The measure of mutual dependence we propose is similar in spirit to the measure of statistical dependence proposed by Fine [3] on the basis of the Solomonoff–Kolmogorov–Chaitin complexity measure. However, Fine's measure is not computable and hence not operational.

VI. Concluding Remarks

We have presented a conceptually new framework for measuring dependence between two time series. Unlike the framework developed by Geweke, which is limited to the AR and ARX class of models, and the framework developed by Fine, which is not operational, our framework is both very general in its applicability and straightforward operationally. All one needs to compute, say, the measure of unidirectional causal dependence from y to x are the two predictive densities $P(x_{t+1}|x^t)$ and $P(x_{t+1}|x^t, y^t)$. These densities need not be fully specified in advance; the parameters of these densities and even the number of parameters is determined as an integral part of the computation of the measure. Obviously, the value of the resulting measure depends critically on the class of predictive densities selected. A good selection gives a sharp measure of causal dependence while a bad one masks a possible causal dependence, which of course is just as it should be.

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A Note on the Shannon Capacity of Run-Length-Limited Codes

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***Abstract*—It is proven that 100-percent efficient fixed-rate codes for run-length-limited (RLL) (d,k) and RLL charge-constrained $(d,k;c)$ channels are possible in only two cases, namely $(d,k;c) = (0,1;1)$ and (1,3;3). Specifically, the binary Shannon capacity of RLL (d,k) constrained systems is shown to be irrational for all values of $(d,k), 0 \le d < k$. For RLL charge-constrained systems with parameters $(d,k;c)$, the binary capacity is irrational for all values of $(d,k;c), 0 \le d < k, 2c \ge k+1$, except (0,1;1) and (1,3;3), which both have binary capacity 1/2.**

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I. Introduction and Background

In this correspondence we show that 100-percent efficient fixed-rate codes are impossible for most run-length-limited (d,k) and run-length-limited charge-constrained $(d,k;c)$ channels. More precisely, we prove that, among these channels, only the (0,1;1) and (1,3;3) have Shannon capacities which permit 100-percent efficient codes.

Run-length-limited (RLL) codes are widely used in magnetic and optical data recording channels. These constrained codes are characterized by two parameters $(d,k), 0 \le d < k$, which specify the minimum and maximum allowable run-lengths of zeros between consecutive ones in the constrained binary sequences.

RLL charge-constrained systems combine run-length constraints with the bounded running digital sum (RDS_0) constraint to ensure a spectral null at dc, as we now describe in more detail.

In data recording, constrained sequences $\{b_j\}, j \ge 0$, with symbols $\{0,1\}$ are typically converted to a two-level channel input signal $\{s_j\}, j \ge 0$, via a precoding convention called non-return-to-zero-index (NRZI), defined by

$$s_j = s_{j-1}(-1)^{b_j}, \qquad j \ge 0, \tag{1}$$

assuming $s_{-1} = 1$. The average power spectrum of the constraint is given by

$$\Phi(f) = \lim_{N \to \infty} E\left\{\frac{\left|\sum_{j=0}^{N} s_j e^{-i2\pi jf}\right|^2}{N}\right\}, \qquad i = \sqrt{-1} \tag{2}$$

where the expected value is taken with respect to the measure of maximal entropy on the ensemble of allowable sequences, and the limit is interpreted in the distribution sense. The power spectrum of maxentropic RLL sequences can be expressed in a simple closed form. See Immink [6]. Specifically, for the RLL (d,k) constraint, let λ be the largest real root of the polynomial $p(x) = x^{k+1} - x^{k-d} - \cdots - x - 1$. Then the spectrum (2) is given by

$$\Phi(f) = \frac{1}{\sin(\pi f)^2 \bar{L}} \frac{1 - |G(2\pi f)|^2}{|1 + G(2\pi f)|^2} \tag{3}$$

where

$$\bar{L} = \sum_{j=d+1}^{k+1} j\lambda^{-j} \tag{4}$$

and

$$G(2\pi f) = \sum_{j=d+1}^{k+1} \lambda^{-j} \exp(2\pi i j f), \qquad i = \sqrt{-1}. \tag{5}$$

