On Codes that Correct Asymmetric Errors with Graded Magnitude Distribution

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***Abstract*—In multi-level flash memories, the dominant cell errors are asymmetric with limited-magnitude. With such an error model in mind, Cassuto et al. recently developed bounds and constructions for codes correcting t asymmetric errors with magnitude no more than ℓ. However, a more refined model of these memory devices reflects the fact that typically only a small number of errors have large magnitude while the remainder are of smaller magnitude.**

In this work, we study such an error model, in which at most t_1 errors of maximum magnitude ℓ_1 and at most t_2 errors of maximum magnitude ℓ_2, with $\ell_1 < \ell_2$, can occur. We adapt the analysis and code construction of Cassuto, et al. for the refined error model and assess the relative efficiency of the new codes. We then consider in more detail specific constructions for the case where $t_1 = t_2 = 1$, $\ell_1 = 1$, and $\ell_2 > 1$.

I. Introduction

The topic of asymmetric error-correcting codes over non-binary alphabets has attracted considerable attention in the past few years, largely due to its relevance in the context of multi-level flash memories. However, research on asymmetric codes has a long history. A number of papers appeared in the 1960's, e.g., [3], [14], [20], [21]. Constructions and upper bounds on such codes were given in, e.g., [2], [8], [9], [12], [13], [16], [23] and constructions of systematic asymmetric error-correcting codes were studied in [4].

Flash memories are comprised of floating gate cells. The charge stored in a cell, also called the cell's level, is used to represent data. While it is possible to increase a cell level by injecting charge to the cell, reducing its level is not possible unless its entire containing block is first erased [5]. One of the dominant error mechanisms of flash memory cells results from over-programming the cells [7], [18], [24]. These errors can not be physically corrected unless the entire containing block is erased and thus it is crucial to design error-correcting codes that correct asymmetric errors of limited-magnitude. Furthermore, the ability to correct such errors can enable the programming of the cells to be less accurate and thus faster.

In [6], Cassuto et al. designed codes which correct t asymmetric errors of limited-magnitude ℓ. In this model, an error can only increase the erroneous symbol by at most ℓ levels. Systematic optimal codes for this model that correct all asymmetric and symmetric errors of limited-magnitude were given by Elarief and Bose [11]. In [17], the case of correcting a single asymmetric error ($t = 1$) of limited-magnitude ℓ was studied, and the results improved upon those given by Cassuto et al. for this scenario. Asymmetric error-correcting codes for binary and non-binary alphabets were recently presented by Dolecek [10]. Codes correcting all unidirectional errors of limited-magnitude were studied in [1]. Another related error model assumes that if the cell level is x then the level can only be reduced to any value less than x. Code constructions were given in [15], and a short survey was given in [16].

These previously proposed codes and bounds for the non-binary case mainly deal with the case of t asymmetric errors of limited-magnitude ℓ. However, it is likely that only a few cells will suffer from an error of large magnitude and that most of the erroneous cells will suffer from an error of a smaller magnitude [24]. In this work, we will present code constructions that correct t_1 asymmetric errors of magnitude at most ℓ_1 and t_2 asymmetric errors of magnitude at most ℓ_2, where $\ell_1 < \ell_2$. This model can be naturally generalized to a wider range of magnitudes as well as for errors in both directions.

The rest of the paper is organized as follows. In Section II, we formally define the error models we discuss in this work. Section III reviews the construction by Cassuto et al. [6] for asymmetric limited-magnitude error-correcting codes and presents a construction of codes that correct asymmetric limited-magnitude errors, where the errors can only be a multiple of some fixed known integer. In Section IV, we present the main code construction of the paper for the correction of asymmetric errors in the new error model and discuss its efficiency with respect to the scheme by Cassuto et al. [6]. Finally, in Section V, we discuss efficient code constructions for $t_1 = t_2 = 1$, $\ell_1 = 1$, and $\ell_2 > 1$.

II. Preliminaries

In this work, the memory elements, called cells, have q states: $0, 1, \ldots, q-1$. For a vector $\boldsymbol{x} = (x_1, \ldots, x_n)$, we let $wt(\boldsymbol{x})$ denote its Hamming weight, i.e. $wt(\boldsymbol{x}) = |\{i \mid x_i \neq 0\}|$. First, let us define asymmetric limited-magnitude errors.

Definition. *An error vector $\boldsymbol{e} = (e_1, e_2, \ldots, e_n)$ is called a **t-asymmetric ℓ-limited-magnitude error** if*

1) $\max_{1 \leqslant i \leqslant n}\{e_i\} \leqslant \ell$,
2) $wt(\boldsymbol{e}) \leqslant t$.

