i$ to encode $\boldsymbol{p}^i_j$ to obtain $\boldsymbol{g}^i_j$, i.e.,

$$\boldsymbol{g}^i_j = \mathcal{E}_{\mathcal{C}''_i}(\boldsymbol{p}^i_j),$$

where $\boldsymbol{p}^i_j$ is treated as a vector of size v_i over $\mathbb{F}_q$. See Fig. 2.

5: For $2 \leqslant i \leqslant m$, represent the ith level shared redundancy $\boldsymbol{r}_i$ in the form of $\boldsymbol{r}_i = (\boldsymbol{g}^i_1, \ldots, \boldsymbol{g}^i_{n'_i-\ell})$.

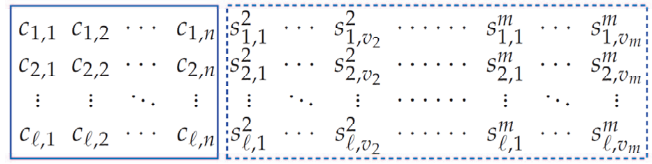

Fig. 1. Step 2 of the encoding procedure in Construction 1.

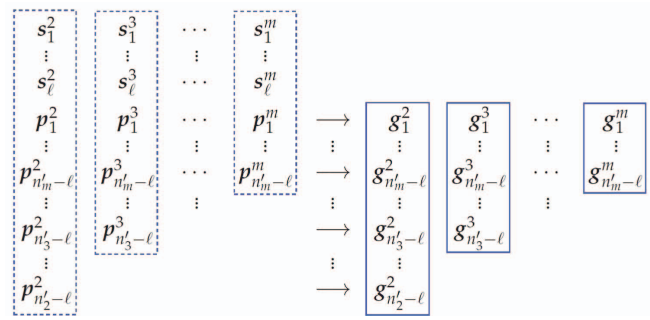

Fig. 2. Steps 3 and 4 of the encoding procedure in Construction 1.

Remark 1. For an m-level ladder code $\mathcal{C}_L$ obtained in Construction 1, its codeword $\boldsymbol{c}_L = (\boldsymbol{c}_1, \ldots, \boldsymbol{c}_\ell, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_m)$ consists of two ingredients: 1) the ℓ sub-blocks $\boldsymbol{c}_i$, $1 \leqslant i \leqslant \ell$, each representing a codeword in $\mathcal{C}_1$, and 2) a total of $m-1$ parts of shared redundancy denoted by $\boldsymbol{r}_i$, $2 \leqslant i \leqslant m$.

Referring to Fig. 1 and Fig. 2, a codeword $\boldsymbol{c}_L$ is comprised of symbols in the regions formed by the solid lines.

Remark 2. As in the construction of tensor product codes [5], [10], Construction 1 for ladder codes is based on operations over the base field $\mathbb{F}_q$ as well as its extension fields; see step 3 in Construction 1. One possible variation of Construction 1 is to modify step 3 by using different component codes so that it is also carried out over the same base field $\mathbb{F}_q$. Such a variation is omitted here due to space constraints. We only focus on Construction 1 in this paper.

B. Minimum Distance of Ladder Codes

The following theorem gives the code parameters of a ladder code $\mathcal{C}_L$ generated by Construction 1.

Theorem 1. *An m-level ladder code $\mathcal{C}_L$ from Construction 1 is a linear code over $\mathbb{F}_q$ of length $n_L = n\ell + \sum_{i=2}^{m} n_i''(n_i' - \ell)$ and dimension $k_L = k_1\ell$. Its minimum distance d_L is lower bounded by d_L^* as*

$$d_L \geqslant d_L^* = \min\left\{\delta_2\tilde{d}_1, \delta_3\tilde{d}_2, \ldots, \delta_m\tilde{d}_{m-1}, d_m\right\},$$

where $\tilde{d}_i = \min\{d_i, d_{i+1}''\}$ for $1 \leqslant i \leqslant m-1$.

Proof: From the code construction procedure, the code length and dimension can be easily determined. In the following, we derive a lower bound on the minimum distance.

Let $\boldsymbol{c}_L$ be a nonzero codeword of the ladder code $\mathcal{C}_L$. Then, $\boldsymbol{c}_L$ contains nonzero vectors $\boldsymbol{c}_i \in \mathcal{C}_1$, $1 \leqslant i \leqslant \ell$. Let the subscripts of all these nonzero vectors form a set $\Phi \subseteq \{1, 2, \ldots, \ell\}$.

Consider the first case that there exists a nonzero vector $\boldsymbol{c}_\lambda \notin \mathcal{C}_2$, $\lambda \in \Phi$. Then, the syndrome $\boldsymbol{s}_\lambda^2 \neq \boldsymbol{0}$, so there are at least δ_2 nonzero symbols in $(\boldsymbol{s}_1^2, \ldots, \boldsymbol{s}_\ell^2, \boldsymbol{p}_1^2, \ldots, \boldsymbol{p}_{n_2'-\ell}^2)$. For any $1 \leqslant j \leqslant \ell$, $\boldsymbol{s}_j^2 \neq \boldsymbol{0}$, we have $w_q(\boldsymbol{c}_j) \geqslant d_1$. For any $1 \leqslant j \leqslant n_2' - \ell$, $\boldsymbol{p}_j^2 \neq \boldsymbol{0}$, we have $w_q(\boldsymbol{g}_j^2) \geqslant d_2''$. Thus, in total, $w_q(\boldsymbol{c}_L) \geqslant \delta_2 \min\{d_1, d_2''\}$.

For the second case, if $\boldsymbol{c}_i \in \mathcal{C}_2$, for all $i \in \Phi$, and there exists a nonzero vector $\boldsymbol{c}_\lambda \notin \mathcal{C}_3$, $\lambda \in \Phi$. Then, the syndrome $\boldsymbol{s}_\lambda^3 \neq \boldsymbol{0}$, so there are at least δ_3 nonzero symbols in $(\boldsymbol{s}_1^3, \ldots, \boldsymbol{s}_\ell^3, \boldsymbol{p}_1^3, \ldots, \boldsymbol{p}_{n_3'-\ell}^3)$. For any $1 \leqslant j \leqslant \ell$, $\boldsymbol{s}_j^3 \neq \boldsymbol{0}$, we have $w_q(\boldsymbol{c}_j) \geqslant d_2$. For any $1 \leqslant j \leqslant n_3' - \ell$, $\boldsymbol{p}_j^3 \neq \boldsymbol{0}$, we have $w_q(\boldsymbol{g}_j^3) \geqslant d_3''$. Thus, in total, $w_q(\boldsymbol{c}_L) \geqslant \delta_3 \min\{d_2, d_3''\}$.

Similarly, for $3 \leqslant i \leqslant m-1$, if $\boldsymbol{c}_j \in \mathcal{C}_i$, for all $j \in \Phi$, and there exists a nonzero vector $\boldsymbol{c}_\lambda \notin \mathcal{C}_{i+1}$, $\lambda \in \Phi$. It can be shown that $w_q(\boldsymbol{c}_L) \geqslant \delta_{i+1} \min\{d_i, d_{i+1}''\}$.

