# Graph-Based Decoding in the Presence of ISI

Mohammad H. Taghavi, *Member, IEEE*, and Paul H. Siegel, *Fellow, IEEE*

***Abstract*—We propose a new graph representation for ISI channels that can be used for combined equalization and decoding by linear programming (LP) or iterative message-passing (IMP) decoding algorithms. We derive this graph representation by linearizing the ML detection metric, which transforms the equalization problem into a classical decoding problem. We observe that the performance of LP and IMP decoding on this model are very similar in the uncoded case, while IMP decoding significantly outperforms LP decoding when low-density parity-check (LDPC) codes are used. In particular, in the absence of coding, for certain classes of channels, both LP and IMP algorithms always find the exact ML solution using the proposed graph representation, without complexity that is exponential in the size of the channel memory. This applies even to certain two-dimensional ISI channels. However, for some other channel impulse responses, both decoders have nondiminishing probability of failure as SNR increases. We provide analytical explanations for many of these observations. In addition, we study the error events of LP decoding in the uncoded case, and derive a measure that can be used to classify ISI channels in terms of the performance of the proposed detection scheme.**

***Index Terms*—Combined equalization and decoding, graph-based decoding, intersymbol interference (ISI) channels, iterative message passing, linear programming, maximum-likelihood detection.**

## I. Introduction

INTERSYMBOL INTERFERENCE (ISI) is a characteristic of many data communications and storage channels. Systems operating on these channels often employ error-correcting codes in conjunction with some form of ISI reduction technique, which, in magnetic recording systems, is often a trellis-based sequence detector. It is known that some gain will be obtained if the equalization and decoding blocks are combined at the receiver by exchanging soft information between them. A possible approach to achieving this gain is to use soft-output equalization methods such as the BCJR algorithm [1] or the soft-output Viterbi algorithm (SOVA) [2] along with iterative decoders. For a survey of performance results with this approach see [3]. However, both BCJR and SOVA suffer from complexity that is exponential in the length of the channel memory.

Kurkoski *et al.* [4] proposed two graph representations of the ISI channel that can be combined with the Tanner graph of the LDPC code for message-passing decoding. Their bit-based representation of the channel contains many 4-cycles, which results in a significant performance degradation compared to maximum-likelihood (ML) detection in the uncoded case. On the other hand, iterative message passing (IMP) on their state-based representation, where messages contain state rather than bit information, has a performance and overall complexity similar to BCJR, while benefiting from a parallel structure and reduced delay. Among other works, Singla *et al.* [5] applied message passing on a bit-based graph representation of a two-dimensional ISI channel combined with an LDPC Tanner graph. However, similar to the case of one-dimensional ISI, the abundance of short cycles prevents the algorithm from achieving performance that is close to optimal.

Linear programming (LP) has been recently applied by Feldman *et al.* [6] to the problem of ML decoding of LDPC codes in memoryless channels, as an alternative to IMP techniques. The idea behind LP decoding is to relax the ML decoding problem, which can be written as an optimization with linear objective function, but nonlinear constraints, into a linear optimization problem. LP decoding performs closely to IMP algorithms such as the sum-product algorithm (SPA) and the min-sum algorithm (MSA), and it is much easier to analyze for deterministic finite-length codes. This approach, however, cannot be directly applied when the channel has ISI, since the objective function of the ML decoding problem will become a quadratic and, generally, nonconvex function.

In this work we study the problem of ML detection in the presence of ISI, with the goal of deriving a graph representation for the channel that allows detection with a complexity that is polynomial in the channel memory size, and that also can be combined with the Tanner graph of an LDPC code for combined decoding. Motivated by LP decoding in the memoryless channel, our approach is based on linearizing the quadratic objective function of the ML detection problem. Using this technique, we convert the detection problem into an equivalent binary decoding problem in a memoryless channel, which can be represented by a factor graph and used for IMP detection, or, after relaxing the binary constraints, LP detection, similar to the way IMP decoding and LP decoding are applied on memoryless channels. Furthermore, decoding an underlying LDPC code can be incorporated into this problem simply by adding the check nodes of the LDPC code to the graph representation. Our simulations indicate that the performance of IMP and LP detection using the proposed graphical model are very similar in the uncoded case. Hence, in our analysis, we focus primarily on LP

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M. H. Taghavi was with the Electrical and Computer Engineering Department, and the Center for Magnetic Recording Research, University of California, San Diego, La Jolla, CA 92093 USA. He is now with Qualcomm, Inc., San Diego, CA 92121 USA (e-mail: mtaghavi@ieee.org).

P. H. Siegel is with the Electrical and Computer Engineering Department, and the Center for Magnetic Recording Research, University of California, San Diego, La Jolla, CA 92093 USA (e-mail: psiegel@ucsd.edu).

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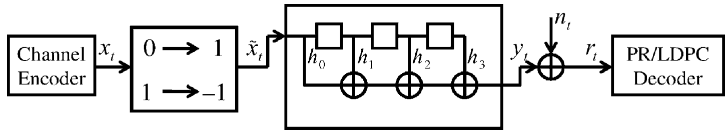


Fig. 1. Binary-input ISI channel.

detection, since it is generally more amenable to deterministic performance analysis.

By a geometric analysis we show that, in the absence of coding, if the impulse response of the ISI channel satisfies certain conditions, LP detection on the proposed graph is guaranteed to produce the ML solution at all SNR values. This means that there are ISI channels, which we call *proper* channels, for which uncoded ML detection can be achieved with a complexity polynomial in the channel memory size. This result becomes more significant in the context of two-dimensional ISI channels, for which the general ML detection problem has been shown to be NP-complete [7].