*An $[n, q, t, \ell]$ error-correcting code $\mathcal{C}$ is called a **t-asymmetric ℓ-limited-magnitude error-correcting code** if it is a q-ary code of length n which can correct all t-asymmetric ℓ-limited-magnitude errors.*

We extend the last definition to error vectors with two different limited-magnitudes.

Definition. *An error vector $\boldsymbol{e} = (e_1, e_1, \ldots, e_n)$ is called a **(t_1, t_2)-asymmetric (ℓ_1, ℓ_2)-limited-magnitude error** if*

1) $\max_{1 \leqslant i \leqslant n}\{e_i\} \leqslant \ell_2$.
2) $wt(\boldsymbol{e}) \leqslant t_1 + t_2$.
3) $|\{i \mid \ell_1 + 1 \leqslant e_i \leqslant \ell_2\}| \leqslant t_2$,

*An $[n, q, (t_1, t_2), (\ell_1, \ell_2)]$ error-correcting code $\mathcal{C}$ is called a **(t_1, t_2)-asymmetric (ℓ_1, ℓ_2)-limited-magnitude error-correcting code** if it is a q-ary code of length n which can* *correct all (t_1,t_2)-asymmetric (ℓ_1,ℓ_2)-limited-magnitude errors.*

That is, the error model is such that there are at most t_1+t_2 errors; at most t_2 of these errors have magnitude between ℓ_1+1 and ℓ_2 and the magnitude of the rest of the errors is at most ℓ_1.

Lemma 1. *Let t_1,t_2,ℓ_1,ℓ_2 be positive integers such that $\ell_1<\ell_2$. Then, the number of (t_1,t_2)-asymmetric (ℓ_1,ℓ_2)-limited-magnitude errors is*

$$\sum_{i=0}^{t_2}\left(\binom{n}{i}(\ell_2-\ell_1)^i\cdot\sum_{j=0}^{t_1+t_2-i}\binom{n-i}{j}\ell_1^j\right).$$

Proof: For any (t_1,t_2)-asymmetric (ℓ_1,ℓ_2)-limited-magnitude error vector, the number of errors of magnitude between ℓ_1+1 and ℓ_2 is at most t_2. Assume this number is i, $0\leqslant i\leqslant t_2$, then the number of error vectors with i such errors is $\binom{n}{i}(\ell_2-\ell_1)^i$. There are at most t_1+t_2-i more errors of magnitude at most ℓ_1 and so for any error vector with i errors between ℓ_1+1 and ℓ_2, the number of (t_1,t_2)-asymmetric (ℓ_1,ℓ_2)-limited-magnitude error vectors is $\sum_{j=0}^{t_1+t_2-i}\binom{n-i}{j}\ell_1^j$. Therefore, the total number of such error vectors is $\sum_{i=0}^{t_2}\left(\binom{n}{i}(\ell_2-\ell_1)^i\cdot\sum_{j=0}^{t_1+t_2-i}\binom{n-i}{j}\ell_1^j\right)$. ■

There are two error models that can be considered. The errors can or cannot wrap-around. That is, in the first case, if the transmitted word is $\boldsymbol{c}$ and the error vector is $\boldsymbol{e}$ then the received word is $(\boldsymbol{c}+\boldsymbol{e}) \bmod q$, while in the latter case we require that $\boldsymbol{c}+\boldsymbol{e}\leqslant(q-1,\ldots,q-1)$. In many practical applications like multi-level flash memories, it is common to assume that errors do not wrap-around. However, the constructions we present can work in some cases for both models.

III. Constructions of t-Asymmetric ℓ-Limited-Magnitude Error-Correcting Codes

The goal of this work is to construct (t_1,t_2)-asymmetric (ℓ_1,ℓ_2)-limited-magnitude error-correcting codes. The construction of such codes is based on a recent construction by Cassuto et al. [6] of t-asymmetric ℓ-limited magnitude error-correcting codes. We now review the construction in [6]. For a vector $\boldsymbol{x}=(x_1,\ldots,x_n)$, and a positive integer m, we define the vector $\boldsymbol{x} \bmod m$ to be

$$\boldsymbol{x} \bmod m=(x_1 \bmod m,\ldots,x_n \bmod m).$$

Construction 1. Let Σ be a t-error-correcting code of size n and redundancy r over an alphabet of size $\ell+1$. Then the q-ary code $\mathcal{C}$ of length n is defined as

$$\mathcal{C}=\{\boldsymbol{c}\in\{0,\ldots,q-1\}^n \mid \boldsymbol{c} \bmod (\ell+1)\in\Sigma\}.$$

The code Σ will be called the *base code* used to construct $\mathcal{C}$. The following theorem was proved in [6].