For the last case, if $\boldsymbol{c}_i \in \mathcal{C}_m$, for all $i \in \Phi$, then it is clear that $w_q(\boldsymbol{c}_L) \geqslant d_m$. ∎

The following corollary follows from Theorem 1. We give a condition under which the exact minimum distance can be determined.

Corollary 2. *For an m-level ladder code $\mathcal{C}_L$ generated by Construction 1,*
1) if $d_i'' = d_{i-1}$ for all $2 \leqslant i \leqslant m$, then

$$d_L \geqslant d_L^* = \min\left\{\delta_2 d_1, \delta_3 d_2, \ldots, \delta_m d_{m-1}, d_m\right\};$$

2) if $\delta_i \min\{d_{i-1}, d_i''\} \geqslant d_m$ for all $2 \leqslant i \leqslant m$, then

$$d_L = d_m.$$

Proof: The first claim is evident, by applying Theorem 1. Here, we prove the second claim. On the one hand, since $\delta_i \min\{d_{i-1}, d_i''\} \geqslant d_m$ for all $2 \leqslant i \leqslant m$, we have $d_L \geqslant d_L^* = d_m$. On the other hand, there exists a codeword $\boldsymbol{c}_L \in \mathcal{C}_L$ with weight d_m. To see this, let $\boldsymbol{c}_1 \in \mathcal{C}_1$ be a codeword with weight d_m and $\boldsymbol{c}_i \in \mathcal{C}_1$, $2 \leqslant i \leqslant \ell$, be the all-zero codeword. It is not hard to verify that the corresponding codeword $\boldsymbol{c}_L \in \mathcal{C}_L$ has weight d_m. Thus, we have $d_L \leqslant d_m$. ∎

Now, we present an example of a two-level ladder code to illustrate the encoding procedure of Construction 1.

Example 1. Let $\mathcal{C}_1$ be the $[8, 7, 2]_2$ single parity code, with parity-check matrix $H_1 = [1\ 1\ 1\ 1\ 1\ 1\ 1\ 1]$. Let $\mathcal{C}_2 \subset \mathcal{C}_1$ be the $[8, 4, 4]_2$ extended Hamming code with parity-check matrix $H_{\mathcal{C}_2}$,

$$H_{\mathcal{C}_2} = \left[\begin{array}{c} H_1 \\ \hline H_2 \end{array}\right] = \left[\begin{array}{cccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{array}\right].$$

Let $\mathcal{C}_2'$ be the $[\ell+1, \ell, 2]_{2^3}$ systematic single parity code. Let $\mathcal{C}_2''$ be the $[4, 3, 2]_2$ systematic single parity code. Thus, the two-level ladder code $\mathcal{C}_L$ is an $[n_L = 8\ell + 4, k_L = 7\ell, d_L = 4]_2$ code. Note that, for $2 \leqslant \ell \leqslant 7$, from the online table [8], $\mathcal{C}_L$ achieves the *optimal* minimum distance.

Suppose that $\ell = 2$ and the two input information vectors are: $\boldsymbol{u}_1 = (1\ 0\ 1\ 0\ 0\ 0\ 0)$ and $\boldsymbol{u}_2 = (1\ 1\ 1\ 1\ 0\ 0\ 0)$. From Construction 1, we have $\boldsymbol{c}_1 = (1\ 0\ 1\ 0\ 0\ 0\ 0\ 0)$ and $\boldsymbol{c}_2 = (1\ 1\ 1\ 1\ 0\ 0\ 0\ 0)$, so $\boldsymbol{s}_1^2 = (0\ 1\ 0)$ and $\boldsymbol{s}_2^2 = (1\ 0\ 0)$. Then, we obtain $\boldsymbol{r}_2 = \boldsymbol{g}_1^2 = (1\ 1\ 0\ 0)$. Thus, the output codeword $\boldsymbol{c}_L = (\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{r}_2) = (1\,0\,1\,0\,0\,0\,0\,0, 1\,1\,1\,1\,0\,0\,0\,0, 1\,1\,0\,0)$. □

III. Correctable Error-Erasure Pattern and Decoding Algorithm

In this section, we study the correctable error-erasure patterns for ladder codes. A decoding algorithm that can correct those patterns is proposed. Explicit results on correctable erasure patterns and correctable error patterns are presented.

A. Correction Capability of A Linear Code and Its Cosets

To study the error-erasure correcting capability of ladder codes, we start by investigating the correction capability of a linear code and its cosets.

Let us introduce the erasure symbol and related operations. Let ? represent an erasure. We extend the addition operation over $\mathbb{F}_q$ to $\mathbb{F}_q \cup \{?\}$ by defining $x + ? = ? + x = ?$ for $x \in \mathbb{F}_q \cup \{?\}$.

Consider a code $\mathcal{C}$ of length n over $\mathbb{F}_q$. The set of its correctable error-erasure patterns is defined as follows.

Definition 3. *Let $\mathcal{T}(\mathcal{C})$ be a set of vectors $\boldsymbol{e}$ of length n over $\mathbb{F}_q \cup \{?\}$. We say that $\mathcal{T}(\mathcal{C})$ is a set of correctable error-erasure patterns for the code $\mathcal{C}$ if for any given $\boldsymbol{e} \in \mathcal{T}(\mathcal{C})$, it satisfies the following condition: for every $\boldsymbol{c} \in \mathcal{C}$, the equation $\boldsymbol{c} + \boldsymbol{e} = \boldsymbol{c}' + \boldsymbol{e}'$, where $\boldsymbol{c}' \in \mathcal{C}$ and $\boldsymbol{e}' \in \mathcal{T}(\mathcal{C})$, implies that $\boldsymbol{c}' = \boldsymbol{c}$.*

Based on $\mathcal{T}(\mathcal{C})$, we define the detectable but uncorrectable error-erasure patterns below.

Definition 4. *A vector $\boldsymbol{e}$ of length n over $\mathbb{F}_q \cup \{?\}$ is a detectable but uncorrectable error-erasure pattern for the code $\mathcal{C}$ if it satisfies the following condition: for every $\boldsymbol{c} \in \mathcal{C}$, $\boldsymbol{y} = \boldsymbol{c} + \boldsymbol{e}$ cannot be expressed as $\boldsymbol{y} = \boldsymbol{c}' + \boldsymbol{e}'$, where $\boldsymbol{c}' \in \mathcal{C}$ and $\boldsymbol{e}' \in \mathcal{T}(\mathcal{C})$. We denote the set of all such detectable but uncorrectable error-erasure patterns by $\Delta(\mathcal{C})$.*

Remark 3. It is clear that the two sets $\mathcal{T}(\mathcal{C})$ and $\Delta(\mathcal{C})$ are disjoint; that is, if an error-erasure pattern $\boldsymbol{e} \in \mathcal{T}(\mathcal{C})$, then we have $\boldsymbol{e} \notin \Delta(\mathcal{C})$.