From (3)–(5), it can be seen that $\Phi(0) \ne 0$ for all $(d,k), 0 \le d < k$. In certain applications, however, it is required that the code power spectrum $\Phi(f)$ have a null at $f = 0$ (dc), that is $\Phi(0) = 0$. It is well-known that the code spectrum has a null at dc if for each finite code sequence $\{b_0, b_1, \cdots, b_N\}$ the running digital sum of the associated channel input sequence $\{s_0, s_1, \cdots, s_N\}$, denoted $RDS_0(s_0, s_1, \cdots, s_N)$, is bounded by a fixed constant. That is,

$$RDS_0(s_0, s_1, \cdots, s_N) = \left|\sum_{j=0}^{N} s_j\right| \le B, \tag{6}$$

for some constant B, and all finite allowable input strings $\{s_j\}$.

In fact, Pierobon [13] recently proved that the bounded RDS_0 condition is necessary as well as sufficient for a constrained system, generated by a finite-state transition diagram (FSTD), to have a spectral null at dc.

The RLL charge-constrained systems are characterized by parameters $(d, k; c)$, where (d, k) are the RLL constraints, and the RDS_0 value of any finite channel input sequence $\{s_j\}$ falls in the range $[-2c, 2c]$. The RLL (d, k) and $(d, k; c)$ constraints are often represented by an FSTD, introduced by Shannon into the study of discrete noiseless channels [15]. For example, Fig. 1 shows the FSTD G for the RLL (0,1) constraint. State diagrams for general (d, k) and $(d, k; c)$ constraints will be discussed in Section III.

Associated to an FSTD G with n vertices is the state-transition matrix T, an $n \times n$ matrix defined by

$$t_{ij} = \text{number of edges in } G \text{ from state } i \text{ to state } j.$$

Thus T is the adjacency matrix of the directed graph underlying the FSTD. For example, for the (0,1) constraint,

$$T = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

is the state-transition matrix associated to the diagram in Fig. 1.

According to Shannon, if the FSTD G has distinct code symbols on outgoing edges from each state, the base b capacity of the constrained system S represented by G is

$$C_b(S) = \log_b \lambda \ b\text{-ary digits/symbol} \tag{7}$$

where λ is the largest real eigenvalue of the matrix T associated to the FSTD G. We will use the notation $C(S)$ to denote the base 2, or binary, capacity of S. As an example, the largest real eigenvalue of T for the (0,1) constraint is the golden mean, $\lambda = (1+\sqrt{5})/2$, so the binary capacity of the (0,1) constraint is $C(\text{RLL}(0,1)) = \log_2 (1+\sqrt{5})/2 \approx 0.6942$.

The Shannon capacity may be thought of as the asymptotic growth rate of the number of constrained strings of length N, denoted x_N:

$$C(S) = \lim_{N \to \infty} \frac{1}{N} \log_2 x_N.$$

The *rate* $r = m/n$ of a code which maps arbitrary binary sequences (binary data) to the system of constrained sequences S is the ratio of the number of data bits to the number of code bits. The Shannon capacity $C(S)$ therefore represents an upper bound on achievable code rates, and *code efficiency* $E = r/C$ is the ratio of the actual code rate to the Shannon capacity.

The construction of efficient codes, that is, codes with E close to one, is an important problem in data recording technology, since recording density is directly proportional to the code efficiency. Marcus [10] and Adler *et al.* [1] devised a systematic code construction algorithm for finite memory constraints, including (d, k) constraints. The algorithm produces a code with finite-state machine encoder and sliding block decoder (ensuring limited error propagation) at any code rate $r = m/n \leq C$. Recently, Karabed and Marcus [9] extended these results to a class of infinite memory constraints, including the $(d, k; c)$ constraints.

These techniques permit the code designer to balance the requirements of high code efficiency, low implementation complexity, and small error propagation. It is therefore of interest to find constraint parameters (d, k) and $(d, k; c)$ with Shannon capacity C equal to, or closely approximated by, a rational rate m/n, with m and n relatively small. In particular, if the capacity is rational, $C = m/n$, the techniques permit the construction of 100-percent efficient practical codes. This correspondence investigates the existence of (d, k) constraints and $(d, k; c)$ constraints with rational capacity.