Theorem 2. *The code $\mathcal{C}$ is an $[n,q,t,\ell]$ error-correcting code if the code Σ corrects t or fewer symmetric errors. If $q>2\ell$, the converse is true as well.*

Decoding: Let $\boldsymbol{c}\in\mathcal{C}$ be the transmitted codeword and $\boldsymbol{y}=\boldsymbol{c}+\boldsymbol{e}$ the received word, where $\boldsymbol{e}$ is a t-asymmetric ℓ-limited-magnitude error vector. Let

$$\boldsymbol{z}=\boldsymbol{y} \bmod (\ell+1)=(\boldsymbol{c}+\boldsymbol{e}) \bmod (\ell+1).$$

Then, since $\boldsymbol{c} \bmod (\ell+1)\in\Sigma$, the word $\boldsymbol{z}$ suffers at most t symbol errors. These errors can be found using the decoder of the code Σ. That is, the value of $\boldsymbol{e} \bmod (\ell+1)$ is found and thus also the error vector $\boldsymbol{e}$.

Remark 1. As mentioned in [6], we will also assume here, for the simplicity of the encoding procedure, that $(\ell+1)|q$, and the construction corrects wrap-around errors as well. However it is possible to modify the encoding procedure also for the case where $(\ell+1)\nmid q$, while sacrificing the ability to correct wrap-around errors.

Encoding: The encoding procedure, as presented in [6], can use any encoding procedure for Σ. However, for our construction, we will require that Σ be systematic[1]. If r is the redundancy of Σ then the encoder's input is a vector $(\boldsymbol{u}_1,\boldsymbol{u}_2)\in\{0,\ldots,q-1\}^{n-r}\times\{0,\ldots,\frac{q}{\ell+1}-1\}^r$. Let $\boldsymbol{v}_1\in\{0,\ldots,\ell\}^r$ be the systematic encoder's output of Σ when applied to $(\boldsymbol{u}_1 \bmod (\ell+1))$. Then the encoder's output of $\mathcal{C}$ is $\boldsymbol{c}$, where

$$\boldsymbol{c}=(\boldsymbol{u}_1,(\ell+1)\cdot\boldsymbol{u}_2+\boldsymbol{v}_1).$$

Note that $\boldsymbol{c}\in\{0,\ldots,q-1\}^n$, $(\boldsymbol{c} \bmod (\ell+1))\in\Sigma$ and distinct input vectors generate distinct output vectors.

In the rest of the paper, we present code constructions that are based on the codes we just described. When we refer to an $[n,q,t,\ell]$ code $\mathcal{C}$, we refer to a code that is designed in Construction 1 which is constructed using a base code Σ. While Σ is constructed over an alphabet of size $\ell+1$ and has to correct t symbol errors, it is possible to use other codes over larger alphabets that correct t-asymmetric ℓ-limited-magnitude errors that wrap around (see Construction 1A in [6]). Either choice of Σ will work in our constructions.

In fact, assume one wants to construct $[n,q,t,1]$ error-correcting codes. According to Construction 1, Σ is a binary code, however if q is an odd integer the construction does not necessarily result in a good code. A different construction of $[n,q,t,1]$ error-correcting codes was recently given by Dolecek [10]. Yet another construction is presented in the next theorem and provides the code Σ to be used as an $[n,p,t,1]$ error-correcting code in order to construct an $[n,q,t,1]$ error-correcting code where p is a prime integer that divides q. We omit the proof due to space limitations.

Theorem 3. *Let p,t,m,n be four positive integers such that p is a prime number, $t\leqslant p-1$, and $n=p^m-1$. Let $\alpha\in GF(p^m)$ be a primitive element. Then, the matrix H,*

$$H=\begin{pmatrix}\alpha^1 & \alpha^2 & \alpha^3 & \cdots & \alpha^n\\ \alpha^2 & \alpha^4 & \alpha^6 & \cdots & \alpha^{2n}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \alpha^t & \alpha^{2t} & \alpha^{3t} & \cdots & \alpha^{tn}\end{pmatrix},$$

is a parity-check matrix of an $[n,p,t,1]$ error-correcting code of dimension $m-t$ over $GF(p)$.

Before we proceed to the next section and construct (t_1,t_2)-asymmetric (ℓ_1,ℓ_2)-limited-magnitude error-correcting codes, we construct a family of codes that correct errors of the following magnitudes:

[1]The restriction that the code Σ has a systematic encoder is not a severe one, as many codes and in particular all linear codes have a systematic encoder.

$$s, 2s, 3s, \ldots, \ell s,$$

for some positive integers s, ℓ.

