The Serial Concatenation of Rate-1 Codes Through Uniform Random Interleavers*

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Abstract

Until the Repeat Accumulate codes of Divsalar, *et al*. [4], few people would have guessed that simple rate-1 codes could play a crucial role in the construction of "good" codes. In this paper, we will construct "good" linear block codes at any rate $r < 1$ by serially concatenating an arbitrary outer code of rate r with a large number of rate-1 inner codes through uniform random interleavers. We derive the average output weight enumerator for this ensemble in the limit as the number of inner codes goes to infinity. Using a probabilistic upper bound on the minimum distance, we prove that long codes from this ensemble will achieve the Gilbert-Varshamov bound with high probability. Finally, by numerically evaluating the probabilistic upper bound, we observe that it is typically achieved with a small number of inner codes.

1 Introduction

The introduction of turbo codes by Berrou, Glavieux, and Thitimajshima [3] is remarkable because it combined simple components together to set a new standard for error-correcting codes. Since then, iterative "turbo" decoding has made it practical to consider a whole new world of concatenated codes while the use of "random" interleavers and recursive convolutional encoders has given us a starting point for choosing new code structures. Many of these concatenated code structures fit into a class that Divsalar, Jin, and McEliece call "turbo-like" codes [4]. This class includes their Repeat Accumulate (RA) codes which consist only of a repetition code, an interleaver, and an accumulator. Still they prove that, for sufficiently low rates and any fixed $\mathrm{E_b}/\mathrm{N_0}$ greater than a threshold, these codes have vanishing word error probability as the block length goes to infinity. This shows that powerful error-correcting codes may be constructed from extremely simple components.

In this paper we consider the serial concatenation of an arbitrary outer code of rate $r < 1$ with m identical rate-1 inner codes where, following the convention of turbo coding literature, we use the term serial concatenation to mean serial concatenation through a "random" interleaver. Any real system must, of course, choose a particular interleaver. Our analysis, however, will make use of the *uniform random interleaver* (URI) [2] which is equivalent to averaging over all possible interleavers. We analyze this system using a probabilistic bound on the minimum distance and show that, in the limit as the number of inner codes m goes to infinity, the minimum distance is bounded by an expression that resembles the Gilbert Bound (GB) [5].

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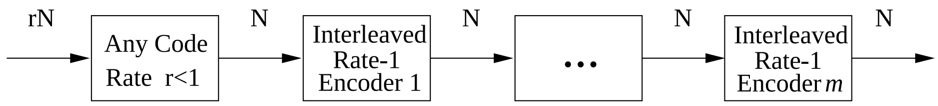


Figure 1: Our system consists of any rate $r < 1$ code followed by m rate-1 codes.

Our work is largely motivated by [4] and by the results of Öberg and Siegel [10]. Both papers consider the effect of a simple rate-1 “Accumulate” code in a serially concatenated system. In [4] a coding theorem is proved for RA codes, while in [10] the “Accumulate” code is analyzed as a precoder for the dicode magnetic recording channel. Benedetto, *et al.* also investigated the design and performance of Double Serially Concatenated Codes in [1].

If the outer code consists of multiple independent copies of a short block code and the inner code is a cascade of m interleaved “Accumulate” codes, we will refer to these codes as Generalized Repeated Accumulated ($\mathrm{GRA}^{\mathrm{m}}$) codes. McEliece has analyzed the maximum likelihood decoding performance of these codes for $m = 1$ [9], and we focus on the minimum distance of these codes for $m \geq 1$.

The outline of the paper is as follows. In Section 2 we review the *weight enumerator* (WE) of linear block codes and the union bound on the probability of error for maximum likelihood decoding. We also review the average weight enumerator for the serial concatenation of two linear block codes through a URI, and relate serial concatenation to matrix multiplication using a normalized form of each code's *input output weight enumerator* (IOWE). In Section 3 we introduce our system, shown in Figure 1, and we compute its average output WE. In Section 4 we derive a probabilistic bound on the minimum distance of any code, taken from a random ensemble, in terms of the ensemble's average WE. Applying this bound to the WE from Section 3 gives an expression very similar to the GB, and examining the bound as the block length goes to infinity produces the Gilbert-Varshamov Bound (GVB). In Section 5 we numerically evaluate our bound on minimum distance for various $\mathrm{GRA}^{\mathrm{m}}$ codes and observe that 3 or 4 “Accumulate” codes seem to be sufficient to achieve the bound corresponding to asymptotically large m. Finally, in Section 6 we discuss some conclusions and directions for future work.