Section II presents the lemma which is the key to the main result. It uses the Perron–Frobenius theorem for irreducible

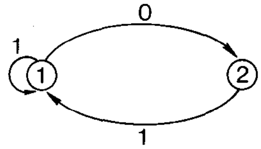


Fig. 1. FSTD for (0,1) channel.

nonnegative matrices [14] to derive a necessary condition on the periodic structure of the FSTD when the Shannon capacity is rational. In Section III, it is shown that the capacity of RLL (d, k) constraints is *never* rational. It is then proved that among charge-constrained RLL $(d, k; c)$ constraints, the only constraints with rational capacity are $(d, k; c) = (0,1;1)$ and $(1,3;3)$, both of which have binary capacity 1/2.

We close the introduction with a few remarks regarding practical codes achieving 100-percent efficiency for the constraints $(0,1;1)$ and $(1,3;3)$. For $(0,1;1)$, a simple rate 1/2 block code is possible. In fact, simple rate 1/2 $(0,1;1)$ codes were early standards in digital magnetic recording applications, and were known variously as frequency modulation (FM), phase encoding (PE), and Manchester code, among other names. See, for example, Jorgensen [7]. For the $(1,3;3)$ constraint, Hong and Ostapko [5] described rate 1/2 codes which, however, suffered from unlimited error propagation. Patel [12] constructed the zero-modulation (ZM) code which limited error propagation by reducing the rate slightly below 1/2. (In the construction, the rate reduction could in fact be made arbitrarily small, although not zero, by use of unboundedly increasing lookahead in the encoder.) Recently, the technique of [9] has been used to construct a $(1,3;3)$ code which has 100-percent efficiency, a finite-state encoder, and a sliding block decoder (limited error propagation) [8].

II. The Key Lemma

This section proves the lemma which is the key ingredient in the determination of the rationality of capacities of the run-length constrained systems discussed in the Introduction. It is an application of the expression (7) for Shannon capacity and the Perron–Frobenius theory of nonnegative matrices [14].

Definition 1: A nonnegative $n \times n$ real matrix T is *irreducible* if, for any $1 \leq i, j \leq n$, there is an integer m (possibly dependent on i and j) such that

$$(T^m)_{ij} > 0.$$

A directed graph G is *irreducible* if its state transition matrix T is irreducible.

We need one more definition before stating the key lemma.

Definition 2: The *period* of an irreducible nonnegative $n \times n$ matrix T is the greatest common divisor (gcd) of cycle lengths, that is,

$$\gcd\{ k | (T^k)_{ii} > 0, \text{ for some } i, 1 \leq i \leq n \}.$$

The *period* of an irreducible directed graph G is the period of its state transition matrix. The matrix T (or graph G) is called *aperiodic* when the period is one.

Lemma 1: Let S be a constrained system of sequences, represented by an irreducible FSTD G with distinct labels on the edges emanating from any single state. If the base b capacity of S satisfies

$$C_b(S) = m/n$$

where $\gcd(m, n) = 1$ and b is not an nth power of an integer, then the period of G is a multiple of n.

Proof: We give a proof for the case $b = 2$. The general proof follows easily. Since $C(S) = m/n$, the expression for Shannon capacity (7) shows that the largest real eigenvalue of the transition matrix T associated to G is given by $\lambda = 2^{m/n}$, where $2^{m/n}$ denotes the real nth root of 2^m. In particular, λ satisfies the polynomial equation

$$x^n - 2^m = 0.$$

The polynomial $x^n - 2^m$ is irreducible over the integers for integral values of $m > 0$ relatively prime to n, since the constant term of any factor of $x^n - 2^m$, being a product of some k roots of $x^n - 2^m = 0$, has modulus $2^{km/n}$, which is not an integer unless $k = n$ or 0. Therefore, $x^n - 2^m$ must be a factor of the characteristic polynomial of T, $p_T(x)$, whose roots constitute the eigenvalues (with multiplicities) of T. In particular, $p_T(x)$ must have among its roots the n values

$$2^{m/n}\omega^j, \qquad j = 0,1,\cdots,n-1$$

where

$$\omega = e^{-i2\pi/n}.$$

The Perron–Frobenius theorem [14] states that an irreducible nonnegative matrix T, of period q, has precisely q eigenvalues of maximum modulus, one of which is a positive real number r. These eigenvalues are $re^{i2\pi k/q}$, $k = 0,1,\cdots,q-1$, which are the q roots of $x^q - r^q = 0$. In particular, $2^{m/n}e^{i2\pi/n} = re^{i2\pi k_0/q}$ for some $0 < k_0 \le q-1$, giving $n = q/k_0$, showing that q is a multiple of n.