Definition. *An error vector* $\boldsymbol{e} = (e_1, \ldots, e_n)$ *is called a* ***t-asymmetric*** $(\boldsymbol{\ell}, \boldsymbol{s})$***-multiple-spaced limited-magnitude error*** *if*

1) $\max_{1 \leqslant i \leqslant n}\{e_i\} \leqslant \ell s$,
2) $wt(\boldsymbol{e}) \leqslant t$,
3) *for all* $1 \leqslant i \leqslant n$, $e_i \equiv 0 (\bmod s)$.

An $[n, q, t, \ell, s]$ *error-correcting-code* $\mathcal{C}$ *is called a* ***t-asymmetric*** $(\boldsymbol{\ell}, \boldsymbol{s})$***-multiple-spaced limited-magnitude error-correcting code*** *if it is a* q*-ary code of length* n *which can correct all* t*-asymmetric* (ℓ, s)*-multiple-spaced limited-magnitude errors.*

The next theorem gives a construction of $[n, q, t, \ell, s]$ error-correcting-codes.

Theorem 4. *Let* n, q, t, ℓ, s *be positive integers and assume that there exists an* $[n, \lceil \frac{q}{s} \rceil, t, \ell]$ *error-correcting code* $\mathcal{C}_1$*. Then, there exists an* $[n, q, t, \ell, s]$ *error-correcting code* $\mathcal{C}_2$ *of the same size.*

Proof: The new code $\mathcal{C}_2$ is defined as follows.

$$\boldsymbol{c} \in \mathcal{C}_2 \text{ if and only if } \left\lfloor \frac{1}{s} \cdot \boldsymbol{c} \right\rfloor \in \mathcal{C}_1.$$

Assume that $\boldsymbol{c} \in \mathcal{C}_2$ and $\boldsymbol{y} = \boldsymbol{c} + \boldsymbol{e}$ is the received word, where $\boldsymbol{e}$ is a t-asymmetric (ℓ, s)-multiple-spaced limited-magnitude error. We use the decoding procedure of $\mathcal{C}_1$, where the input is $\left\lfloor \frac{1}{s} \cdot \boldsymbol{y} \right\rfloor$. Note that,

$$\left\lfloor \frac{1}{s} \cdot \boldsymbol{y} \right\rfloor = \left\lfloor \frac{1}{s} \cdot (\boldsymbol{c} + \boldsymbol{e}) \right\rfloor = \left\lfloor \frac{1}{s} \cdot \boldsymbol{c} \right\rfloor + \frac{1}{s} \cdot \boldsymbol{e}.$$

Since $\left\lfloor \frac{1}{s} \cdot \boldsymbol{c} \right\rfloor \in \mathcal{C}_1$, we can consider $\left\lfloor \frac{1}{s} \cdot \boldsymbol{c} \right\rfloor + \frac{1}{s} \cdot \boldsymbol{e}$ to be the input to the decoder of $\mathcal{C}_1$, where $\frac{1}{s} \cdot \boldsymbol{e}$ is a t-asymmetric ℓ-limited-magnitude error. Thus, the decoder of $\mathcal{C}_1$ can decode the error vector $\frac{1}{s} \cdot \boldsymbol{e}$ and multiplying it by s gives with the original error vector $\boldsymbol{e}$. ■

A code will be called *perfect* if it attains the sphere packing bound for t-asymmetric ℓ-limited-magnitude errors [6].

Theorem 5. *If the code* $\mathcal{C}_1$ *is perfect and* $s|q$*, then the code* $\mathcal{C}_2$ *is perfect as well.*

Proof: If the code $\mathcal{C}_1$ is perfect then

$$|\mathcal{C}_1| \cdot \sum_{i=0}^{t} \binom{n}{i} \ell^i = (q')^n,$$

where $q' = \frac{q}{s}$. The size of the code $\mathcal{C}_2$ is $|\mathcal{C}_2| = s^n \cdot |\mathcal{C}_1|$ and the number of errors is $\sum_{i=0}^{t} \binom{n}{i} \ell^i$. Therefore,

$$|\mathcal{C}_2| \cdot \sum_{i=0}^{t} \binom{n}{i} \ell^i = s^n \cdot |\mathcal{C}_1| \cdot \sum_{i=0}^{t} \binom{n}{i} \ell^i = s^n \cdot (q')^n = q^n$$

and the code $\mathcal{C}_2$ is perfect as well. ■

Theorem 4 gives us the construction as well as the decoding procedure for the new code $\mathcal{C}_2$. Its encoding procedure is derived from the encoding procedure of $\mathcal{C}_1$. Assume that $\mathcal{C}_1$ is constructed as described earlier in this section using a base code Σ of length n and redundancy r which corrects t symbol errors over an alphabet of size $\ell + 1$ and it has a systematic encoder. Then, Σ is also the base code for $\mathcal{C}_2$. For the simplicity of the encoder we assume that $s(\ell + 1)|q$. The encoder's input is a vector

$$(\boldsymbol{u}_1, \boldsymbol{u}_2) \in \{0, \ldots, q-1\}^{n-r} \times \left\{0, \ldots, \frac{q}{\ell+1} - 1\right\}^r.$$