Based on $\mathcal{T}(\mathcal{C})$, we can also define a decoder $\mathcal{D}_{\mathcal{C}}$ for $\mathcal{C}$:

$$\mathcal{D}_{\mathcal{C}} : (\mathbb{F}_q \cup \{?\})^n \to \mathcal{C} \cup \{\text{“e”}\},$$

where “e” is a decoding failure indicator. For a received word $\boldsymbol{y} = \boldsymbol{c} + \boldsymbol{e}$, where $\boldsymbol{c} \in \mathcal{C}$ and $\boldsymbol{e} \in (\mathbb{F}_q \cup \{?\})^n$, the decoder $\mathcal{D}_{\mathcal{C}}$ of $\mathcal{C}$ searches for a codeword $\hat{\boldsymbol{c}} \in \mathcal{C}$ and an error-erasure pattern $\hat{\boldsymbol{e}} \in \mathcal{T}(\mathcal{C})$ such that $\boldsymbol{y} = \hat{\boldsymbol{c}} + \hat{\boldsymbol{e}}$:

1) If such $\hat{\boldsymbol{c}}$ and $\hat{\boldsymbol{e}}$ exist, then they are unique and the decoder $\mathcal{D}_{\mathcal{C}}$ outputs $\hat{\boldsymbol{c}}$.

2) If such $\hat{\boldsymbol{c}}$ and $\hat{\boldsymbol{e}}$ do not exist, then the decoder $\mathcal{D}_{\mathcal{C}}$ outputs a decoding failure indicator “e”.

Remark 4. If $\boldsymbol{e} \in \mathcal{T}(\mathcal{C})$, the decoder $\mathcal{D}_{\mathcal{C}}$ will output the correct codeword; if $\boldsymbol{e} \in \Delta(\mathcal{C})$, the decoder $\mathcal{D}_{\mathcal{C}}$ will detect the error and output a decoding failure “e”.

In this paper, we will use the cosets of a linear code. For an $[n, k, d]_q$ linear code $\mathcal{C}$ with a parity-check matrix H, let $\mathcal{C}(\boldsymbol{s})$ be the set of all vectors in $\mathbb{F}_q^n$ that have syndrome $\boldsymbol{s}$, namely,

$$\mathcal{C}(\boldsymbol{s}) = \{\boldsymbol{x} \in \mathbb{F}_q^n : \boldsymbol{x}H^T = \boldsymbol{s}\}.$$

The code $\mathcal{C}(\boldsymbol{s})$ is a coset of $\mathcal{C}$. For $\boldsymbol{s} = \boldsymbol{0}$, we have $\mathcal{C}(\boldsymbol{s}) = \mathcal{C}$. It is easy to verify the following two claims:

1) A set of correctable error-erasure patterns for $\mathcal{C}$ is also a set of correctable error-erasure patterns for its coset $\mathcal{C}(\boldsymbol{s})$, i.e., $\mathcal{T}(\mathcal{C}) = \mathcal{T}\big(\mathcal{C}(\boldsymbol{s})\big)$.

2) A set of detectable but uncorrectable error-erasure patterns for $\mathcal{C}$ is also a set of detectable but uncorrectable error-erasure patterns for $\mathcal{C}(\boldsymbol{s})$, i.e., $\Delta(\mathcal{C}) = \Delta\big(\mathcal{C}(\boldsymbol{s})\big)$.

B. A General Result on Correctable Error-Erasure Patterns

Now we are ready to investigate the correctable error-erasure patterns of a ladder code $\mathcal{C}_L$ which is generated by Construction 1.

Suppose a codeword $\boldsymbol{c}_L \in \mathcal{C}_L$ is transmitted, and the corresponding received word is $\boldsymbol{y} = \boldsymbol{c}_L + \boldsymbol{e}$, $\boldsymbol{e} \in (\mathbb{F}_q \cup \{?\})^{n_L}$. More specifically, we use the following notation:

1) the transmitted codeword is $\boldsymbol{c}_L = (\boldsymbol{c}_1, \ldots, \boldsymbol{c}_\ell, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_m)$, where $\boldsymbol{r}_i = (\boldsymbol{g}_1^i, \ldots, \boldsymbol{g}_{n_i'-\ell}^i)$ for $2 \leqslant i \leqslant m$;

2) the error-erasure vector is $\boldsymbol{e} = (\boldsymbol{e}_1, \ldots, \boldsymbol{e}_\ell, \boldsymbol{e}_2'', \ldots, \boldsymbol{e}_m'')$, where $\boldsymbol{e}_i'' = (\boldsymbol{e}_1^i, \ldots, \boldsymbol{e}_{n_i'-\ell}^i)$ for $2 \leqslant i \leqslant m$;

3) the received word is $\boldsymbol{y} = (\boldsymbol{y}_1, \ldots, \boldsymbol{y}_\ell, \boldsymbol{y}_2'', \ldots, \boldsymbol{y}_m'')$, where $\boldsymbol{y}_i'' = (\boldsymbol{y}_1^i, \ldots, \boldsymbol{y}_{n_i'-\ell}^i)$ for $2 \leqslant i \leqslant m$.

Given the error-erasure vector $\boldsymbol{e}$, we define the following new error-erasure vector $\boldsymbol{e}_i' = (e_{i,1}', e_{i,2}', \ldots, e_{i,n_i'}')$, $2 \leqslant i \leqslant m$. For $1 \leqslant j \leqslant \ell$,

$$e_{i,j}' = \begin{cases} 0 & \boldsymbol{e}_j \in \mathcal{T}(\mathcal{C}_{i-1}) \\ ? & \boldsymbol{e}_j \in \Delta(\mathcal{C}_{i-1}) \\ w & \boldsymbol{e}_j \notin \mathcal{T}(\mathcal{C}_{i-1}) \cup \Delta(\mathcal{C}_{i-1}); \end{cases} \tag{1}$$

for $1 \leqslant j \leqslant n_i' - \ell$,

$$e_{i,j+\ell}' = \begin{cases} 0 & \boldsymbol{e}_j^i \in \mathcal{T}(\mathcal{C}_i'') \\ ? & \boldsymbol{e}_j^i \in \Delta(\mathcal{C}_i'') \\ w & \boldsymbol{e}_j^i \notin \mathcal{T}(\mathcal{C}_i'') \cup \Delta(\mathcal{C}_i''), \end{cases} \tag{2}$$

where w is an indeterminate symbol. The above assignment for $e_{i,j}'$, $1 \leqslant j \leqslant \ell$, can be interpreted as follows: $e_{i,j}' = 0$ if $\boldsymbol{e}_j$ is correctable, $e_{i,j}' =?$ if $\boldsymbol{e}_j$ is detectable but uncorrectable, and $e_{i,j}' = w$ if $\boldsymbol{e}_j$ is miscorrected. The same interpretation holds for $e_{i,j+\ell}'$, $1 \leqslant j \leqslant n_i' - \ell$.