On the other end of the spectrum, we observe that, for some channels, which we call *improper*, both IMP and LP decoders results in a nonintegral solution with a probability bounded away from zero at all SNR, even in the absence of noise. Moreover, there are some intermediate channels, which we designate as *asymptotically proper*, for which the performance asymptotically converges to that of ML detection at high SNR. In order to explain this behavior, we analyze the error events of LP detection, and derive a closed-form parameter that can be used to classify different ISI channel impulse responses in terms of the performance of the proposed graph-based detection scheme. In particular, using this measure, we prove sufficient conditions for a channel to be improper.

When LDPC decoding is incorporated in the detector, IMP detection often significantly outperforms LP detection. With LP detection, proper and asymptotically proper channels achieve very good performance; for some other channels, the word error rate (WER) cannot go below a certain value. On the other hand, IMP detection often produces reasonable results, even on some improper channels.

The rest of this paper is organized as follows. In Section II, we describe the channel, and introduce the graph-based formulation of ML detection. The performance analysis and simulation results of uncoded graph-based detection are presented in Section III. In Section IV, we study the combination of graph-based detection and LDPC decoding, and Section V concludes the paper.

## II. Graph-Based Detection

### A. Channel Model

We consider a partial-response (PR) channel with bipolar (BPSK) inputs, as described in Fig. 1, and use the following notation for the transmitted symbols.

*Notation 1:* The bipolar version of a binary symbol, $b \in \{0,1\}$, is denoted by $\tilde{b} \in \{-1,1\}$, and is given by

$$\tilde{b} = 1 - 2b. \tag{1}$$

The partial-response channel transfer polynomial is $h(D) = \sum_{i=0}^{\mu} h_i D^i$, where $\mu$ is the channel memory size. Thus, the output sequence of the PR channel in Fig. 1 before adding the white Gaussian noise can be written as

$$y_t = \sum_{i=0}^{\mu} h_i \tilde{x}_{t-i}. \tag{2}$$

### B. Maximum-Likelihood (ML) Detection

Given the vector of received samples $\underline{r} = [r_1\ r_2\ \ldots\ r_n]^T$, the ML detector solves the optimization problem

$$\begin{aligned} &\text{Minimize} \quad \|\underline{r} - \underline{y}\|_2 \\ &\text{Subject to} \quad \underline{x} \in C \end{aligned} \tag{3}$$

where $C \subset \{0,1\}^n$ is the codebook and $\|\cdot\|_2$ denotes the $L_2$-norm. By expanding the square of the objective function, the problem becomes equivalent to minimizing

$$\begin{aligned} \sum_t (r_t - y_t)^2 &= \sum_t \left[ r_t^2 - 2r_t \sum_i h_i \tilde{x}_{t-i} \right. \\ &\qquad \left. + \left( \sum_i h_i \tilde{x}_{t-i} \right)^2 \right] \\ &= \sum_t \left[ r_t^2 - 2r_t \sum_i h_i \tilde{x}_{t-i} + \sum_i h_i^2 \tilde{x}_{t-i}^2 \right. \\ &\qquad \left. + \sum_{i \neq j}\sum h_i h_j \tilde{x}_{t-i} \tilde{x}_{t-j} \right] \end{aligned} \tag{4}$$

where, for simplicity, we have dropped the limits of the summations. Equivalently, we can write the problem in a general matrix form

$$\begin{aligned} &\text{Minimize} \quad -\underline{q}^T \underline{\tilde{x}} + \frac{1}{2} \underline{\tilde{x}}^T P \underline{\tilde{x}} \\ &\text{Subject to} \quad \underline{x} \in C \end{aligned} \tag{5}$$

where in this problem $q_t = \sum_i h_i r_{t+i}$, and $P = H^T H$, with $H$ defined as the $n \times n$ Toeplitz matrix

$$H = \begin{bmatrix} h_0 & 0 & \cdots & & & \\ \vdots & \ddots & & & & \\ h_\mu & \cdots & h_0 & 0 & & \\ 0 & h_\mu & & h_0 & 0 & \\ \vdots & & \ddots & & \ddots & \\ 0 & \cdots & 0 & h_\mu & \cdots & h_0 \end{bmatrix}. \tag{6}$$

Here we have assumed that $\mu$ zeros are padded at the beginning and the end of the transmitted sequence, so that the trellis diagram corresponding to the ISI channel starts and ends at the zero state. If the signals are uncoded, i.e., $C = \{0,1\}^n$, and $\underline{q}$ and $P$ are chosen arbitrarily, (5) will represent the general form of an integer quadratic programming (IQP) problem, which is, in general, NP-hard. In the specific case of a PR channel, where we have the Toeplitz structure of (6), the problem can be solved by the Viterbi algorithm with a complexity linear in $n$, but exponential in $\mu$. However, this model can also be used to describe other problems such as detection in MIMO or two-dimensional ISI channels. Also, when the source symbols belong to a nonbinary alphabet with a regular lattice structure such as the QAM and PAM alphabets, the problem can be reduced to the binary problem of (5) by introducing some new variables.

### C. Detection on a Graph

A common approach for solving the IQP problem is to first convert it to an integer LP (ILP) problem by introducing a new variable for each quadratic term, and then to relax the integrality condition; e.g., see [8]. While this relaxed problem does not necessarily have an integer solution, it can be used along with branch-and-cut techniques to solve integer problems of reasonable size. A more recent method is based on dualizing the IQP problem twice to obtain a convex relaxation in the form of a semidefinite program (SDP) [9], [10].