Let $\boldsymbol{v}_2 \in \{0, \ldots, \ell\}^n$ be the systematic encoder's output of Σ when applied to $\left(\left\lfloor \frac{\boldsymbol{u}_1}{s} \right\rfloor \bmod (\ell+1)\right)$. The encoder's output of $\mathcal{C}_2$ is $\boldsymbol{c} = (\boldsymbol{c}_1, \boldsymbol{c}_2)$, where $\boldsymbol{c}_1 = \boldsymbol{u}_1$ and

$$\boldsymbol{c}_2 = s \cdot \left((\ell+1) \left\lfloor \frac{\boldsymbol{u}_2}{s} \right\rfloor + \boldsymbol{v}_2 \right) + (\boldsymbol{u}_2 \bmod s).$$

The vector $\boldsymbol{c}$ satisfies $\boldsymbol{c} \in \{0, \ldots, q-1\}^n$, $\left\lfloor \frac{1}{s} \cdot \boldsymbol{c} \right\rfloor \in \mathcal{C}_1$ and the outputs of two different input vectors are different. Thus, the encoding procedure follows the construction of $\mathcal{C}_2$.

Remark 2. Let us explain the intuition behind this construction. Assume that q is a power of two and every cell level is represented as a sequence of $\log_2 q$ bits. If we construct asymmetric error-correcting codes where $\ell = 1$, then the base code Σ is binary and the encoding and decoding of the q-ary code are implemented on the LSB of each cell. For asymmetric (ℓ, s)-multiple-spaced limited-magnitude error-correcting codes, assume that $\ell = 2$ and s is also a power of two, say the i-th power, where $2 \leqslant i < \log_2 q - 1$, then the base code Σ is again binary and the encoding and decoding of the q-ary code are implemented on the i-th digit of each cell.

IV. A Construction of (t_1, t_2)-Asymmetric (ℓ_1, ℓ_2)-Limited-Magnitude Error-Correcting Codes

In this section, we present a construction of (t_1, t_2)-asymmetric (ℓ_1, ℓ_2)-limited-magnitude error-correcting codes. The construction uses the codes proposed by Cassuto et al. [6] which were reviewed in Section III. We will describe the encoding procedure and then show its correctness by the success of its decoding procedure.

Construction 2. Let t_1, t_2, ℓ_1, ℓ_2 be positive integers such that $\ell_1 < \ell_2$, and let $\ell_2' = \left\lfloor \frac{\ell_2}{\ell_1+1} \right\rfloor$. Let $\mathcal{C}_1$ be an $[n, q, t_1 + t_2, \ell_1]$ error-correcting code and let $\mathcal{C}_2$ be an $[n, q, t_2, \ell_2', \ell_1 + 1]$ error-correcting code. Let Σ_1 and Σ_2 be the base codes that are used to generate the codes $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. Both base codes are of length n, and they have redundancy r_1 and r_2, respectively. They also have systematic encoders. We construct the code $\mathcal{C}$ by means of the following encoding procedure. The input to the encoder is a vector

$$(\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3) \in \{0, \ldots, q-1\}^{n-r_1-r_2} \times \left\{0, \ldots, \frac{q}{\ell_2'+1} - 1\right\}^{r_2} \times \left\{0, \ldots, \frac{q}{\ell_1+1} - 1\right\}^{r_1}.$$

The encoding of these information symbols is carried out in two steps. First, let $\boldsymbol{v}_2$ be the systematic encoder's output of Σ_2 applied to the vector

$$\left(\left\lfloor \frac{\boldsymbol{u}_1}{\ell_1+1} \right\rfloor \bmod (\ell_2'+1), \left\lfloor \frac{\boldsymbol{u}_3}{\ell_1+1} \right\rfloor \bmod (\ell_2'+1) \right),$$

and let

$$\boldsymbol{u}_2' = (\ell_1+1) \left((\ell_2'+1) \left\lfloor \frac{\boldsymbol{u}_2}{\ell_1+1} \right\rfloor + \boldsymbol{v}_2 \right) + (\boldsymbol{u}_2 \bmod (\ell_1+1)).$$

Then, we calculate $\boldsymbol{v}_3$ to be the systematic encoder's output of Σ_1 applied to $(\boldsymbol{u}_1 \bmod (\ell_1+1), \boldsymbol{u}_2' \bmod (\ell_1+1))$. Finally, the encoder's output is $\boldsymbol{c} = (\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3)$, where $\boldsymbol{c}_1 = \boldsymbol{u}_1, \boldsymbol{c}_2 = \boldsymbol{u}_2'$ and $\boldsymbol{c}_3 = (\ell_1+1) \cdot \boldsymbol{u}_3 + \boldsymbol{v}_3$.