With the error-erasure vector $\boldsymbol{e}_i'$ defined above in (1) and (2), we say that $\boldsymbol{e}_i'$ is correctable if $\boldsymbol{e}_i' \in \mathcal{T}(\mathcal{C}_i')$ for all $w \in \mathbb{F}_{q^{v_i}}$. In other words, the set of vectors obtained by replacing each w in $\boldsymbol{e}_i'$ by all possible elements in $\mathbb{F}_{q^{v_i}}$ are in $\mathcal{T}(\mathcal{C}_i')$.

The following theorem describes the correctable error-erasure patterns for a ladder code $\mathcal{C}_L$.

Theorem 5. *An m-level ladder code $\mathcal{C}_L$ from Construction 1 corrects any error-erasure pattern $\boldsymbol{e} = (\boldsymbol{e}_1, \ldots, \boldsymbol{e}_\ell, \boldsymbol{e}_2'', \ldots, \boldsymbol{e}_m'')$, $\boldsymbol{e} \in (\mathbb{F}_q \cup \{?\})^{n_L}$, that satisfies the following two conditions:*
1) for $1 \leqslant i \leqslant \ell$, the error-erasure pattern $\boldsymbol{e}_i$ is correctable by $\mathcal{C}_m$, i.e., $\boldsymbol{e}_i \in \mathcal{T}(\mathcal{C}_m)$;
2) for $2 \leqslant i \leqslant m$, the ith level error-erasure pattern $\boldsymbol{e}_i' = (e_{i,1}', e_{i,2}', \ldots, e_{i,n_i'}')$, defined in (1) and (2), is correctable by $\mathcal{C}_i'$, i.e., $\boldsymbol{e}_i' \in \mathcal{T}(\mathcal{C}_i')$ for all $w \in \mathbb{F}_{q^{v_i}}$.

C. A Decoding Algorithm for Ladder Codes

To prove Theorem 5, we present a decoding algorithm, referred to as Algorithm 1, for a ladder code $\mathcal{C}_L$. It employs the following decoders for different component codes used in Construction 1:

a) The decoder $\mathcal{D}_{\mathcal{C}_i}$ for a coset of the code $\mathcal{C}_i$ with syndrome $\boldsymbol{s}$, for $1 \leqslant i \leqslant m$, is defined by

$$\mathcal{D}_{\mathcal{C}_i} : (\mathbb{F}_q \cup \{?\})^n \times (\mathbb{F}_q \cup \{?\})^{n-k_i} \to \mathcal{C}_i(\boldsymbol{s}) \cup \{\text{“e”}\}.$$

For a length-n input vector $\boldsymbol{y}$ and a length-$(n - k_i)$ syndrome $\boldsymbol{s}$ without erasures, the decoder $\mathcal{D}_{\mathcal{C}_i}(\boldsymbol{y}, \boldsymbol{s})$ searches for a codeword $\hat{\boldsymbol{c}} \in \mathcal{C}_i(\boldsymbol{s})$ and an error-erasure pattern $\hat{\boldsymbol{e}} \in \mathcal{T}(\mathcal{C}_i)$ [1] such that $\boldsymbol{y} = \hat{\boldsymbol{c}} + \hat{\boldsymbol{e}}$. If such $\hat{\boldsymbol{c}}$ and $\hat{\boldsymbol{e}}$ exist, the decoder outputs $\hat{\boldsymbol{c}}$; otherwise, the decoder returns a decoding failure “e”.

b) The decoder $\mathcal{D}_{\mathcal{C}_i''}$ for the code $\mathcal{C}_i''$, for $2 \leqslant i \leqslant m$, is defined by

$$\mathcal{D}_{\mathcal{C}_i''} : (\mathbb{F}_q \cup \{?\})^{n_i''} \to \mathcal{C}_i'' \cup \{\text{“e”}\}.$$

For a length-n_i'' input vector $\boldsymbol{y}$, the decoder $\mathcal{D}_{\mathcal{C}_i''}(\boldsymbol{y})$ searches for a codeword $\hat{\boldsymbol{c}} \in \mathcal{C}_i''$ and an error-erasure pattern $\hat{\boldsymbol{e}} \in \mathcal{T}(\mathcal{C}_i'')$ such that $\boldsymbol{y} = \hat{\boldsymbol{c}} + \hat{\boldsymbol{e}}$. If such $\hat{\boldsymbol{c}}$ and $\hat{\boldsymbol{e}}$ exist, the decoder outputs $\hat{\boldsymbol{c}}$; otherwise, the decoder returns a decoding failure “e”.

[1] Here, we use $\mathcal{T}(\mathcal{C}_i)$, since we have $\mathcal{T}\big(\mathcal{C}_i(\boldsymbol{s})\big) = \mathcal{T}(\mathcal{C}_i)$.

c) The decoder $\mathcal{D}_{\mathcal{C}_i'}$ for the code $\mathcal{C}_i'$, for $2 \leqslant i \leqslant m$, is defined by

$$\mathcal{D}_{\mathcal{C}_i'} : (\mathbb{F}_{q^{v_i}} \cup \{?\})^{n_i'} \to \mathcal{C}_i' \cup \{\text{"e"}\}.$$

For a length-n_i' input vector $\boldsymbol{y}$, the decoder $\mathcal{D}_{\mathcal{C}_i'}(\boldsymbol{y})$ searches for a codeword $\hat{\boldsymbol{c}} \in \mathcal{C}_i'$ and an error-erasure pattern $\hat{\boldsymbol{e}} \in \mathcal{T}(\mathcal{C}_i')$ such that $\boldsymbol{y} = \hat{\boldsymbol{c}} + \hat{\boldsymbol{e}}$. If such $\hat{\boldsymbol{c}}$ and $\hat{\boldsymbol{e}}$ exist, the decoder outputs $\hat{\boldsymbol{c}}$; otherwise, the decoder returns a decoding failure "e".

Note that each decoder defined above is merely based on the correctable set of the corresponding code.

The decoding algorithm for $\mathcal{C}_L$ is outlined as follows.

Algorithm 1: Decoding Procedure for $\mathcal{C}_L$

Input: received word $\boldsymbol{y} = (\boldsymbol{y}_1, \ldots, \boldsymbol{y}_\ell, \boldsymbol{y}_2'', \ldots, \boldsymbol{y}_m'')$.
Output: information vectors $\boldsymbol{u}_i$, $1 \leqslant i \leqslant \ell$, or a decoding failure indicator "e".