In this work, we use a method similar to [8] to derive an ILP form of the ML detection problem. In [8], the original IQP problem is given in terms of 0-1 variables, and each of a number of 0-1 auxiliary variables is defined as the product of two original 0-1 variables. To apply this method, here we can first convert the bipolar (i.e., $\pm 1$) IQP problem in (5) into an equivalent 0-1 problem, and then define the auxiliary variables as in [8] to linearize the objective function. As an alternative, we can also define the auxiliary variables in the $\pm 1$ domain as the pairwise products of original $\pm 1$ variables. If we then convert the linearized problem from the $\pm 1$ domain to the 0-1 domain, the binary versions of the auxiliary variables will each become the modulo-2 sum of two binary variables, rather than the product. In this work, we use the latter approach, since modulo-2 additive constraints are similar to parity-check constraints; thus, LP and message-passing decoders designed for decoding binary linear codes can be applied without any modification. However, in the Appendix we show that the LP relaxations resulting from these two approaches are equivalent.

To linearize (4), we define

$$\tilde{z}_{t,j} = \tilde{x}_t \cdot \tilde{x}_{t-j}$$
$$j = 1,\ldots,\mu, \quad t = j+1,\ldots,n. \quad (7)$$

In the binary domain, this will be equivalent to

$$z_{t,j} = x_t \oplus x_{t-j} \quad (8)$$

where $\oplus$ stands for modulo-2 addition. Hence, the right-hand side (RHS) of (4) is a linear combination of $\{x_t\}$ and $\{z_{t,j}\}$, plus a constant, given that $\tilde{x_i}^2 = 1$ is a constant. With some simplifications, the IQP in (5) can be rewritten as

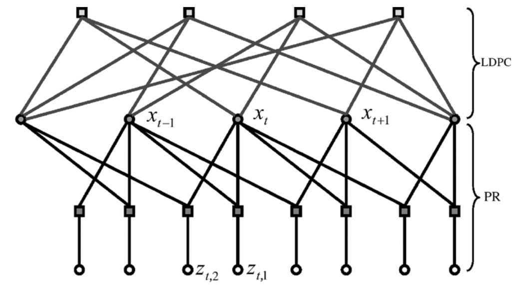


Fig. 2. PR channel and LDPC code represented by a Tanner graph.

$$\begin{aligned} \text{Minimize} \quad & \sum_t q_t x_t + \sum_t \sum_j \lambda_{t,j} z_{t,j} \\ \text{Subject to} \quad & \underline{x} \in C, \\ & z_{t,j} = x_t \oplus x_{t-j},\ j = 1,\ldots,\mu, \\ & t = j+1,\ldots,n \end{aligned} \quad (9)$$

where, in the equalization problem

$$\lambda_{t,j} = -P_{t,t-j} = -\sum_{i=0}^{\min(\mu-j,n-t)} h_i h_{i+j}. \quad (10)$$

In this optimization problem, we call $\{x_i\}$ the information bits, and $\{z_{t,j}\}$ the state bits. It can be seen from (10) that $\lambda_{t,j}$ is independent of $t$, except for indices near the two ends of the block; i.e., $1 \le t \le \mu$ and $n-\mu+1 \le t \le n$. In practice, this "edge effect" can be neglected due to the zero padding at the transmitter. For clarity, we sometimes drop the first subscript in $\lambda_{t,j}$ when the analysis is specific to the PR detection problem.

The combined equalization and decoding problem (9) has the form of a single decoding problem, which can be represented by a low-density Tanner graph. It is worth emphasizing that there is no approximation in this derivation, and hence this decoding problem is equivalent to ML detection. Fig. 2 shows an example of the combination of a PR channel of memory size 2 with an LDPC code. We call the upper and lower layers of this Tanner graph the code layer and the PR layer (or the PR graph), respectively. The PR layer of the graph consists of $\mu n$ check nodes $c_{t,j}$ of degree 3, each connected to two information bit nodes $x_t$, $x_{t-j}$, and one distinct state bit node, $z_{t,j}$. Also, the PR layer can contain cycles of length 6 and higher. If a coefficient, $\lambda_{t,j}$, is zero, its corresponding state bit node, $z_{t,j}$, and the check node it is connected to can be eliminated from the graph, as they have no effect on the detection process.

It follows from (10) that the coefficients of the state bits in the objective function, $\{\lambda_{t,j}\}$, are only a function of the PR channel impulse response, while the coefficients of the information bits are the results of matched filtering the noisy received signal by the channel impulse response, and therefore dependent on the noise realization. Once the variable coefficients in the objective function are determined, IMP or LP decoding can be directly applied to decode on this Tanner graph. We call this method *graph-based detection* or, more specifically, *IMP detection* and *LP detection*, when IMP and LP decoding, respectively, are used.

For graph-based detection by means of LP decoding, we use the relaxation introduced in [6], where the binary parity-check constraint corresponding to each check node $c$ is relaxed as follows. Let $N_c$ be the index set of neighbors of check node $c$, i.e., the variable nodes it is directly connected to in the Tanner graph. Then, we include the following constraints:

$$\sum_{i\in V} x_i - \sum_{i\in N_c\setminus V} x_i \le |V| - 1 \qquad \forall\, V \subset N_c \text{ s.t. } |V| \text{ is odd.} \quad (11)$$

In addition, the integrality constraints $x_i \in \{0,1\}$ are relaxed to box constraints $0 \le x_i \le 1$. This relaxation has the *ML certificate property*; i.e., if the solution of the relaxed LP is integral, it will also be the solution of (9).

For graph-based detection using IMP decoding, note that the ML detection problem (9) is equivalent to ML decoding of a linear code on a memoryless channel, where the variables $\{x_t, z_{t,j},\ \forall t,j\}$ are the bit variables, coefficients $\{q_t, \lambda_{t,j},\ \forall t,j\}$ are their log-likelihood ratio (LLR) values—albeit up to a scalar multiplicative factor—and the constraints of the problem are the parity-check constraints defining the linear code. Hence, IMP decoding algorithms commonly used for linear codes can be directly applied to this equivalent decoding problem. In particular, the Min-Sum Algorithm (MSA) can be used to solve this problem with no modification to the coefficients $\{q_t, \lambda_{t,j},\ \forall t,j\}$, since MSA is not sensitive to a uniform scaling of the LLR values. On the other hand, to apply the Sum-Product Algorithm (SPA), proper scaling should be applied to the coefficients to derive suitable values to serve as equivalent LLRs of the bit variables.