III. Capacity or RLL (d,k) and $(d,k;c)$ Constraints

In this section, we apply Lemma 1 to investigate the rationality of the capacity of (d,k) and $(d,k;c)$ constraints. The base b of the capacity is taken to be $b = 2$ when not otherwise specified. An FSTD $G_{2,5}$ for (2,5) constraints is shown in Fig. 2. In general, we define an FSTD $G_{d,k}$ for (d,k) constraints containing $k+1$ states, denoted $1,2,\cdots,k+1$, with state transitions described by the associated matrix $T_{d,k}$ defined by

$$t_{i,i+1} = 1, \qquad 1 \le i \le k$$
$$t_{i,1} = 1, \qquad d+1 \le i \le k+1$$
$$t_{i,j} = 0, \qquad \text{otherwise.}$$

Denote by $e(i,j)$ the label on the edge from state i to state j, when there is a transition from i to j. Then labels on $G_{d,k}$ are given by

$$e(i,i+1) = 0, \qquad 1 \le i \le k$$
$$e(i,1) = 1, \qquad d+1 \le i \le k+1.$$

It is easy to verify that $G_{d,k}$ is irreducible. For example, from state $i, 1 \le i \le k+1$, there is a path of length no more than $d+1$ to state 1, and from state 1 to state $i, 1 \le i \le k+1$ there is a path of length no more than $k+1$.

Moreover, $G_{d,k}$ is aperiodic. This follows from the fact that $G_{d,k}$ contains cycles of length $d+1$ and $d+2$, and gcd $(d+1, d+2) = 1$, for $d \ge 0$.

Denote the binary capacity of the (d,k) channel by $C(d,k)$.

Proposition 1: $C(d,k)$ is irrational for all $(d,k), 0 \le d < k < \infty$.

Proof: By Lemma 1 and the aperiodicity of $G_{d,k}$, if $C(d,k)$ is rational, it must be of the form $C(d,k) = \log_2 2^{m/1} = m$ for some positive integer m. However, the (d,k) constrained system is a proper shift invariant subset of the system of unconstrained binary sequences, which has capacity 1. Therefore, $C(d,k) < 1 < m$, all $m \ge 1$. See, for example, Coven and Paul [2]. It follows that $C(d,k)$ is irrational.

We can extend this result to the class of RLL constraints with parameters (d,∞). These constraints have been discussed by Franaszek [3] and Freiman and Wyner [4]. Fig. 3 shows an FSTD $G_{3,\infty}$ for the $(3,\infty)$ constraint. In general, for $d \ge 1$ an FSTD $G_{d,\infty}$ can be defined using $d+1$ states, $1,2,\cdots,d+1$, with state

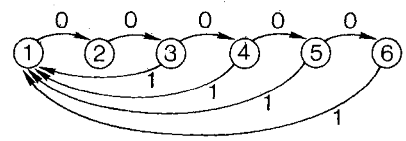


Fig. 2. FSTD for (2,5) channel.

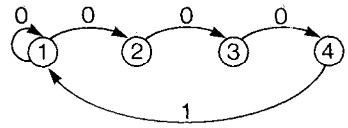


Fig. 3. FSTD for $(3,\infty)$ channel.

transitions given by

$$t_{1,1} = 1$$
$$t_{d+1,1} = 1$$
$$t_{i,i+1} = 1, \qquad 1 \le i \le d$$
$$t_{i,j} = 0, \qquad \text{otherwise.}$$

The labels on $G_{d,\infty}$ are given by

$$e(1,1) = 0$$
$$e(d+1,1) = 1$$
$$e(i,i+1) = 0, \qquad 1 \le i \le d.$$

Corollary 1: $C(d,\infty)$ is irrational for $d \ge 1$.

Proof: Note that unconstrained binary sequences constitute the RLL $(0,\infty)$ constraint, which has capacity exactly one. For the (d,∞) constraint with $d \ge 1$, we have

$$C(d,\infty) = \log_2 \mu$$

where μ is the largest real root of

$$P_{d,\infty}(X) = X^{d+1} - X^d - 1.$$

The capacity of the RLL (d,k) constraint, $k < \infty$, is given by

$$C(d,k) = \log_2 \lambda$$

where λ is the largest real root of

$$P_{d,k}(X) = X^{k+1} - X^{k-d} - X^{k-d-1} - \cdots - X - 1$$

(the characteristic polynomial of the matrix $T_{d,k}$).