// **Level** 1**:**
1: Let the 1st level syndrome $\hat{\mathbf{s}}_i^1 = \mathbf{0}$, $1 \leqslant i \leqslant \ell$.
2: Let $\mathcal{F} = \{i : \mathcal{D}_{\mathcal{C}_1}(\boldsymbol{y}_i, \hat{\mathbf{s}}_i^1) == \text{"e"}, 1 \leqslant i \leqslant \ell\}$.
3: **for** $1 \leqslant i \leqslant \ell$ and $i \notin \mathcal{F}$ **do**
4: $\quad \hat{\boldsymbol{c}}_i \leftarrow \mathcal{D}_{\mathcal{C}_1}(\boldsymbol{y}_i, \hat{\mathbf{s}}_i^1)$.
5: **end for**
// **Level** 2 $-$ **Level** m**:**
1: **for** $\mu = 2, 3, \ldots, m$ **do**
2: $\quad$ Let $\mathcal{F}_\mu = \{i : \mathcal{D}_{\mathcal{C}_\mu''}(\boldsymbol{y}_i^\mu) == \text{"e"}, 1 \leqslant i \leqslant n_\mu' - \ell\}$.
3: $\quad$ **for** $1 \leqslant i \leqslant n_\mu' - \ell$ and $i \notin \mathcal{F}_\mu$ **do**
4: $\quad\quad \hat{\boldsymbol{g}}_i^\mu \leftarrow \mathcal{D}_{\mathcal{C}_\mu''}(\boldsymbol{y}_i^\mu)$, and $\hat{\boldsymbol{p}}_i^\mu \leftarrow \mathcal{E}_{\mathcal{C}_\mu''}^{-1}(\hat{\boldsymbol{g}}_i^\mu)$.
5: $\quad$ **end for**
6: $\quad$ Let $\mathbf{X}_\mu = (x_1, x_2, \ldots, x_{n_\mu'})$ be the word over $\mathbb{F}_{q^{v_\mu}} \cup \{?\}$ that is defined as follows:
$\quad$ for $1 \leqslant i \leqslant \ell$,

$$x_i = \begin{cases} \hat{\boldsymbol{c}}_i H_\mu^T & \text{if } i \notin \mathcal{F} \\ ? & \text{otherwise;} \end{cases}$$

$\quad$ for $1 \leqslant j \leqslant n_\mu' - \ell$,

$$x_{j+\ell} = \begin{cases} \hat{\boldsymbol{p}}_j^\mu & \text{if } j \notin \mathcal{F}_\mu \\ ? & \text{otherwise.} \end{cases}$$

7: $\quad$ **if** $\mathcal{D}_{\mathcal{C}_\mu'}(\mathbf{X}_\mu) == \text{"e"}$ **then**
8: $\quad\quad$ Go to step 17.
9: $\quad$ **else**
10: $\quad\quad$ Get syndromes $(\hat{\mathbf{s}}_1^\mu, \ldots, \hat{\mathbf{s}}_\ell^\mu)$ by:

$$(\hat{\mathbf{s}}_1^\mu, \ldots, \hat{\mathbf{s}}_\ell^\mu, \hat{\boldsymbol{p}}_1^\mu, \ldots, \hat{\boldsymbol{p}}_{n_\mu'-\ell}^\mu) \leftarrow \mathcal{D}_{\mathcal{C}_\mu'}(\mathbf{X}_\mu).$$

11: $\quad$ **end if**
12: $\quad$ Update the index list $\mathcal{F}$:

$$\mathcal{F} = \{i : \mathcal{D}_{\mathcal{C}_\mu}(\boldsymbol{y}_i, (\hat{\mathbf{s}}_i^1, \hat{\mathbf{s}}_i^2, \ldots, \hat{\mathbf{s}}_i^\mu)) == \text{"e"}, 1 \leqslant i \leqslant \ell\}.$$

13: $\quad$ **for** $1 \leqslant i \leqslant \ell$ and $i \notin \mathcal{F}$ **do**
14: $\quad\quad$ Update $\hat{\boldsymbol{c}}_i$: $\hat{\boldsymbol{c}}_i \leftarrow \mathcal{D}_{\mathcal{C}_\mu}(\boldsymbol{y}_i, (\hat{\mathbf{s}}_i^1, \hat{\mathbf{s}}_i^2, \ldots, \hat{\mathbf{s}}_i^\mu))$.
15: $\quad$ **end for**
16: **end for**
// Decoding Output:
17: **if** $\mathcal{F} == \emptyset$ **then**
18: $\quad$ **for** $1 \leqslant i \leqslant \ell$ **do**
19: $\quad\quad \boldsymbol{u}_i \leftarrow \mathcal{E}_{\mathcal{C}_1}^{-1}(\hat{\boldsymbol{c}}_i)$, and output $\boldsymbol{u}_i$.
20: $\quad$ **end for**
21: **else**
22: $\quad$ Output a decoding failure "e".
23: **end if**

Claim 6. *For a ladder code $\mathcal{C}_L$, Algorithm 1 corrects any error-erasure pattern $\boldsymbol{e} = (\boldsymbol{e}_1, \ldots, \boldsymbol{e}_\ell, \boldsymbol{e}_2'', \ldots, \boldsymbol{e}_m'')$ that satisfies the two conditions in Theorem 5.*

Proof: The proof follows from the decoding procedure of Algorithm 1. At level 1, we obtain the correct syndromes $\hat{\mathbf{s}}_i^1 = \mathbf{0}$, for $1 \leqslant i \leqslant \ell$. Then, these syndromes are used in decoding for the received words $\boldsymbol{y}_i$, $1 \leqslant i \leqslant \ell$.

In the loop $\mu = 2$, since the error-erasure pattern $\boldsymbol{e}_2'$ satisfies condition 2) in Theorem 5, the vector $\mathbf{X}_2$ obtained in step 6 can be decoded successfully. Thus, we obtain the correct syndromes $\hat{\mathbf{s}}_i^2$, $1 \leqslant i \leqslant \ell$. Then, the syndromes $\hat{\mathbf{s}}_i^1$ and $\hat{\mathbf{s}}_i^2$, $1 \leqslant i \leqslant \ell$, will be used to help decode the received words $\boldsymbol{y}_i$, $1 \leqslant i \leqslant \ell$.

Similarly, for each loop $3 \leqslant \mu \leqslant m$, since the error-erasure pattern $\boldsymbol{e}_\mu'$ satisfies condition 2) in Theorem 5, we can obtain the correct syndromes $\hat{\mathbf{s}}_i^\mu$, $1 \leqslant i \leqslant \ell$.

Therefore, when the decoding runs until the last loop, i.e., $\mu = m$, we have obtained all the correct syndromes $\hat{\mathbf{s}}_i^1, \hat{\mathbf{s}}_i^2, \ldots, \hat{\mathbf{s}}_i^m$, $1 \leqslant i \leqslant \ell$. Since the error-erasure patterns $\boldsymbol{e}_i$, $1 \leqslant i \leqslant \ell$, satisfy condition 1) in Theorem 5, using all these correct syndromes for coset decoding, the error-erasure patterns $\boldsymbol{e}_i$, $1 \leqslant i \leqslant \ell$, can be corrected. Thus, in this last loop, the decoder is guaranteed to return the correct information vectors $\boldsymbol{u}_i$, $1 \leqslant i \leqslant \ell$. ■

Remark 5. The decoding procedure of Algorithm 1 that utilizes the multi-level shared redundancy successively from the lowest level to the highest level mimics climbing up a ladder, which suggests the name of ladder codes.

D. More Explicit Results on Correctable Patterns

In the previous Section III-B, Theorem 5 gives a very general result on correctable error-erasure patterns for ladder codes. Now, we present more *explicit* results on erasure patterns and error patterns separately.