2 Weight Enumerators and Serial Concatenation

2.1 The Union Bound

In this section, we review the weight enumerator of a linear block code and the union bound on error probability for maximum likelihood decoding. The IOWE $A_{w,h}$ of an (n, k) block code is the number of codewords with input weight w and output weight h, and the WE A_h is the number of codewords with output weight h and any input weight. Using these definitions, the probability of word error is upper bounded by

$$P_w \leq \sum_{h=1}^{n} \sum_{w=1}^{k} A_{w,h} z^h,$$

and the probability of bit error is upper bounded by

$$P_b \leq \sum_{h=1}^{n} \sum_{w=1}^{k} \frac{w}{k} A_{w,h} z^h.$$

The term z^h represents an upper bound on the pairwise error probability, between any two codewords differing in h positions, for the channel of interest. The constant z is defined for many memoryless channels [7, Section 5.3], and for the AWGN channel it is $z = e^{-(k/n)(E_b/N_0)}$.

2.2 Serial Concatenation through a Uniform Interleaver

In this section, we review the serial concatenation of codes through a uniform random interleaver. The introduction of URI in the analysis of turbo codes by Benedetto and Montorsi [2] has made the analysis of complex concatenated coding systems relatively straightforward, and using the URI for analysis is equivalent to averaging over all possible interleavers. The important property of the URI is that the distribution of output sequences is a function only of the weight distribution of input sequences. More precisely, an input sequence of weight w produces all possible output sequences of weight w, each with equal probability.

Consider any (n,k) block code with IOWE $A_{w,h}$ preceded by a URI. We will refer to such a code as a *uniformly interleaved code* (UIC). The probability of the combined system mapping an input sequence of weight w to an output sequence of weight h is

$$Pr\left(w \rightarrow h\right) = \frac{A_{w,h}}{\binom{k}{w}}. \tag{1}$$

We can now consider an (n,k) block code formed by first encoding with an (n_1,k) outer code with IOWE $A^{(o)}_{w,h}$, then permuting the output bits with a URI, and finally encoding again with an (n,n_1) inner code with IOWE $A^{(i)}_{w,h}$. The average number of codewords with input weight w and output weight h is then given by

$$\begin{aligned}\overline{A}_{w,h} &= \sum_{h_1=0}^{n_1} A^{(o)}_{w,h_1}\, Pr\left(h_1 \rightarrow h\right) \\ &= \sum_{h_1=0}^{n_1} A^{(o)}_{w,h_1}\, \frac{A^{(i)}_{h_1,h}}{\binom{n_1}{h_1}}.\end{aligned} \tag{2}$$

The average IOWE for the serial concatenation of two codes may also be written as the matrix product of the IOWE for the outer code and a normalized version of the IOWE for the inner code. Let us define, for any code, the *input output weight transition probability* (IOWTP) $P_{w,h}$ as the probability that a uniform random input sequence of weight w is mapped to an output sequence of weight h. From (1), we can see that

$$P^{(i)}_{w,h} = \frac{A^{(i)}_{w,h}}{\binom{k}{w}}. \tag{3}$$

Substituting (3) into (2), we have

$$\overline{A}_{w,h} = \sum_{h_1=0}^{n_1} A^{(o)}_{w,h_1}\, P^{(i)}_{h_1,h} = \mathbf{A}^{(o)}\mathbf{P}^{(i)}.$$

where $\mathbf{A}^{(o)}$ is the matrix representation of the outer code IOWE and $\mathbf{P}^{(i)}$ is the matrix representation of the inner code IOWTP. By inductively applying this to multiple inner code IOWTP matrices, one can see that matrix multiplication computes the overall $\overline{A}_{w,h}$ for an arbitrary number of serial concatenations. It is also clear from (3) that IOWTP matrices are stochastic (i.e. all rows sum to 1).