In the following section, we use the MSA directly, treating the coefficients in (9) as the equivalent LLR values. In Section IV, we describe a reasonable scaling of these coefficients that can be used for IMP detection with the SPA.

## III. Performance Analysis of Uncoded Detection

In this section, we study the performance of graph-based detection using LP relaxation in the absence of coding, i.e., solving (5) with $C = \{0,1\}^n$. It is known that if the off-diagonal elements of $P$ are all nonpositive; i.e., $\lambda_{t,j} \ge 0,\ \forall j \ne 0, t$, the 0-1 problem is solvable in polynomial time by reducing it to the MIN-CUT problem; e.g., see [11]. As an example, Sankaran and Ephremides [12] argued using this fact that when the spreading sequences in a synchronous CDMA system have nonpositive cross-correlations, optimal multiuser detection can be done in polynomial time. Here, we derive a condition, slightly weaker than the nonnegativity of the $\lambda_{t,j}$, that is necessary and sufficient for the LP relaxation to have an integer solution for any value of $\underline{q}$ in (9). The derivation also sheds some light on the question of how the algorithm behaves in cases where this condition is not satisfied. For a check node in the Tanner graph connecting information bit nodes $x_t$ and $x_{t-j}$ and state bit node $z_{t,j}$, the constraints (11) can be summarized as

$$\begin{aligned} z_{t,j} &\ge \max[x_t - x_{t-j}, x_{t-j} - x_t] \\ z_{t,j} &\le \min[x_t + x_{t-j}, 2 - x_t - x_{t-j}] \end{aligned} \quad (12)$$

which can be represented more simply as

$$|x_t - x_{t-j}| \le z_{t,j} \le 1 - |x_t + x_{t-j} - 1|. \quad (13)$$

Since there is exactly one such pair of upper and lower bounds for each state bit $z_{t,j}$, its value in the solution vector will be equal to either the lower or upper bound, depending on the sign of its coefficient in the linear objective function $\lambda_{t,j}$. Hence, the cost of $z_{t,j}$ in the objective function can be written as

$$\lambda_{t,j} z_{t,j} = \begin{cases} \lambda_{t,j}|x_t - x_{t-j}| & \text{if } \lambda_{t,j} \ge 0, \\ \lambda_{t,j} - \lambda_{t,j}|x_t + x_{t-j} - 1| & \text{if } \lambda_{t,j} < 0, \end{cases} \quad (14)$$

where the first term in the second line is constant and does not affect the solution. Consequently, by substituting (14) in the objective function, the LP problem will be projected into the original $n$-dimensional space, giving the equivalent minimization problem

$$\begin{aligned} \text{Minimize} \quad & f(\underline{x}) = \sum_t q_t x_t \\ & + \sum_{t,j:\lambda_{t,j}>0} \sum |\lambda_{t,j}||x_t - x_{t-j}| \\ & + \sum_{t,j:\lambda_{t,j}<0} \sum |\lambda_{t,j}||x_t + x_{t-j} - 1|, \\ \text{Subject to} \quad & 0 \le x_t \le 1,\ \forall t = 1, \ldots, n \end{aligned} \quad (15)$$

which has a convex and piecewise-linear objective function. Each absolute value term in this expression corresponds to a check node in the PR layer of the Tanner graph representation of the problem.

### A. Proper Channels: Guaranteed ML Performance

We now discuss a class of channels, which we call *proper channels*, for which the proposed LP relaxation of uncoded graph-based detection always gives the ML solution. We also provide a criterion for recognizing a proper channel.

*Definition 1:* An LP relaxation of an optimization problem[1] is called *exact* if whenever the original optimization problem has a unique solution, the relaxed problem has the same unique solution, and vice versa.

Thus, a proper channel is one whose LP relaxation (15) is exact.

[1]With some abuse of terminology, here what we refer to as an optimization problem is in fact a class of optimization problems all having the same set of constraints, but different objective functions. An example is the class of decoding problems corresponding to a given code representation and channel model, but different transmitted codewords and/or noise realizations.

*Definition 2:* We call a check node $c_{t,j}$ in the PR layer of the Tanner graph representation of the graph-based detection problem a *positive (respectively, negative) check node* if the state bit node $z_{t,j}$ connected to this check node has a positive (respectively, negative) coefficient.

We now present a property of the PR Tanner graph that, as we will demonstrate, can be used to identify a proper channel.

**Cycle Condition (CC):** Every cycle in the channel graph contains an even number of negative check nodes.

The following theorem shows that satisfaction of the CC is a necessary and sufficient condition for a channel to be proper.

*Theorem 1:* Consider a PR channel with ML detection problem defined by (9), where $C = \{0,1\}^n$. If the associated channel graph satisfies the CC, then the channel is proper. Conversely, if the CC is not satisfied, there are instances of the corresponding LP relaxation that do not have an integer vector solution, implying that the channel is not proper.

*Proof:* If a bounded LP has a unique solution, it is always at one of the vertices of the feasible region. Hence, the proposed LP relaxation of problem (9) is exact if and only if all the vertices of its feasible region—i.e., all its potential unique solutions—are integer-valued. We first prove sufficiency: the CC guarantees that all the vertices of the feasible region of the LP relaxation are integral. We then prove necessity: if the CC is not satisfied, then some of the coordinates in the LP solution have fractional values.

*1) Sufficiency:* The objective function in (15) is a piecewise linear function $f : \mathbb{R}^n \mapsto \mathbb{R}$. We call $\underline{a}$ a *breakpoint* of $f$ if the derivative of $f(\underline{a} + s\underline{v})$ with respect to $s$ changes at $s = 0$, for any nonzero vector $\underline{v} \in \mathbb{R}^n$.