For $d = 1$, we find

$$P_{1,\infty}(x) = P_{0,1}(x).$$

It is easy to verify that, for $d \ge 2$,

$$P_{d-1,2d-1}(x) = P_{d,\infty}(x)\left(\sum_{j=0}^{d-1} x^j\right). \tag{8}$$

The second factor on the right-hand side of (8) factors over the complex numbers as

$$\left(\sum_{j=0}^{d-1} x^j\right) = \prod_{j=1}^{d-1}(x - \alpha^j)$$

where $\alpha = e^{i2\pi/d}$. Since all of the α^j have modulus 1 and the largest real root of the polynomial $P_{d-1,2d-1}$ is greater than 1 (because $C(d-1,2d-1) > 0$), it follows that $P_{d-1,2d-1}(x)$ and $P_{d,\infty}(x)$ have the same largest real root. Therefore,

$$C(d,\infty) = C(d-1,2d-1).$$

(For example, the constrained channels represented by Figs. 2 and 3 have the same Shannon capacity.) The corollary now follows directly from Proposition 1.

The charge-constrained RLL systems with parameters $(0,1;1)$ and $(1,3;3)$ are described by the irreducible FSTD's $G_{(0,1;1)}$ and $G_{(1,3;3)}$ shown in Fig. 4. We now define an FSTD $G_{(d,k;c)}$ for $(d,k;c)$ constraints with $0 \le d < k < \infty$ and $2c \ge k+1$. Note that the restriction on c is not artificial; any smaller value of c would restrict the maximum achievable run-length of zeros to strictly less than k.

The states in $G_{(d,k;c)}$ are described by two parameters (s,t), where s may be thought of as the "accumulated charge" and t represents the run-length of zeros. Specifically, the FSTD $G_{(d,k;c)}$ contains states (s,t) with

$$-c+1 \le s-t \le c-d$$

subject to

$$-c+1 \le s \le c$$

and

$$0 \le t \le k.$$

The state transitions are described by the rules

$$(s,t) \to (s+1,t+1),$$

with edge label $e((s,t);(s+1,t+1))=0$, whenever the destination state is a valid state, and

$$(s,t) \to (-s+1,0),$$

with edge label $e((s,t);(-s+1,0))=1$, whenever $d \le t \le k$.

Lemma 2: The graph $G_{(d,k;c)}$ has period 2.

Proof: Let z_1 be the sequence of length $d+1$ consisting of d 0's followed by a 1:

$$z_1 = \underbrace{(0,0,0,\cdots,0,1)}_{d+1}.$$

Let z_2 be the sequence of length $d+2$ consisting of $d+1$ 0's followed by a 1:

$$z_2 = \underbrace{(0,0,0,\cdots,0,1)}_{d+2}.$$

If d is odd, then the sequence z_1 is generated by a cycle starting at state $(-(d-1)/2,0)$, and the sequence z_2z_2 (that is, z_2 repeated twice) is generated by a cycle starting at state $(c-(d+1),0)$. The lengths of these two cycles are

$$l_1 = \text{length}(z_1) = d+1$$

$$l_2 = \text{length}(z_2z_2) = 2(d+2) = 2(d+1)+2$$

with greatest common divisor

$$\gcd(l_1,l_2) = \gcd((d+1),2(d+1)+2) = 2.$$

If d is even, we use the cycles z_1z_1, starting at $(c-d,0)$, and z_2, starting at $(-d/2,0)$, with lengths