2.3 A Simple Example

In this section, we will compute the IOWE and IOWTP of the rate-1 “Accumulate” code [4]. The “Accumulate” code is a block code formed by truncating the simplest recursive convolutional code possible, having generator matrix $G(D) = 1/(1 \oplus D)$, after n symbols. The

Input Sequence	000	001	010	100	011	101	110	111
Input Weight	0	1	1	1	2	2	2	3
Output Sequence	000	001	011	111	010	110	100	101
Output Weight	0	1	2	3	1	2	1	2

Table 1: Input-output sequences and weight mappings for $n = 3$ "Accumulate" code.

generator matrix for this block code is an $n \times n$ matrix with all 1's in the upper triangle and all 0's elsewhere. In the example, we will look at the case $n = 3$. The generator matrix is

$$G = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Using Table 1, we see that a uniform random input of weight 1 maps to output weights 1, 2, and 3 with equal probability, and cannot be mapped to output weight 0. So the $w = 1$ row of the IOWTP matrix is $\begin{bmatrix} 0 & 1/3 & 1/3 & 1/3 \end{bmatrix}$. Filling in the rest of the entries, we give both the IOWE $A_{w,h}$ and the associated IOWTP $P_{w,h}$ in matrix form:

$$A_{w,h} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}_{w,h}, \quad P_{w,h} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/3 & 1/3 & 1/3 \\ 0 & 2/3 & 1/3 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}_{w,h}.$$

3 Multiple Rate-1 Serial Concatenations

3.1 The Input Output Weight Enumerator

In this section, we will consider a code formed by encoding $m + 1$ times. The first (outer) encoder is for an (n, k) block code with IOWE $A^{(o)}_{w,h}$. The next m (inner) encoders are for identical rate-1 UICs of block length n with IOWE $A^{(i)}_{w,h}$. If we let $\mathbf{P}$ be the IOWTP matrix associated with $A^{(i)}_{w,h}$, then we can write the average IOWE $\overline{A}_{w,h}$ for this code as

$$\overline{A}_{w,h} = \sum_{h_1=0}^{n} A^{(o)}_{w,h_1} \left[\mathbf{P}^m\right]_{h_1 h}. \tag{4}$$

The linearity of the code guarantees that the matrix $\mathbf{P}$ will be block diagonal with at least two blocks because inputs of weight 0 will always be mapped to outputs of weight 0 and inputs of weight greater than 0 will always be mapped to outputs of weight greater than 0. So let the first block be the 1×1 submatrix associated with $w = h = 0$, and let the second block $\mathbf{Q}$ be the $n \times n$ submatrix formed by deleting the first row and column of $\mathbf{P}$. Writing $\mathbf{P}^m$as the product of block diagonal matrices, we see that

$$\mathbf{P}^m = \begin{bmatrix} 1 & 0 \\ 0 & \mathbf{Q}^m \end{bmatrix}.$$

3.2 Stationary Distributions and Markov Chains

In this section, we will discuss the stationary distributions of a Markov Chain (MC) and how they relate to the stationary weight distributions of a rate-1 UIC. This discussion is based on the observation that if $\mathbf{P}$ is a finite dimensional stochastic matrix, then there is an associated MC with state transition matrix $\mathbf{P}$. Applying this to the IOWTP matrix of any UIC, we see that all UICs have an associated MC.

A MC, with state transition matrix $\mathbf{P}$, has a stationary distribution $\boldsymbol{\pi} = [\pi_0, \ldots, \pi_n]$ if $\boldsymbol{\pi}\mathbf{P} = \boldsymbol{\pi}$ and $\sum \pi_i = 1$. Accordingly, a rate-1 UIC has a stationary weight distribution $\boldsymbol{\pi}$ if $\boldsymbol{\pi}$ is a stationary distribution of the code's associated MC. Recall that a MC is *irreducible* if there is a path from any state to any other state with a finite number of steps. Using these definitions, we can draw upon some well-known results from the theory of non-negative matrices and MCs [11].