Now, consider (15) without the box constraints. The unique optimum point $\underline{x}^*$ of the objective function has to occur either at infinity or at a breakpoint of $f$. Since the nonlinearity in the function comes from the terms involving the absolute value function, each breakpoint is determined by making $n$ of these absolute value terms *active*, i.e., setting their arguments equal to zero. These $n$ terms should have the property that the linear system of equations obtained by setting their arguments to zero has a unique solution.

When the feasible region is restricted to the unit cube $[0,1]^n$, the optimum can also occur at the boundaries of this region. Without loss of generality, we can assume that the optimum point lies on $k$ hyperplanes corresponding to the box constraints in (15), where $k = 0, \ldots, n$. This will make exactly $k$ variables, $x_i,\ i \in I$, equal to either 0 or 1, for some index set $I \subseteq \{1, \ldots, n\}$ with $|I| = k$. In addition, at least $n - k$ other equations are needed to determine the remaining $n-k$ fractional variables. Each of these equations will be the result of making some number of absolute value terms active, each having the form $x_t = x_{t-j}$ or $x_t + x_{t+j} = 1$, depending on whether $\lambda_{t,j} > 0$ or $\lambda_{t,j} < 0$, respectively. When an absolute value term in (15) is active, either both or neither of its variables can be integer-valued. Since the former case does not provide an equation in terms of the fractional variables, we can assume that all of these active absolute value terms involve only fractional variables.

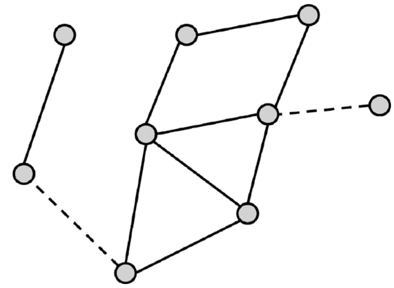

Fig. 3. The dependence graph of the system of linear equations with one cluster of cycles. Solid lines represent positive edges and dashed lines represent negative edges.

We now examine the question of when such equations can have a unique and nonintegral solution. To determine the answer, we represent the system of equations by a *dependence graph*. The vertices correspond to the unknowns, i.e., the $n-k$ fractional variable nodes. Between vertices $x_s$ and $x_{s-i}$, there is an edge if a check node $c_{s,i}$ connects them in the PR Tanner graph. We refer to the edge as a *positive* edge if $c_{s,i}$ is a positive check node (i.e., $\lambda_{s,i} > 0$) and a *negative* edge if $c_{s,i}$ is a negative check node (i.e., $\lambda_{s,i} < 0$). An example of a dependence graph is shown in Fig. 3, where a a solid line denotes a positive edge and a dotted line denotes a negative edge. Note that the corresponding Tanner graph satisfies the CC. If two vertices are connected in the dependence graph by a positive edge, the corresponding variables will have the same value in the solution of the system of equations. Hence, we can collapse the positive edge and merge these two vertices into a single vertex, and the value that this combined vertex takes will be shared by the two original vertices. If we perform this collapsing for every positive edge, we will be left with a reduced dependence graph that has only negative edges.

We claim that, if the PR Tanner graph satisfies the CC, the reduced dependence graph will not contain any cycles of odd length. To see this, consider a cycle in the original dependence graph, passing through a negative edge $e_{t,j}$, which connects vertex $x_t$ on its "left side" to vertex $x_{t-j}$ on its "right side." The CC ensures that any path other than $e_{t,j}$ from vertex $x_t$ to vertex $x_{t-j}$ passes through an odd number of negative edges. This property will be inherited by the reduced graph, since only positive edges collapse and no new path will be created between $x_t$ and $x_{t-j}$. This proves the claim.

Now consider a cycle of length $l$ in the reduced dependence graph. The system of linear equations corresponding to the edges in this cycle, possibly after some row and column permutations in the coefficient matrix, will have the form

$$\begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ \vdots & & \ddots & & \\ 0 & 0 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{i_1} \\ x_{i_2} \\ \vdots \\ x_{i_l} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \tag{16}$$

where the $l \times l$ coefficient matrix contains exactly two ones in each row. Since $l$ is even, the coefficient matrix is rank deficient, and we can remove any one of its rows without losing any useful equation. Hence, we can break any cycle of the reduced dependence graph by removing an edge from it without changing the space of system solutions. By repeating this for every cycle, we will be left with an equivalent dependence graph which is a tree. Since trees have fewer edges than vertices, the system of equations for determining the unknown variables will be underdetermined, and none of the nodes in the dependence graph will have a unique solution. Consequently, the only case where we have a unique solution for any of the variables will be $k = n$, which means that all of the variables $\{x_i\}$ are integral. This proves the sufficiency of the CC.

*2) Necessity:* We now prove the necessity of the CC by means of a counterexample. Consider a case where the realization of the noise sequence is such that the received sequence is zero. This will make $\{q_t\}$, the coefficients of the linear term in (15), equal to zero. Hence we are left with the positive-weighted sum of a number of absolute value terms, each of them being greater than or equal to zero. The objective function will become zero if and only if all these terms are zero, which is satisfied if $\underline{x} = [\frac{1}{2}, \ldots, \frac{1}{2}]^T$. We need to show that if the CC is not satisfied, equating all of the absolute value terms to zero will determine a unique, fractional value for at least one of the elements of $\underline{x}$, thereby implying that the relaxation is not exact.