The following notation will be used. For a length-n vector $\boldsymbol{x} \in (\mathbb{F}_q \cup \{?\})^n$, let $\mathcal{N}_\sigma(\boldsymbol{x})$ denote the number of erasures ? in $\boldsymbol{x}$, and let $\mathcal{N}_\tau(\boldsymbol{x})$ denote the number of nonzero elements that belong to $\mathbb{F}_q/\{0\}$ in $\boldsymbol{x}$.

a) *Correctable Erasure Patterns*

Let us consider the case when only erasures occur.

We first choose the following correctable sets for the component codes in Construction 1.

1) For $1 \leqslant i \leqslant m$, choose the correctable set for $\mathcal{C}_i$ as

$$\mathcal{T}(\mathcal{C}_i) = \{\boldsymbol{x} : \boldsymbol{x} \in \{0, ?\}^n \text{ and } \mathcal{N}_\sigma(\boldsymbol{x}) \leqslant d_i - 1\}.$$

Based on $\mathcal{T}(\mathcal{C}_i)$, we have

$$\Delta(\mathcal{C}_i) = \{\boldsymbol{x} : \boldsymbol{x} \in \{0, ?\}^n \text{ and } \mathcal{N}_\sigma(\boldsymbol{x}) \geqslant d_i\}.$$

2) For $2 \leqslant i \leqslant m$, choose the correctable set for $\mathcal{C}_i''$ as

$$\mathcal{T}(\mathcal{C}_i'') = \{\boldsymbol{x} : \boldsymbol{x} \in \{0, ?\}^{n_i''} \text{ and } \mathcal{N}_\sigma(\boldsymbol{x}) \leqslant d_i'' - 1\}.$$

Based on $\mathcal{T}(\mathcal{C}_i'')$, we have

$$\Delta(\mathcal{C}_i'') = \{\boldsymbol{x} : \boldsymbol{x} \in \{0,?\}^{n_i''} \text{ and } \mathcal{N}_\sigma(\boldsymbol{x}) \geqslant d_i''\}.$$

3) For $2 \leqslant i \leqslant m$, choose the correctable set for $\mathcal{C}_i'$ as

$$\mathcal{T}(\mathcal{C}_i') = \{\boldsymbol{x} : \boldsymbol{x} \in \{0,?\}^{n_i'} \text{ and } \mathcal{N}_\sigma(\boldsymbol{x}) \leqslant \delta_i - 1\}.$$

By applying Theorem 5, we directly obtain the following explicit result on correctable erasure patterns.

Lemma 7. *An m-level ladder code $\mathcal{C}_L$ from Construction 1 corrects any erasure pattern $\boldsymbol{e} = (\boldsymbol{e}_1, \ldots, \boldsymbol{e}_\ell, \boldsymbol{e}_2'', \ldots, \boldsymbol{e}_m'')$, $\boldsymbol{e} \in \{0,?\}^{n_L}$, that satisfies the following two conditions:*
1) for $1 \leqslant i \leqslant \ell$, $\mathcal{N}_\sigma(\boldsymbol{e}_i) \leqslant d_m - 1$;
2) for $2 \leqslant i \leqslant m$, $a_i^1 + a_i^2 \leqslant \delta_i - 1$, where
$a_i^1 = |\{j : 1 \leqslant j \leqslant \ell, \mathcal{N}_\sigma(\boldsymbol{e}_j) \geqslant d_{i-1}\}|$,
$a_i^2 = |\{j : 1 \leqslant j \leqslant n_i' - \ell, \mathcal{N}_\sigma(\boldsymbol{e}_j^i) \geqslant d_i''\}|$.

Recall that d_L^* is a lower bound on the minimum distance of $\mathcal{C}_L$. The following lemma shows that $\mathcal{C}_L$ corrects any $d_L^* - 1$ erasures.

Lemma 8. *An m-level ladder code $\mathcal{C}_L$ from Construction 1 corrects any erasure pattern of less than d_L^* erasures.*

Proof: We only need to show that any erasure pattern of $d_L^* - 1$ erasures satisfies the two conditions in Lemma 7, so it can be corrected. To see this, first let us assume that condition 1) in Lemma 7 is violated. Then, for some integer j, $1 \leqslant j \leqslant \ell$, we have $\mathcal{N}_\sigma(\boldsymbol{e}_j) \geqslant d_m \geqslant d_L^*$, which violates the assumption that there are only $d_L^* - 1$ erasures. Second, let us assume that condition 2) is violated; that is, for some integer i, $2 \leqslant i \leqslant m$, we have $a_i^1 + a_i^2 \geqslant \delta_i$. It means that $\sum_{j=1}^{\ell} \mathcal{N}_\sigma(\boldsymbol{e}_j) + \sum_{j=1}^{n_i'-\ell} \mathcal{N}_\sigma(\boldsymbol{e}_j^i) \geqslant \delta_i \min\{d_{i-1}, d_i''\} \geqslant d_L^*$, which violates the assumption that there are only $d_L^* - 1$ erasures. ■

Let us give a simple example on correctable erasure patterns. It shows that ladder codes can correct more than $d_L^* - 1$ erasures in some cases.

Example 2. Consider the $[n_L = 8\ell + 4, k_L = 7\ell, d_L = 4]_2$ ladder code $\mathcal{C}_L$ constructed in Example 1. According to Lemma 8, any erasure pattern of 3 erasures can be corrected. In addition, using Lemma 7, it is easy to verify that some erasure patterns with more than 3 erasures can be corrected, but some are not. For instance, assume that the codeword $\boldsymbol{c}_L = (1\ 0\ 1\ 0\ 0\ 0\ 0\ 0, 1\ 1\ 1\ 1\ 0\ 0\ 0\ 0, 1\ 1\ 0\ 0)$ is sent. The received word $\boldsymbol{y} = (1\ ?\ 1\ ?\ 0\ 0\ 0\ 0, 1\ ?\ 1\ ?\ 0\ 0\ 0\ 0, 1\ 1\ 0\ 0)$ with 4 erasures cannot be corrected. In contrast, the received word $\boldsymbol{y} = (?\ 0\ ?\ ?\ 0\ 0\ 0\ 0, 1\ ?\ 1\ 1\ 0\ 0\ 0\ 0, ?\ 1\ 0\ 0)$ with 5 erasures can be decoded. □

b) *Correctable Error Patterns*

Now, let us consider the case when the transmitted codeword $\boldsymbol{c}_L \in \mathcal{C}_L$ only suffers from errors.

We choose the following correctable sets for the component codes in Construction 1.