THEOREM 1. *An irreducible Markov Chain has a unique stationary distribution.*

We define a rate-1 UIC to be *irreducible* if the $\mathbf{Q}$ submatrix of its IOWTP matrix $\mathbf{P}$ can be associated with an irreducible MC. Similarly, this implies that there is a path from any weight to any other weight in a finite number of encodings. We now apply Theorem 1 to the $\mathbf{Q}$ submatrix of an irreducible rate-1 UIC. Since the matrix $\mathbf{Q}$ does not include inputs and outputs of weight 0, we must assume $\pi_0 = 0$ to make the following stationary weight distribution unique.

PROPOSITION 1. *The unique stationary weight distribution $\boldsymbol{\pi} = [\pi_0, \ldots, \pi_n]$ of an irreducible rate-1 UIC with $\pi_0 = 0$ is*

$$\pi_h = \frac{\binom{n}{h}}{2^n - 1} \text{ for } 1 \le h \le n.$$

The example from Section 2.3 is irreducible, and applying Proposition 1 gives

$$\begin{bmatrix} 0 & \frac{3}{7} & \frac{3}{7} & \frac{1}{7} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/3 & 1/3 & 1/3 \\ 0 & 2/3 & 1/3 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1+2}{7} & \frac{1+1+1}{7} & \frac{1}{7} \end{bmatrix}.$$

An irreducible MC is *primitive* if its state transition matrix has a unique eigenvalue of maximum modulus. Accordingly, we define an irreducible rate-1 UIC to be *primitive* if the MC associated with the $\mathbf{Q}$ submatrix of the IOWTP matrix is primitive. The following theorem from the theory of MCs [11] will allow to examine the asymptotic behavior of (4) as m goes to infinity.

THEOREM 2. *A primitive Markov Chain with state transition matrix $\mathbf{P}$ and unique stationary distribution $\boldsymbol{\pi}$ satisfies the limit*

$$\lim_{m \to \infty} \mathbf{P}^m = \begin{bmatrix} \boldsymbol{\pi} \\ \vdots \\ \boldsymbol{\pi} \end{bmatrix}.$$

The example from Section 2.3 is also primitive, and applying Theorem 2 gives

$$\lim_{m \to \infty} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/3 & 1/3 & 1/3 \\ 0 & 2/3 & 1/3 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}^m = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3/7 & 3/7 & 1/7 \\ 0 & 3/7 & 3/7 & 1/7 \\ 0 & 3/7 & 3/7 & 1/7 \end{bmatrix}.$$

3.3 Asymptotic Behavior for Many Concatenations

In this section, we use (4) and Theorem 2 to compute the average WE of any rate $r < 1$ outer code serially concatenated with m primitive rate-1 UICs, in the limit as m goes to infinity. The intriguing part of this result is that the WE is independent of the particular outer code and primitive rate-1 UIC chosen. We note that this is essentially a new construction for a uniform random ensemble of linear codes.

THEOREM 3. *Consider a rate-1 code formed by serially concatenating m primitive rate-1 UICs. For non-zero input weights and in the limit as m goes to infinity, the output weight distribution is independent of the input weight distribution and is*

$$\pi_h = \frac{\binom{n}{h}}{2^n - 1}.$$

COROLLARY. *The ensemble averaged WE for non-zero output weights for any rate $r < 1$ code serially concatenated with m primitive UICs, in the limit as m goes to infinity, is*

$$\begin{aligned} \overline{A}_h &= \sum_{w=1}^{rn} \sum_{h_1=1}^{n} A^{(o)}_{w,h_1} \left[\mathbf{P}^m\right]_{h_1 h} \\ &= \left(\sum_{w=1}^{rn} \sum_{h_1=1}^{n} A^{(o)}_{w,h_1} \right) \frac{\binom{n}{h}}{2^n - 1} = \left(2^{rn} - 1\right) \frac{\binom{n}{h}}{2^n - 1}. \end{aligned} \tag{5}$$

4 Bounds on the Minimum Distance

4.1 A General Bound on the Distribution of $\mathbf{d_{min}}$ from $\overline{\mathbf{A}}_\mathbf{h}$

In this section, we derive an upper bound on the probability that a randomly chosen code from some ensemble has $d_{min} < d$. This upper bound can be computed using only the average WE of the ensemble. A similar bound was used by Gallager to bound the minimum distance of Low Density Parity Check Codes [6].