Remark 3. We assume here that $r_1 + r_2 \leqslant n$. However, if this is not the case we can modify the construction to be applicable in this scenario.

Before we show the correctness of this construction, let us prove a few properties of (t_1,t_2)-asymmetric (ℓ_1,ℓ_2)-limited-magnitude errors. Assume that $\boldsymbol{e}$ is a (t_1,t_2) asymmetric (ℓ_1,ℓ_2)-limited-magnitude error. First note that this error vector can be written as $\boldsymbol{e} = \boldsymbol{e}_1 + \boldsymbol{e}_2$, where

$$\boldsymbol{e}_1 = \boldsymbol{e} \bmod (\ell_1+1), \quad \boldsymbol{e}_2 = \boldsymbol{e} - \boldsymbol{e}_1.$$

Lemma 6. *For all $\boldsymbol{c} \in \mathcal{C}_1$, $\boldsymbol{c}+\boldsymbol{e}_2 \in \mathcal{C}_1$.*

Proof: The proof follows from the observation that

$$\boldsymbol{e}_2 \bmod (\ell_1+1) = \mathbf{0}.$$

■

Lemma 7. *The error vector $\boldsymbol{e}_1$ is a (t_1+t_2)-asymmetric ℓ_1-limited-magnitude error.*

Proof: Since $wt(\boldsymbol{e}) \leqslant t_1+t_2$ so is $wt(\boldsymbol{e}_1) \leqslant t_1+t_2$ and clearly $\max_{0\leqslant i\leqslant n-1}\{e_{1,i}\} \leqslant \ell_1$. ■

Lemma 8. *The error vector $\boldsymbol{e}_2$ is a t_2-asymmetric (ℓ_2', ℓ_1+1)-multiple-spaced limited-magnitude error.*

Proof: For each i, $1 \leqslant i \leqslant n$, if $e_i \leqslant \ell_1$ then $e_{2,i} = 0$ and therefore $wt(\boldsymbol{e}_2) \leqslant t_2$. Furthermore,

$$e_{2,i} = e_i - e_{1,i} = e_i - (e_i \bmod (\ell_1+1)),$$

and thus $e_{2,i} \equiv 0 \bmod (\ell+1)$. Finally,

$$e_{2,i} = e_i - (e_i \bmod (\ell_1+1)) = \left\lfloor \frac{e_i}{\ell_1+1} \right\rfloor \cdot (\ell_1+1)$$
$$\leqslant \left\lfloor \frac{\ell_2}{\ell_1+1} \right\rfloor \cdot (\ell_1+1) = \ell_2' \cdot (\ell_1+1)$$

■

In the next theorem we will prove the correctness of this construction by showing the success of its decoding procedure.

Theorem 9. *The code $\mathcal{C}$ generated by Construction 2 is an $[n,q,(t_1,t_2),(\ell_1,\ell_2)]$ error-correcting code.*

Proof: The proof follows from the decoding procedure of the code $\mathcal{C}$. Assume the received word is $\boldsymbol{y} = \boldsymbol{c}+\boldsymbol{e}$ where $\boldsymbol{e}$ is a (t_1,t_2)-asymmetric (ℓ_1,ℓ_2)-limited-magnitude error vector. As mentioned above, we write the error vector in the form $\boldsymbol{e} = \boldsymbol{e}_1+\boldsymbol{e}_2$, where $\boldsymbol{e}_1 = \boldsymbol{e} \bmod (\ell_1+1)$ and $\boldsymbol{e}_2 = \boldsymbol{e}-\boldsymbol{e}_1$. From Lemma 6, $\boldsymbol{c}+\boldsymbol{e}_2 \in \mathcal{C}_1$ and from Lemma 7, $\boldsymbol{e}_1$ is a (t_1+t_2)-asymmetric ℓ_1-limited magnitude error vector. Therefore, by applying the decoder of $\mathcal{C}_1$ to the received word $\boldsymbol{y}$, the error vector $\boldsymbol{e}_1$ is decoded and the decoder's output is $\boldsymbol{y}' = \boldsymbol{c}+\boldsymbol{e}_2$.