Consider the dependence graph of this system of equations, and for simplicity assume that all of the positive edges have been collapsed in order to obtain a reduced dependence graph with negative edges only, following the procedure described in the proof of sufficiency. Note that, in contrast to what was assumed in the sufficiency proof, here there is at lease one cycle of odd length. For any such cycle, the corresponding coefficient matrix in (16) has full rank, and therefore the unique solution to this system is $x_{i_k} = \frac{1}{2}, k = 1, \ldots, l$. This means that no integral vector $\underline{x}$ can make the objective function in (15) zero, and therefore no integral vector can be a solution to the problem. This completes the proof of the necessity of the CC for the LP relaxation to be exact, and for the channel to be proper. ■

*Corollary 1:* The unique solutions of the LP relaxation of the uncoded graph-based detection problem are in the space $\{0, \frac{1}{2}, 1\}^n$.

*Proof:* The values of the fractional elements of $\underline{x}$ are the unique solutions of a system of linear equations, each of the form $x_t = x_{t+i}$ or $x_t + x_{t+i} = 1$. The vector $[\frac{1}{2}, \ldots, \frac{1}{2}]$ satisfies all these equations, and, hence, it must be the unique solution. ■

### *B. Implications of the CC*

In the uncoded PR detection problem, validity of either of the following conditions is sufficient for the CC to be satisfied and therefore, by Theorem 1, for the channel to be proper.

1) **Acyclicity Condition (AC):** The PR Tanner graph is acyclic.
2) **Nonnegativity Condition (NC):** All state variables have nonnegative coefficients; i.e., $\lambda_{t,j} \geq 0 \; \forall t, j$.

The following lemma shows that the Acyclicity Condition also has implications for the coefficients of state variables.

*Lemma 1:* If the PR channel satisfies the Acyclicity Condition, the number of nonzero state-variable coefficients $\{\lambda_{t,j} : t = 1, \ldots, n, \;\; j = 1, 2, \ldots\}$ must be less than $n$.

*Proof:* Let $\kappa$ be the number of nonzero elements of $\{\lambda_{t,j} : t = 1, \ldots, n, \;\; j = 1, 2, \ldots\}$. Then, it is easy to see that the PR Tanner graph will have $3\kappa$ edges and $n + 2\kappa$ vertices (i.e., variable or check nodes). But the graph can be acyclic only if $3\kappa \leq n + 2\kappa - 1$, which means that $\kappa < n$. ■

*Example 1:* It is easy to see that the memory-1 PR channel $h(D) = h_0 + h_1 D$, with $h_i \neq 0, \; i = 0, 1$ satisfies the AC. The state-variable coefficients are given by $\lambda_{t,1} = -h_0 h_1, \; 2 \leq t \leq n$. Lemma 1 is of course satisfied. On the other hand, these coefficients are not necessarily nonnegative. Therefore the NC may not hold. The memory-2 channel $h(D) = 1 + D - D^2$ also satisfies the AC, as well as Lemma 1. However, the state-variable coefficients satisfy $\lambda_{t,1} = -h_0 h_1 - h_1 h_2 = 0, \; 2 \leq t \leq n-1$ and $\lambda_{t,2} = -h_0 h_2 = 1, \; 3 \leq t \leq n$. Therefore this channel satisfies the NC, as well.

In a one-dimensional ISI channel, where the state coefficients are given by (10), the NC implies that the autocorrelation function of the discrete-time impulse response of the channel should be nonpositive everywhere except at time zero. As the memory size of the channel increases, this condition becomes more restrictive, so that a long and randomly-chosen impulse response is not very likely to satisfy the NC. However, in some applications, such as magnetic recording, the impulse response of the channel is first equalized to an optimized, constrained-length target response in order to simplify the subsequent sequence detection algorithm, which is often a trellis-based, Viterbi-like algorithm with complexity exponential in the target length. A possible alternative, based upon the linear relaxation of graph-based detection, might be to optimize the target channel subject to the NC. This should make it possible to achieve performance comparable to that of the trellis-based detector, without incurring the exponential complexity penalty.

This alternative approach can be applied to two-dimensional (2-D) ISI channels, for which there is no tractable extension of the Viterbi algorithm [7]. In a 2-D channel, the received signal $r_{t,s}$ at coordinate $(t, s)$ in terms of the transmitted symbol sequence $\{\tilde{x}_{t,s}\}$ has the form

$$r_{t,s} = \sum_{i=0}^{\mu} \sum_{j=0}^{\nu} h_{i,j} \tilde{x}_{t-i,s-j} + n_{t,s}. \quad (17)$$

Hence, following the same procedure that resulted in (10), one can obatain the coefficients of the state variables in the 2-D channel. In particular, the state variable defined as $z_{(t,s),(k,l)} = x_{t,s} \oplus x_{t-k,s-l}$ will have the coefficient

$$\gamma_{k,l} = -\sum_{i=0}^{\mu} \sum_{j=0}^{\nu} h_{i,j} h_{i+l,j+l}. \quad (18)$$

For simplicity, we have dropped the $(t, s)$ index in (18) because $\gamma$ is independent of $t$ and $s$, except near the boundaries of the transmitted array, where appropriate truncation may be used. Theorem 1 guarantees that ML detection can be achieved by LP detection if the NC is satisfied, i.e., $\gamma_{k,l} \geq 0$ for any $k, l > 0$. An example of a 2-D channel satisfying the NC is given by the impulse response matrix

$$[h_{i,j}] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \quad (19)$$

### C. Error Event Characterization: Asymptotically Proper and Improper Channels

In the preceding subsection, we showed that if the CC is not satisfied, there are noise configurations that result in the failure of LP detection to find the ML sequence. We now present a probabilistic analysis of the performance of LP detection for general, uncoded ISI channels and determine conditions under which such failure occurs. Specifically, we develop a notion of "distance" for detector error events that is helpful in understanding the behavior of channels that are not proper. We focus here on stationary 1-D PR channels, so we can assume that $\lambda_{t,j} = \lambda_j$ is independent of $t$, as discussed in Section II.C. However, we note that, with some modifications, a similar analysis can be applied to 2-D ISI channels.

We begin the error event analysis with a useful definition.