THEOREM 4. *The probability that a code, randomly chosen from an ensemble with average WE $\overline{A}_h$, has $d_{min} < d$ is bounded by*

$$Pr(d_{min} < d) \leq \sum_{h=1}^{d-1} \overline{A}_h.$$

Proof. Let an ensemble of linear codes with average WE $\overline{A}_h$ be defined by a set of WEs $\left\{ A_h^{(1)}, A_h^{(2)}, \ldots, A_h^{(M)} \right\}$ each chosen with equal probability. Further, let $d_{min}^{(i)}$ be the minimum distance of the code associated with $A_h^{(i)}$. We can upper bound the probability of choosing a code with $d_{min} < d$ from this ensemble. First we define an indicator function

$$I(\text{condition}) = \begin{cases} 0 & \text{if condition false} \\ 1 & \text{if condition true} \end{cases}$$

and we note that for all non-negative integers x,

$$I(x > 0) \leq x. \tag{6}$$

First counting the number of codes with $d_{min} < d$, and then substituting an equivalent condition in the indicator function we have

$$Pr(d_{min} < d) = \frac{1}{M} \sum_{i=1}^{M} I(d_{min}^{(i)} < d) = \frac{1}{M} \sum_{i=1}^{M} I\left(\left(\sum_{h=1}^{d-1} A_h^{(i)}\right) > 0\right).$$

Upper bounding the indicator function with (6) and then summing over i gives

$$Pr(d_{min} < d) \leq \frac{1}{M} \sum_{i=1}^{M} \sum_{h=1}^{d-1} A_h^{(i)} = \sum_{h=1}^{d-1} \overline{A}_h.$$

□

4.2 An Application of the Bound

We now apply Theorem 4 to the WE in (5). This leads to a statement that, for a given block length n and rate r, upper bounds the probability of picking a code with minimum distance less than some threshold. Let $d^*(n, r, \epsilon)$ be the largest d which satisfies

$$\sum_{h=0}^{d-1} \binom{n}{h} \leq \frac{2^n - 1}{2^{rn} - 1}\epsilon + 1 \tag{7}$$

for block length n, rate r, and $0 \leq \epsilon \leq 1$. For this d^*, we can rearrange terms to get

$$\sum_{h=0}^{d^*-1} \binom{n}{h} - 1 \leq \frac{2^n - 1}{2^{rn} - 1}\epsilon.$$

Changing the lower limit of the sum and rearranging we have

$$\sum_{h=1}^{d^*-1} \frac{2^{rn} - 1}{2^n - 1} \binom{n}{h} \leq \epsilon.$$

Substituting for the expression of the WE $\overline{A}_h$ given in (5), we have

$$\sum_{h=1}^{d^*-1} \overline{A}_h \leq \epsilon$$

which, when combined with Theorem 4, implies that

$$P(d_{min} < d^*) \leq \epsilon. \tag{8}$$

So with probability at least $1 - \epsilon$, a randomly chosen code from this ensemble will have minimum distance at least $d^*(n, r, \epsilon)$.

The Gilbert Bound (GB) for binary codes [5] says that there exists at least one code with block length n, rate r, and minimum distance d if

$$2^{rn} \sum_{h=0}^{d-1} \binom{n}{h} \leq 2^n. \tag{9}$$

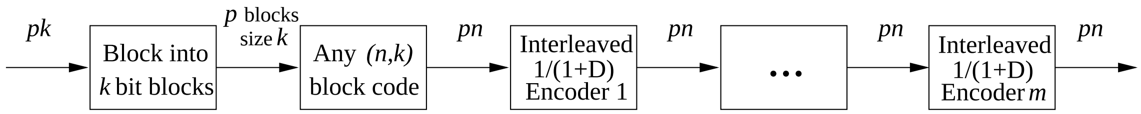


Figure 2: Encoder for a GRA^{m} Code with the block size indicated at each stage.

If we substitute $\epsilon = (2^{(1-r)n} - 1)(2^{rn} - 1)/(2^n - 1)$ into (7), then we have an expression identical to (9). Since this ϵ is strictly less than one, it follows from (8) that there exists at least one code in our ensemble with $d_{min} \geq d$. So we have qualitatively the same result as the GB.