$$l_3 = \text{length}(z_1z_1) = 2d+2$$

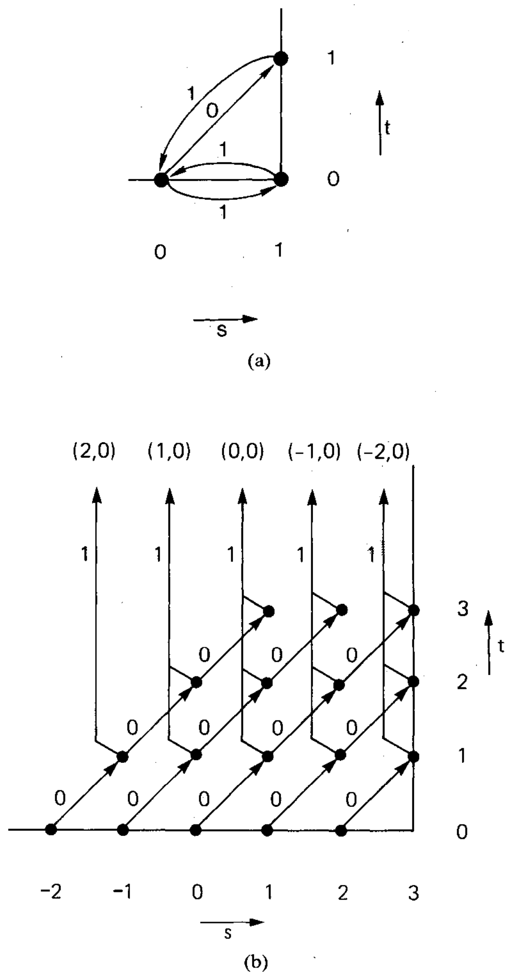
and

$$l_4 = \text{length}(z_2) = d+2$$

satisfying

$$\gcd(l_3,l_4) = \gcd(2d+2,d+2) = \gcd(2(d+1),d+2) = 2.$$

We have shown, therefore, that the period of $G_{(d,k;c)}$ is a divisor of 2. However, it is easy to see that the period is at least 2, since the states $\mathcal{S}$ of $G_{(d,k;c)}$ can be partitioned into two subsets $\mathcal{S}_0 = \{(s,t) \in \mathcal{S} | s \equiv 0 (\text{mod } 2)\}$ and $\mathcal{S}_1 = \{(s,t) \in \mathcal{S} | s \equiv 1 (\text{mod } 2)\}$ such that any transition from a state $(s,t) \in \mathcal{S}_0$ ends in a state $(s',t') \in \mathcal{S}_1$, and vice versa. Therefore, $G_{(d,k;c)}$ has period exactly 2.



(a)

(b)

Fig. 4. (a) FSTD for $(0,1;1)$ channel. (b) FSTD for $(1,3;3)$ channel.

Corollary 2: The only rational binary capacity of a $(d,k;c)$ constrained system is $C=1/2$.

Proof: By Lemma 1, the only possible rational capacity of a period 2 system of binary sequences is $C=1$ or $C=1/2$. Only the unconstrained binary system achieves capacity $C=1$, so the only possible rational capacity of a constrained $(d,k;c)$ channel is $1/2$. One can check that the $(0,1;1)$ and $(1,3;3)$ constraints provide examples which achieve $C=1/2$. It is computationally easier to verify this by using a variable-length symbol presentation of the $(d,k;c)$ constraint based on states $(-c+1,0)$, $(-c+2,0),\cdots,(c-d,0)$, with the edge labels corresponding to allowed runs of zeros followed by a one. Shannon [15] gives a determinantal equation for such a diagram whose largest real root λ may be used to compute the capacity, as in (7).

Finally, we show that the two constraints $(0,1;1)$ and $(1,3;3)$ are the only $(d,k;c)$ systems with binary capacity $1/2$.

Proposition 2: The only constrained $(d,k;c)$ systems with capacity $1/2$—and therefore the only ones with rational capacity—are $(d,k;c) = (0,1;1)$ and $(1;3;3)$.

Proof: We make use of the table of computed $(d,k;c)$ capacities in [11]. Denote the capacity of the $(d,k;c)$ constraint by $C(d,k;c)$. Note that for a given value of d, the capacities $C(d,k;c)$ in any column of the table, corresponding to a fixed value of c and increasing value of k, are strictly increasing since the $(d,k;c)$ constraint is a proper subsystem of the $(d,k+1;c)$

constraint. Similarly, the capacities in any row of the table, corresponding to a fixed value of d and k and increasing value of c, are strictly increasing since the $(d,k;c)$ constraint is contained in the $(d,k;c+1)$ constraint.