*Definition 3:* Given the solution, $\underline{\hat{x}}$, of LP detection on a PR, the *fractional set*, $F = \{i_1, \ldots, i_{n-k}\} \subset \{1, \ldots, n\}$, is the set of indices of the information bit nodes in the Tanner graph of the PR channel that have fractional values in the solution, $\underline{\hat{x}}$.

Our analysis is based upon the following assumption, which is supported by extensive computer simulation at high SNR.

*Assumption:* If the ML solution, which we denote by $\underline{w}$, is correct, then the integer-valued elements of the LP detection solution, $\underline{\hat{x}}$, are also correct.

We know from Corollary 1 that the fractional elements in the LP solution, $\underline{\hat{x}}$, are all equal to $\frac{1}{2}$. Therefore, the assumption implies that we have

$$\hat{x}_i = \begin{cases} \frac{1}{2} & \text{if } i \in F, \\ w_i & \text{if } i \notin F. \end{cases} \tag{20}$$

Since $\underline{\hat{x}}$ minimizes the objective function $f(\underline{x})$ of (15), we can associate with it a nonnegative penalty defined by

$$g(\underline{x}, \underline{\hat{x}}) \triangleq f(\underline{x}) - f(\underline{\hat{x}}) \geq 0. \tag{21}$$

Expanding $f(\underline{x})$ using the expression in (15), we can write this inequality in terms of $\{\lambda_{t,j}\}$, $\underline{x}$, and $\underline{q}$. To help simplify the resulting expression, we make use of the following lemma.

*Lemma 2:* Let $x_t$ and $x_s$ be binary variables and $\tilde{x}_t$ and $\tilde{x}_s$ be their bipolar versions, respectively. Define

$$h(x, y, \lambda) \triangleq \begin{cases} |\lambda||x - y| & \text{if } \lambda \geq 0, \\ |\lambda||x + y - 1| & \text{if } \lambda < 0. \end{cases} \tag{22}$$

Then, the following equations hold:
1) $h(x_t, x_s, \lambda_{|t-s|}) = \frac{1}{2}|\lambda_{|t-s|}| - \frac{1}{2}\lambda_{|t-s|}\tilde{x}_t\tilde{x}_s$,
2) $h(x_t, \frac{1}{2}, \lambda_{|t-s|}) = \frac{1}{2}|\lambda_{|t-s|}|$,
3) $h(\frac{1}{2}, \frac{1}{2}, \lambda_{|t-s|}) = 0$.

*Proof:* The equations can be verified by using (1), and testing all possible values for $x_t$ and $x_s$. ∎

By using Lemma 2, and cancelling the common terms in $f(\underline{w})$ and $f(\underline{\hat{x}})$, we can write $g(\underline{w}, \underline{\hat{x}})$ as

$$\begin{aligned} g(\underline{w}, \underline{\hat{x}}) = \sum_{t \in F} \Bigg[ & q_t \left( w_t - \frac{1}{2} \right) \\ & + \sum_{s \in F,\, s<t} \left( \frac{1}{2}|\lambda_{|t-s|}| - \frac{1}{2}\lambda_{|t-s|}\tilde{w}_t\tilde{w}_s \right) \\ & - \sum_{s \notin F} \frac{1}{2}\lambda_{|t-s|}\tilde{w}_t\tilde{w}_s \Bigg] \end{aligned} \tag{23}$$

where $\lambda_d = -\sum_i h_i h_{i+d}$ is defined to be zero if $d > \mu$. Since we assumed that $\underline{w}$ is equal to the transmitted sequence, we can expand $q_t$ as

$$\begin{aligned} q_t &= \sum_{i=0}^{\mu} h_i r_{t+i} \\ &= \sum_{i=0}^{\mu}\sum_{i=0}^{\mu} h_i h_j \tilde{w}_{t+i-j} + \sum_{i=0}^{\mu} h_i n_{t+i} \\ &= -\sum_{s=t-\mu}^{t+\mu} \lambda_{|t-s|}\tilde{w}_s + \eta_t \end{aligned} \tag{24}$$

where $\eta_t \triangleq \sum_i h_i n_{t+i}$. By substituting (24) into (23), and using the fact that $w_t - \frac{1}{2} = -\frac{1}{2}\tilde{w}_t$, we obtain

$$\begin{aligned} g(\underline{w}, \underline{\hat{x}}) &= \sum_{t \in F} \Bigg[ \frac{1}{2}\lambda_0 + \sum_{s \in F,\, s<t} \Bigg( \frac{1}{2}|\lambda_{|t-s|}| \\ &\qquad + \frac{1}{2}\lambda_{|t-s|}\tilde{w}_t\tilde{w}_s \Bigg) - \frac{1}{2}\eta_t\tilde{w}_t \Bigg] \\ &= \frac{1}{2}\left[ c_F + \frac{1}{2}\underline{\tilde{w}}_F^T \bar{P}_F \underline{\tilde{w}}_F + \underline{\eta}_F^T \underline{\tilde{w}}_F \right]. \end{aligned} \tag{25}$$

In this equation, $\underline{\tilde{w}}_F$ and $\underline{\eta}_F$ are obtained, respectively, from $\underline{\tilde{w}}$ and $\underline{\eta}$ by keeping only the elements with indices in $F$, $\bar{P}_F$ is a submatrix of $P$ (defined in Section II-B) consisting of the elements of $P$ with column and row indices in $F$ and its diagonal elements made equal to zero, and

$$c_F \triangleq \sum_{t \in F} \lambda_0 + \sum_{t,\, s \in F,\, s<t} |\lambda_{|t-s|}|. \tag{26}$$

Equations (21) and (25) lead to the following theorem, describing a condition for LP detector failure (in the assumed scenario where the ML sequence $\underline{w}$ equals the transmitted sequence and $\underline{\hat{x}}$ agrees with $\underline{w}$ on integer-valued elements).