The Gilbert-Varshamov Bound (GVB) takes its name from the GB and from a related bound due to Varshamov [5]. The GVB is the form of both bounds in the limit as n goes to infinity, and it says that there is a code with rate r and normalized minimum distance $\delta = d_{min}/n$ if

$$H(\delta) \leq 1 - r \tag{10}$$

where $H(x) = -x \log_2 x - (1-x)\log_2(1-x)$ is the binary entropy function.

If we let

$$\delta^*(r,\epsilon) = \lim_{n\to\infty} \frac{1}{n} d^*(n,r,\epsilon)$$

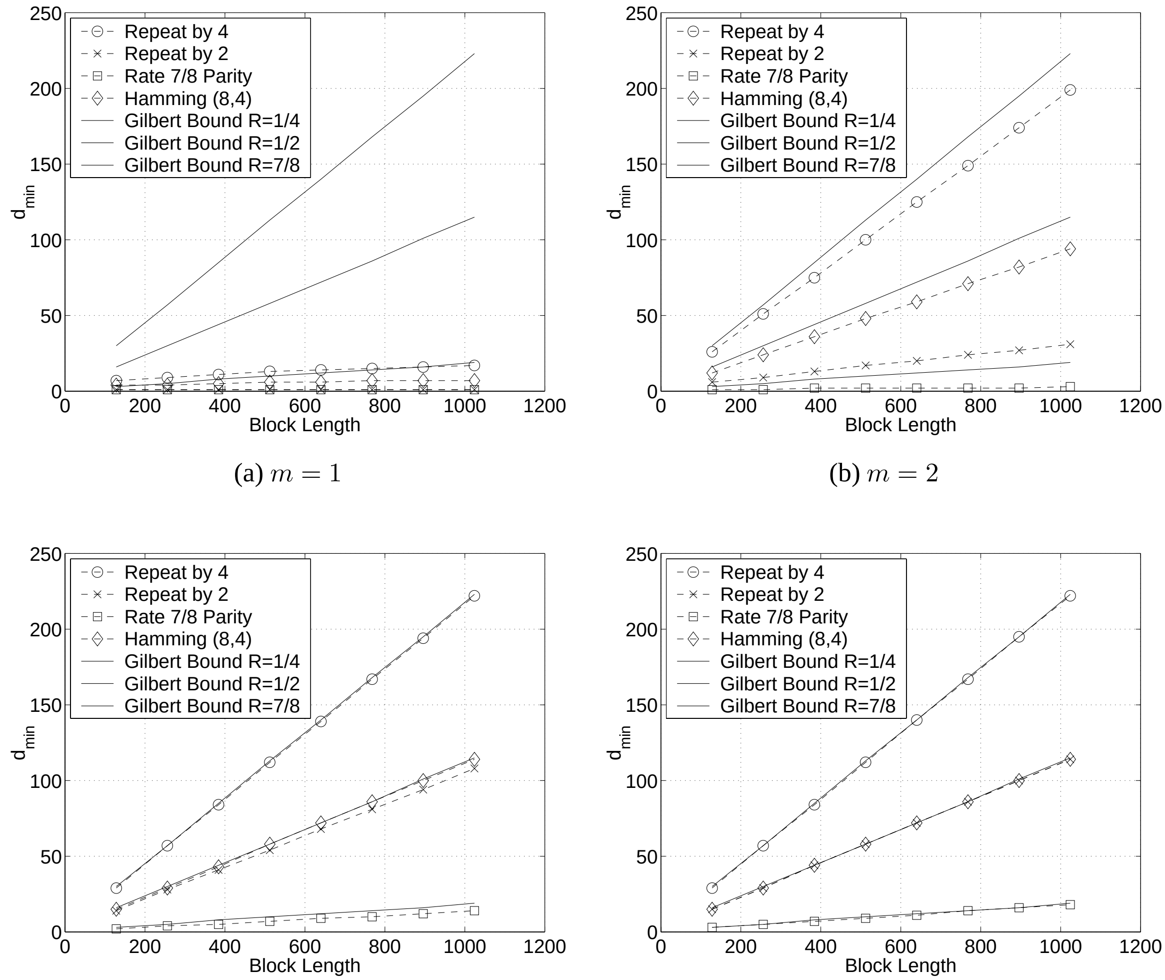
and examine (7) in the limit as n goes to infinity, we find that our bound says something even stronger than the GVB. In fact, we find that $\delta^*(r,\epsilon)$ is equal to the largest δ that satisfies (10) for any $\epsilon > 0$. This implies that in the limit as n goes to infinity, almost all of our codes will have a normalized minimum distance δ satisfying (10). This makes our codes "good" in the sense that, for a fixed rate as the block length goes to infinity, almost all of the codes in our ensemble have a normalized minimum distance that is bounded away from zero. It should be noted that this behavior is well-known for long random codes.