According to Lemma 8, the error vector $\boldsymbol{e}_2$ is a t_2-asymmetric (ℓ_2',ℓ_1+1)-multiple-spaced limited-magnitude error. However, note that $\boldsymbol{c}$ is not a codeword of $\mathcal{C}_2$. In fact, $\boldsymbol{c}$ is the encoded codeword, which is the output of the encoder of $\mathcal{C}_1$. Its input, which is the output of the encoder of $\mathcal{C}_2$, is $\boldsymbol{c}'$, where for all $1 \leqslant i \leqslant n-r_1$, $c_i' = c_i$, and for $n-r_1+1 \leqslant i \leqslant n$, $c_i' = c_i - (c_i \bmod (\ell_1+1))$. But, the decoder of $\mathcal{C}_2$ decodes a t_2-asymmetric $(\ell+1)$-multiple ℓ_2'-limited-magnitude error by calculating

$$\left\lfloor \frac{1}{\ell_1+1} \cdot \boldsymbol{y}' \right\rfloor = \left\lfloor \frac{1}{\ell_1+1} \cdot (\boldsymbol{c}+\boldsymbol{e}_2) \right\rfloor$$
$$= \left\lfloor \frac{1}{\ell_1+1} \cdot \boldsymbol{c} \right\rfloor + \frac{1}{\ell_1+1} \cdot \boldsymbol{e}_2 = \left\lfloor \frac{1}{\ell_1+1} \cdot \boldsymbol{c}' \right\rfloor + \frac{1}{\ell_1+1} \cdot \boldsymbol{e}_2.$$

Therefore, the error output of the decoder of $\mathcal{C}_2$ is $\boldsymbol{e}_2$ and we successfully receive the transmitted codeword $\boldsymbol{c}$. ■

In order to evaluate this code construction we compare it to the codes by Cassuto el al. [6]. Clearly, for all positive integers t_1,t_2,ℓ_1,ℓ_2 such that $\ell_1 \leqslant \ell_2$, every $[n,q,t_1+t_2,\ell_2]$ error-correcting code is also an $[n,q,(t_1,t_2),(\ell_1,\ell_2)]$ error-correcting code. In case that t_1 and t_2 are roughly the same, it turns out that our construction is inferior. The reason is that the number of errors found by $\mathcal{C}_1$ is t_1+t_2 and the number of errors found by $\mathcal{C}_2$ is t_2. Even though the magnitude of the errors is smaller than ℓ_2, the total number of errors found by the two codes is t_1+2t_2, as opposed to t_1+t_2 errors corrected by an $[n,q,t_1+t_2,\ell_2]$ code. Since the sizes of the two codes depend on the sizes of their base codes, in order to give an accurate comparison, one needs to know the exact sizes of these base codes. If all the base codes were perfect or close to be perfect then it is possible to show that, approximately, if $\frac{t_1}{t_2} > \frac{\log n}{\log \ell_2 - \log \ell_1}$, then our scheme is superior. For example, if $n = 1000$ and $\ell_1 = 1, \ell_2 = 4, t_1 = 6, t_2 = 1$, our construction yields better codes. Consider another example of $[n,q,(n-1,1),(1,2)]$ error-correcting codes, where $12|q$. Then, the size of the best $[n,q,n,2]$ error-correcting codes will be $\left(\frac{q}{3}\right)^n$, while our construction achieves codes of size $\frac{1}{2^{\lceil \log n \rceil}} \cdot \left(\frac{q}{2}\right)^n > \left(\frac{q}{3}\right)^n$.

V. A Construction of $(1,1)$-Asymmetric $(1,\ell)$-Limited-Magnitude Error-Correcting Codes

We saw in the previous section that if the values of t_1 and t_2 are roughly the same then our construction does not necessarily outperform the construction by Cassuto et al. Here, we consider one case where it is possible to achieve better code constructions. We start with a construction of an $[n,q,(1,1),(1,2)]$ error-correcting code.

Theorem 10. *Let q,m be positive integers such that $m > 1$ and $3|q$, and $\mathcal{C}_1$ is the code constructed in Theorem 3 where $n = 3^m-1, p = 3, t = 2$. Then, the code $\mathcal{C}$, defined as,*

$$\mathcal{C} = \{\boldsymbol{c} \in \{0,\ldots,q-1\}^n | \boldsymbol{c}(\bmod 3) \in \mathcal{C}_1, \sum_{i=1}^{n} c_i \equiv 0(\bmod 2)\}.$$

is an $[n,q,(1,1),(1,2)]$ error-correcting code.

Proof: Let $\boldsymbol{c}$ be the transmitted codeword, $\boldsymbol{y} = \boldsymbol{c}+\boldsymbol{e}$ the received word where $\boldsymbol{e}$ is a $(1,1)$-asymmetric $(1,2)$-limited magnitude error, and $\boldsymbol{s}_1 = \sum_{i=1}^n y_i\alpha^i, \boldsymbol{s}_2 = \sum_{i=1}^n y_i\alpha^{2i}$. The sum of the received symbols can have either odd or even parity, modulo 2.