For $(d,k;c)$ constraints with $d=0,1$, or 2, it then follows easily from the table that only the (0,1;1) and (1,3;3) constraints can have capacity 1/2. For $(d,k;c)$ constraints with $d \geq 3$, note that

$$C(3,\infty)=C(2,5)\approx 0.4057<\frac{1}{2}.$$

Since $C(d,\infty)$ is a strictly decreasing function of d, and $C(d,k;c)$ is strictly less than $C(d,\infty)$ for finite k and c, we conclude that no $(d,k;c)$ constraint with $d \geq 3$ has capacity 1/2. This completes the proof.

Remark: Similar arguments show that the base b capacities of the (d,k) constrained systems are irrational for all b and that the only $(d,k;c)$ constrained systems with rational base b capacities are the (0,1;1) and (1,3;3) systems where the base is a power of two.

IV. Conclusion

This correspondence has investigated the rationality of the binary capacity of constrained noiseless channels relevant to data storage applications. Specifically, the binary capacity of run-length-limited (d,k) constrained channels is shown to be irrational for all parameters $d=0, 0<k<\infty$ and all $1 \leq d<k \leq \infty$ (that is, only the unconstrained binary channel with parameters $(d,k)=(0,\infty)$ has rational capacity, namely $C=1$). For charge-constrained RLL channels with parameters $(d,k;c)$, it is shown that only the (0,1;1) and (1,3;3) constraints have rational capacity, both exactly 1/2.

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Remarks on Codes from Hermitian Curves

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***Abstract*—Parameters and generator matrices are given for the codes obtained by applying Goppa's algebraic-geometric construction method to Hermitian curves in PG(2, q), where $q=2^{2s}$ for some $s \in \mathbb{N}$. Automorphisms of these codes are also discussed, and some results on self-duality and weak self-duality are given.**

I. Introduction

In [1] van Lint and Springer calculate the parameters of codes constructed from Hermitian curves. The remark that these codes are usually better than the corresponding Reed–Solomon codes with the same rate makes it appropriate to study these codes in greater detail. In Section II we will give the necessary algebraic-geometric tools. These tools will be used in Section III to prove some nice results for the codes constructed from Hermitian curves. In Section IV we will give a worked out example.

II. Basic Facts about Hermitian Curves and Their Points

For the definition of Hermitian curves we refer to [4, p. 146] and for the definition of genus of a curve, rational points, uniformizing parameter, differential, residue, $L(D)$, and $\Omega(D)$, we refer to [2] or [3]. Let $q=r^2$ be a power of two, and α be a primitive element of $\mathbb{F}_q$. The Hermitian curve H is given by the equation $X^{r+1}+Y^{r+1}+Z^{r+1}=0$ (cf. [4, p. 146]). Since H has no multiple points, its genus is $g=(r-1)r/2$. The number of rational points (over $\mathbb{F}_q$) on H is r^3+1 (cf. [4, p. 147]). In what follows y,z will be defined as $y=Y/X$, $z=Z/X$.

Lemma 1: The only rational points (over $\mathbb{F}_q$) on H with corresponding uniformizing parameters are

a) uniformizing parameter $t=1/y$;
points
$$Q=(0,1,1),\quad P_{1/y,i}=(0,1,\alpha^{(r-1)i}),\qquad (i=1,\cdots,r);$$

b) uniformizing parameter $t=y$;
points
$$P_{y,i}=(1,0,\alpha^{(r-1)i}),\qquad (i=0,\cdots,r);$$

c) uniformizing parameter $t=z$;
points
$$P_{z,i}=(1,\alpha^{(r-1)i},0),\qquad (i=0,\cdots,r);$$

d) uniformizing parameter $t=\beta y+z$;
points
$$P_{t,i_0,i,j}=(1,\alpha^{(r-1)i+i_0},\alpha^{(r-1)j+j_0}),$$
$$(i_0=1,\cdots,r-2;\ i=0,\cdots,r;\ j=0,\cdots,r),$$

j_0 and β are uniquely determined by the equations
$$1+\alpha^{i_0(r+1)}=\alpha^{j_0(r+1)},$$
$$\beta=\alpha^{(r-1)(j-i)+(j_0-i_0)}.$$

Proof: The fact that the equation $1+\alpha^{i_0(r+1)}=\alpha^{j_0(r+1)}$ uniquely determines j_0 follows from the fact that $1+\alpha^{i_0(r+1)}$ is

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