5 Generalized Repeated Accumulated (GRA^{m}) Codes

In this section, we describe GRA^{m} codes and apply Theorem 4 to some specific examples. GRA^{m} codes are formed by the serial concatenation of a simple outer code, which consists of p independent copies of a short (n,k) block code, and m interleaved rate-1 "Accumulate" codes. The encoder for GRA^{m} codes is shown in Figure 2. The performance of long GRA^{1} codes with maximum likelihood decoding was reported in [9], but cases with $m > 1$ were not considered. So we give results pertaining to the minimum distance GRA^{m} codes using a few examples.

In order to apply Theorem 4 to a specific ensemble, we must compute its average WE and choose an ϵ. For the following results, we computed the average WEs numerically and chose $\epsilon = 1/2$. This means that at least half of the codes in our ensemble have a minimum distance at least as large as the values shown in Figure 3. For the short block codes, we chose: a repeat by 2 (R2), a repeat by 4 (R4), a rate 7/8 single parity check (P8), and the $(8,4)$ extended Hamming code (H8).

It is important to note that, at a fixed rate, a "good" code is defined by a minimum distance which grows linearly with the block length. When examining these results, we will focus on whether or not the minimum distance appears to be growing linearly with block length and how close it is to the GB. For $m = 1$, it is known that the typical minimum distance grows $O(n^{(d^o-2)/d^o})$ where d^o is the minimum distance of the repeated outer code [8]. Examining Figure 3 for $m = 1$, we see that the minimum distance grows slowly for R4 and H8 and not all for R2 and P8. While for $m = 2$, the minimum distance growth of R4, H8, and R2



(a) $m = 1$

(b) $m = 2$

(c) $m = 3$

(d) $m = 4$

Figure 3: Probabilistic lower bound on the minimum distance of various GRA$^{\text{m}}$ codes.

appears distinctly linear. It is difficult to determine the growth rate of P8 with $m = 2$ from these results. At $m = 3$, all of the codes appear to have a minimum distance growing linearly with block length and the rates are very close to the GB. Finally, with $m = 4$, the bound on minimum distance and the GB are almost indistinguishable. These results are very encouraging and suggest that, over a wide range of rates, only a few “Accumulate” codes are sufficient to approach the GB on minimum distance.

6 Conclusions and Future Work

In this paper, we began by showing the relationship between serial concatenation through a uniform random interleaver and matrix multiplication of input output weight transition probability (IOWTP) matrices. We then introduced an ensemble of codes consisting of any rate $r < 1$ outer code followed by an infinite number of rate-1 primitive uniformly interleaved codes, and computed the ensemble’s average weight enumerator. This was done by introducing a correspondence between IOWTP matrices and Markov Chains (MCs), and drawing on some well-known limit theorems from MC theory. Next, we derived a probabilistic bound on the minimum distance of codes from this ensemble and noted that this bound is almost identical to both the finite block length Gilbert Bound (GB) and the infinite block length Gilbert-Varshamov Bound

(GVB). This implies that the ensemble of codes is “good” because, for long block lengths and fixed rate, almost all of the codes in our ensemble have a normalized minimum distance meeting the GVB. Finally, by evaluating our bound on minimum distance for specific outer codes and a small number of “Accumulate” codes, we observed that a small number of inner codes may be sufficient to approach the bound for an infinite number.

We are currently evaluating the iterative decoding of GRA^m codes and working to prove a coding theorem similar to [4] for these codes.

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