1) For $1 \leqslant i \leqslant m$, choose the correctable set for $\mathcal{C}_i$ as

$$\mathcal{T}(\mathcal{C}_i) = \{\boldsymbol{x} : \boldsymbol{x} \in \mathbb{F}_q^n \text{ and } \mathcal{N}_\tau(\boldsymbol{x}) \leqslant \rho_i\},$$

where ρ_i is an integer that satisfies $0 \leqslant \rho_i \leqslant \lfloor \frac{d_i - 1}{2} \rfloor$. The value of ρ_i can be interpreted as the decoding radius of the bounded-distance decoding. Based on $\mathcal{T}(\mathcal{C}_i)$, we can express the detectable but uncorrectable set as

$$\Delta(\mathcal{C}_i) = \left\{\boldsymbol{x} : \boldsymbol{x} \in \mathbb{F}_q^n \text{ and } \boldsymbol{x} \notin \bigcup_{\boldsymbol{y} \in \mathcal{T}(\mathcal{C}_i)} \{\mathcal{C}_i + \boldsymbol{y}\}\right\}.$$

2) For $2 \leqslant i \leqslant m$, choose the correctable set for $\mathcal{C}_i''$ as

$$\mathcal{T}(\mathcal{C}_i'') = \{\boldsymbol{x} : \boldsymbol{x} \in \mathbb{F}_q^{n_i''} \text{ and } \mathcal{N}_\tau(\boldsymbol{x}) \leqslant \rho_i''\},$$

where ρ_i'' is an integer satisfying $0 \leqslant \rho_i'' \leqslant \lfloor \frac{d_i'' - 1}{2} \rfloor$. Based on $\mathcal{T}(\mathcal{C}_i'')$, we have

$$\Delta(\mathcal{C}_i'') = \left\{\boldsymbol{x} : \boldsymbol{x} \in \mathbb{F}_q^{n_i''} \text{ and } \boldsymbol{x} \notin \bigcup_{\boldsymbol{y} \in \mathcal{T}(\mathcal{C}_i'')} \{\mathcal{C}_i'' + \boldsymbol{y}\}\right\}.$$

3) For $2 \leqslant i \leqslant m$, choose the correctable set for $\mathcal{C}_i'$ as

$$\mathcal{T}(\mathcal{C}_i') = \{\boldsymbol{x} : \boldsymbol{x} \in (\mathbb{F}_{q^{v_i}} \cup \{?\})^{n_i'} \text{ and } 2\mathcal{N}_\tau(\boldsymbol{x}) + \mathcal{N}_\sigma(\boldsymbol{x}) \leqslant \delta_i - 1\}.$$

Now, using Theorem 5, it is not hard to obtain the following lemma.

Lemma 9. *An m-level ladder code $\mathcal{C}_L$ from Construction 1 corrects any error pattern $\boldsymbol{e} = (\boldsymbol{e}_1, \ldots, \boldsymbol{e}_\ell, \boldsymbol{e}_2'', \ldots, \boldsymbol{e}_m'')$, $\boldsymbol{e} \in \mathbb{F}_q^{n_L}$, that satisfies the following two conditions:*
1) for $1 \leqslant i \leqslant \ell$, $\mathcal{N}_\tau(\boldsymbol{e}_i) \leqslant \rho_m$;
2) for $2 \leqslant i \leqslant m$, $2\big(a_i^1(\rho_{i-1}) + a_i^2(\rho_i'')\big) + \big(b_i^1(\rho_{i-1}) + b_i^2(\rho_i'')\big) \leqslant \delta_i - 1$, where
$a_i^1(\rho_{i-1}) = |\{j : 1 \leqslant j \leqslant \ell, \boldsymbol{e}_j \notin \mathcal{T}(\mathcal{C}_{i-1}) \cup \Delta(\mathcal{C}_{i-1})\}|$,
$b_i^1(\rho_{i-1}) = |\{j : 1 \leqslant j \leqslant \ell, \boldsymbol{e}_j \in \Delta(\mathcal{C}_{i-1})\}|$,
$a_i^2(\rho_i'') = |\{j : 1 \leqslant j \leqslant n_i' - \ell, \boldsymbol{e}_j^i \notin \mathcal{T}(\mathcal{C}_i'') \cup \Delta(\mathcal{C}_i'')\}|$,
$b_i^2(\rho_i'') = |\{j : 1 \leqslant j \leqslant n_i' - \ell, \boldsymbol{e}_j^i \in \Delta(\mathcal{C}_i'')\}|$.

From Lemma 9, it is clear that the error correcting capability of $\mathcal{C}_L$ depends on the choices of the integers ρ_i and ρ_i''. Note that in Lemma 9, we use the notation such as $a_i^1(\rho_{i-1})$ to explicitly indicate that $a_i^1(\rho_{i-1})$ depends on ρ_{i-1}.

Based on Lemma 9, we have the following theorem. Its proof is omitted due to space constraints.

Theorem 10. *For any length-n_L error pattern $\boldsymbol{e}$ whose Hamming weight is less than $d_L^*/2$, there exist ρ_i, $1 \leqslant i \leqslant m$, and ρ_j'', $2 \leqslant j \leqslant m$, such that the two conditions in Lemma 9 are satisfied.*

Remark 6. Theorem 10 indicates that any received word $\boldsymbol{y}$ with number of errors less than $d_L^*/2$ can be corrected.

IV. Two-Level Ladder Codes Versus Concatenated Codes

In this section, we study the similarity and difference between two-level ladder codes and concatenated codes [7]. It will be shown that compared to a concatenated code, a two-level ladder code can achieve a higher rate for a given minimum distance.

In the following, we denote a two-level (i.e., $m = 2$) ladder code by $\mathcal{C}_L^2$. We also assume that $\mathcal{C}_i''$, $2 \leqslant i \leqslant m$, has minimum distance $d_i'' = d_{i-1}$.

The following corollary on the code parameters of $\mathcal{C}_L^2$ is directly concluded from Corollary 2.

Corollary 11. *A two-level ladder code $\mathcal{C}_L^2$ is a linear code over $\mathbb{F}_q$ of length $n_L = n\ell + n_2''(n_2' - \ell)$, dimension $k_L = k_1\ell$, and minimum distance $d_L \geqslant \min\{\delta_2 d_1, d_2\}$.*

A concatenated code $\mathcal{C}_{cont}$ over $\mathbb{F}_q$ is formed from an inner code $\mathcal{C}_{in}$ and an outer code $\mathcal{C}_{out}$ [7]. Here, let the inner code be an $[n,k,d]_q$ code and the outer code be an $[N,K,D = N-K+1]_{q^k}$ MDS code, which exists whenever $N \leqslant q^k$. Thus, the corresponding $\mathcal{C}_{cont}$ is an $[nN, kK, \geqslant dD]_q$ code.

First, we show that a two-level ladder code $\mathcal{C}_L^2$ can have the same code parameters as those of a corresponding concatenated code $\mathcal{C}_{cont}$. To this end, we choose the following component codes in Construction 1 for constructing $\mathcal{C}_L^2$.

Design I:

1) Let $\mathcal{C}_1$ be an $[n, k_1 = k, d_1 = d]_q$ code, and $\mathcal{C}_2$ be the $[n, 0, \infty]_q$ code with only an all-zero codeword.

2) Let $\mathcal{C}_2'$ be an $[n_2' = N, \ell = K, \delta_2 = D]_{q^k}$ MDS code, where $N \leqslant q^k$.