Odd sum-parity: $\sum_{i=1}^n y_i \equiv 1(\bmod 2)$.

There are two possible cases.

1) The weight of $\boldsymbol{e}$ is one and the error magnitude is one.
2) The weight of $\boldsymbol{e}$ is two, one error is of magnitude one and the other one is of magnitude two.

In the first case, we get $\boldsymbol{s}_1 = \alpha^i, \boldsymbol{s}_2 = \alpha^{2i} = \boldsymbol{s}_1^2$, where i is the error location. In the second case, $\boldsymbol{s}_1 = \alpha^{i_1}+2\alpha^{i_2}, \boldsymbol{s}_2 = \alpha^{2i_1}+2\alpha^{2i_2}$, where i_1,i_2 are the error locations and

$$\boldsymbol{s}_1^2 = (\alpha^{i_1}+2\alpha^{i_2})^2 = \alpha^{2i_1}+\alpha^{2i_2}+\alpha^{i_1+i_2}$$
$$= \alpha^{2i_1}+2\alpha^{2i_2} - (\alpha^{2i_2}-\alpha^{i_1+i_2}) = \boldsymbol{s}_2 + \alpha^{i_2}(\alpha^{i_2}-\alpha^{i_1}) \neq \boldsymbol{s}_2.$$

Hence we can distinguish between these two cases. The error location error in the first case is easy to find. In the second case, we decode as follows:

$$\begin{aligned}\frac{s_2}{s_1}+s_1 &= \frac{\alpha^{2i_1}+2\alpha^{2i_2}}{\alpha^{i_1}+2\alpha^{i_2}}+\alpha^{i_1}+2\alpha^{i_2}\\ &= \frac{\alpha^{2i_1}-\alpha^{2i_2}}{\alpha^{i_1}-\alpha^{i_2}}+\alpha^{i_1}-\alpha^{i_2}=2\alpha^{i_1}.\end{aligned}$$

From this we can determine the location of the error with magnitude one and, therefore, also the location of the error with magnitude two.

Even sum-parity: $\sum_{i=1}^{n} y_i \equiv 0(\bmod 2)$. There are three cases:

1) There is no error.
2) The weight of $\boldsymbol{e}$ is one and the error magnitude is two.
3) The weight of $\boldsymbol{e}$ is two and both errors have magnitude one.

In the first case, we get $s_1 = s_2 = 0$. In the second case, $s_1 = 2\alpha^i, s_2 = 2\alpha^{2i} = 2s_1^2$, where i is the error location. In the third case, we can show as before that $s_2 \neq 2s_1^2$. Hence, we can distinguish between the three cases and the error locations in each case can again be easily determined. ■

Assume q is even. The size of the code $\mathcal{C}_1$ is 3^{n-2m} and, as defined, the size of $\mathcal{C}$ is $\frac{q^n}{2\cdot 9^m}$. On the other hand, suppose that we use Construction 1 to design an $[n, q, 2, 2]$ error correcting code $\mathcal{C}'$ with the same parameters n and q. If the base code Σ is an optimal linear code that corrects two errors over $GF(3)$, its redundancy is at least $\lceil \log_3(2n^2+1) \rceil = 2m+1$, and therefore the size of the code $\mathcal{C}'$ is at most $\frac{q^n}{3^{2m+1}} < \frac{q^n}{2\cdot 9^m}$.

The last construction can be extended to $[n, q, (1,1), (1,\ell)]$ error-correcting codes for arbitrary ℓ. Here, we will use *location-correcting codes*, introduced by Roth and Seroussi [19]. These codes find the locations of errors whose values are known. For the case $t = 2$, such a code over $F = GF(p^m)$ is constructed by the parity check-matrix

$$H = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_n \\ \alpha_1^{-1} & \alpha_2^{-1} & \alpha_3^{-1} & \cdots & \alpha_n^{-1} \end{pmatrix},$$

where $S = \{\alpha_1, \ldots, \alpha_n\}$ is a weak Sidon set of the multiplicative group F^* of F. We omit the details due to the lack of space and leave them to an extended version of this work.

VI. Conclusion

In this paper, we studied a new error model for multi-level flash memories based upon a graded distribution of asymmetric errors of limited magnitudes. Using a recent construction by Cassuto et al. [6] of asymmetric limited-magnitude error-correcting codes, we developed a family of codes that correct asymmetric errors with magnitudes a multiple of some fixed integer. We then utilized these two classes of codes to construct codes that correct t_1 asymmetric errors of magnitude no more than ℓ_1 and t_2 errors of magnitude no more than ℓ_2, where $\ell_1 < \ell_2$. Finally, we discussed efficient constructions for the special case where $t_1 = t_2 = 1$, $\ell_1 = 1$, and $\ell_2 > 1$.

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