3) Let $\mathcal{C}_2''$ be an $[n_2'' = n, k_2'' = k, d_2'' = d]_q$ code.

Lemma 12. *From Construction 1 with Design I, the corresponding two-level ladder code $\mathcal{C}_L^2$ over $\mathbb{F}_q$ has code length $n_L = nN$, dimension $k_L = kK$, and minimum distance $d_L \geqslant dD$.*

Proof: From Design I and Corollary 11, the code length, dimension, and minimum distance are obtained. ■

Second, for some cases, a two-level ladder code $\mathcal{C}_L^2$ can even outperform a concatenated code $\mathcal{C}_{cont}$ in the sense of possessing a higher rate but the same minimum distance. To see this, in Construction 1, we choose the following component codes to construct $\mathcal{C}_L^2$.

Design II:

1) Let $\mathcal{C}_1$ be an $[n, k_1 = k, d_1 = d]_q$ code, and $\mathcal{C}_2 \subset \mathcal{C}_1$ be an $[n, k_2, d_2 = dD]_q$ code (here, we assume that $\mathcal{C}_2$ exists with positive dimension $k_2 > 0$ and finite minimum distance $d_2 = dD < \infty$).

2) Let $\mathcal{C}_2'$ be an $[n_2' = N, \ell = K, \delta_2 = D]_{q^{k-k_2}}$ MDS code, which exists whenever $N \leqslant q^{k-k_2}$.

3) Let $\mathcal{C}_2''$ be an $[n_2'' \leqslant n, k_2'' = k - k_2, d_2'' = d]_q$ code. Note that here we can choose $n_2'' \leqslant n$, since an $[n,k,d]_q$ inner code $\mathcal{C}_{in}$ of $\mathcal{C}_{cont}$ exists and $k_2'' < k$.

Lemma 13. *From Construction 1 with Design II, the corresponding two-level ladder code $\mathcal{C}_L^2$ over $\mathbb{F}_q$ has code length $n_L = nK + (N-K)n_2''$, dimension $k_L = kK$, and minimum distance $d_L = dD$.*

Proof: The code parameters are obtained directly from Design II, Corollary 2, and Corollary 11. ■

From Lemma 13, the rate of $\mathcal{C}_L^2$ is $R_L = \frac{kK}{nK+(N-K)n_2''}$. Denote the rate of $\mathcal{C}_{cont}$ by $R_{cont} = \frac{kK}{nN}$. Since $n_2'' \leqslant n$, we have $R_L \geqslant R_{cont}$, where the inequality is strict for many cases, one simple example of which is as follows.

Example 3. Consider a concatenated code $\mathcal{C}_{cont}$ with an $[n = 8, k = 7, d = 2]_2$ inner code $\mathcal{C}_{in}$ and an $[N = \ell+1, K = \ell, D = 2]_{2^7}$ outer code $\mathcal{C}_{out}$. Thus, $\mathcal{C}_{cont}$ is an $[8(\ell+1), 7\ell, 4]_2$ code with rate $R_{cont} = \frac{7\ell}{8(\ell+1)}$.

For comparison, we choose the corresponding ladder code $\mathcal{C}_L^2$ from Design II as the code constructed in Example 1. It is an $[n_L = 8\ell+4, k_L = 7\ell, d_L = 4]_2$ code with rate $R_L = \frac{7\ell}{8\ell+4}$.

Thus, in this case, $\mathcal{C}_L^2$ has a higher rate than that of $\mathcal{C}_{cont}$, while their minimum distances are the same. □

In addition, we briefly compare two-level ladder codes with two-level generalized tensor product codes [5], [12] by using the following example.

Example 4. Let $\mathcal{C}_1$ be the $[16, 15, 2]_2$ single parity code and $\mathcal{C}_2 \subset \mathcal{C}_1$ be the $[16, 11, 4]_2$ extended Hamming code. Choose $\mathcal{C}_2'$ to be the $[\ell+1, \ell, 2]_{2^4}$ single parity code and $\mathcal{C}_2''$ to be the $[5, 4, 2]_2$ single parity code. From Construction 1, the resulting two-level ladder code $\mathcal{C}_L^2$ is an $[n_L = 16\ell+5, k_L = 15\ell, d_L = 4]_2$ code.

In the construction of generalized tensor product codes, we use the same component codes $\mathcal{C}_1$ and $\mathcal{C}_2$. From [5], [12], we can construct a two-level generalized tensor product code $\mathcal{C}_{GTP}$ with parameters $[16\ell+16, 15\ell+11, 4]_2$. By shortening $\mathcal{C}_{GTP}$ by 11 information symbols, we obtain a $[16\ell+5, 15\ell, 4]_2$ code, which has the same code parameters as the above ladder code $\mathcal{C}_L^2$. □

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REFERENCES

[1] K. A. Abdel-Ghaffar and M. Hassner, "Multilevel error-control codes for data storage channels," *IEEE Trans. Inf. Theory*, vol. 37, no. 3, pp. 735–741, May 1991.

[2] R. Bez, E. Camerlenghi, A. Modelli, and A. Visconti, "Introduction to flash memory," *Proc. of the IEEE*, vol. 91, no. 4, pp. 489–502, April 2003.

[3] M. Hassner, K. Abdel-Ghaffar, A. Patel, R. Koetter, and B. Trager, "Integrated interleaving-a novel ECC architecture," *IEEE Trans. Magn.*, vol. 37, no. 2, pp. 773–775, Mar. 2001.

[4] P. Huang, E. Yaakobi, and P. H. Siegel, "Multi-erasure locally recoverable codes over small fields," in *Proc. 55th Annual Allerton Conference on Communication, Control, and Computing*, Oct. 2017, pp. 1123–1130.

[5] H. Imai and H. Fujiya, "Generalized tensor product codes," *IEEE Trans. Inf. Theory*, vol. 27, no. 2, pp. 181–187, Mar. 1981.

[6] A. M. Patel, "Two-level coding for error control in magnetic disk storage products," *IBM Journal of Research and Development*, vol. 33, no. 4, pp. 470–484, July 1989.

[7] R. Roth, *Introduction to Coding Theory*. Cambridge University Press, 2006.

[8] W. C. Schmid and R. Schürer. (2014) MinT: Table for Linear Codes. [Online]. Available: http://mint.sbg.ac.at/index.php

[9] X. Tang and R. Koetter, "A novel method for combining algebraic decoding and iterative processing," in *Proc. IEEE Int. Symp. Inf. Theory*, July 2006, pp. 474–478.

[10] J. Wolf, "On codes derivable from the tensor product of check matrices," *IEEE Trans. Inf. Theory*, vol. 11, no. 2, pp. 281–284, Apr. 1965.

[11] ——, "An introduction to tensor product codes and applications to digital storage systems," in *Proc. IEEE Inf. Theory Workshop*, Oct. 2006, pp. 6–10.

[12] Y. Wu, "Generalized integrated interleaved codes," *IEEE Trans. Inf. Theory*, vol. 63, no. 2, pp. 1102–1119, Feb. 2017.