*Theorem 2:* Uncoded LP detection fails to find the transmitted (and ML) sequence $\underline{x}$ if there is an index set $F \subset \{1, \ldots, n\}$ for which

$$c_F + \frac{1}{2}\underline{\tilde{x}}_F^T \bar{P}_F \underline{\tilde{x}}_F + \underline{\eta}_F^T \underline{\tilde{x}}_F > 0. \tag{27}$$

∎

If the transmitted sequence, $\underline{x}$, is given, we can estimate the probability that the failure condition given by Theorem 2 is satisfied, and determine the dominant error event causing such a failure. In order to do that, for any given $F$, we define a "distance" for the corresponding error event as

$$d_F = -c_F - \frac{1}{2}\underline{\tilde{x}}_F^T \bar{P}_F \underline{\tilde{x}}_F. \tag{28}$$

The corresponding noise variance is given by

$$\begin{aligned} \sigma_F^2 &\triangleq \text{var}\left[ \underline{\eta}_F^T \underline{\tilde{x}}_F \right] \\ &= \text{var}\left[ \sum_s n_s \sum_{t=s-\mu,\, t \in F}^{s} \tilde{x}_t h_{s-t} \right] \\ &= \sigma^2 \sum_s \left[ \sum_{t=s-\mu,\, t \in F}^{s} \tilde{x}_t h_{s-t} \right]^2 \end{aligned} \tag{29}$$

where $\sigma^2$ is the variance of each noise sample $n_t$. The probability that the error event corresponding to $F$ occurs will then be equal to

$$p(F, \underline{\tilde{x}}_F) = Q\left(\frac{d_F}{\sigma_F}\right) \tag{30}$$

where $Q(x)$ is the Gaussian $Q$-function. In order to find the dominant error event over all transmitted sequences, for every choice of the index set $F$, we should find the vector $\underline{\tilde{x}}_F \in \{-1, 1\}^{|F|}$ that maximizes the probability in (30). However, this will require an exhaustive search over all $\underline{\tilde{x}}_F$. As an alternative, we can upper bound this probability by finding the smallest distance $d_F^{\min} \triangleq \min_{\underline{\tilde{x}}_F} d_F$ and the largest standard deviation $\sigma_F{}^{\max} \triangleq \max_{\underline{\tilde{x}}_F} \sigma_F$, and computing $Q(d_F^{\min}/\sigma_F^{\max})$. Fortunately, each of these two optimization problems can be solved by dynamic programming (in particular, the Viterbi algorithm) using a trellis of at most $2^\mu$ states.

Referring to (28), we note that the minimum distance can be negative. In that case, the probability that the corresponding sequence $\underline{\tilde{x}}_F$ exists in the transmitted block is independent of the SNR. This means that the probability of failure $p(F, \underline{\tilde{x}}_F)$ will be greater than $\frac{1}{2}$ for any SNR value. Therefore, there will be a nondiminishing probability of failure as the SNR goes to infinity. This reasoning leads to the following corollary.

*Corollary 2:* If for an ISI channel, there is a index set $F \subset \{1, \ldots, n\}$ and a vector $\underline{\tilde{x}}_F \in \{-1, 1\}^{|F|}$ for which $d_F$, as defined in (28), is negative, LP detection on this channel will have a nondiminishing word error rate (WER) as SNR grows. ■

We call a channel for which LP detection has nondiminishing probability of error as the SNR increases an *improper* channel. Thus, the corollary provides a condition that implies the channel is improper.

For some ISI channels, for sufficiently large SNR, the probability of failure becomes dominated by the probability that the ML sequence is different from the one that was transmitted. For such a channel, the WER of graph-based detection is asymptotically equal to that of ML detection as the SNR increases, so we call it *asymptotically proper*.

*Remark 1:* The error events considered in the analysis do not represent all possible modes of detector failure. Rather, our analysis is intended to provide an estimate of the gap between the performance of LP and ML detection methods by estimating the probability that graph-based detection fails to find the ML solution when the ML detector is successful. Moreover, as mentioned before, we only studied the events where a sequence satisfying (20) has a lower cost than the transmitted sequence. Therefore, even if (27) does not hold for any $F$, it is theoretically possible that LP decoding for graph-based detection has a fractional solution. However, a substantial body of empirical evidence suggests that this is not very likely.

### D. *The Fully Fractional Solution*

Of particular interest among the possible error events is the one where the all-$\frac{1}{2}$ sequence has a lower cost than the correct solution; i.e., $F = \{1, \ldots, n\}$ in (27). For a given transmitted sequence $\underline{x}$, this event is not necessarily the most likely one. However, studying this event provides us with another, simpler sufficient condition for the failure of LP detection, further clarifying the distinction between the different classes of ISI channels.

The distance, $\delta$, corresponding to this event is obtained by putting $F = \{1, \ldots, n\}$ in (28). If the block length $n$ is much larger than the channel memory length $\mu$, we can neglect the "edge effects" caused by the indices that are within a distance $\mu$ of one of the two ends of the sequence. We will then have

$$\begin{aligned}\delta &= -n\lambda_0 - n\sum_{j=1}^{\mu}|\lambda_j| - \sum_{j=1}^{\mu}\lambda_j\sum_t \tilde{x}_t\tilde{x}_{t+j} \\ &= -n\lambda_0 - n\sum_{j=1}^{\mu}|\lambda_j| - \sum_{j=1}^{\mu}\lambda_j\rho_j \end{aligned}\tag{31}$$

where $\rho_j$ is the autocorrelation of $\underline{\tilde{x}}$ corresponding to a shift equal to $j$. On the other hand, for the noise variance, $\zeta^2$, corresponding to this event we see from (29) that

$$\begin{aligned}\zeta^2 &= \sigma^2\sum_t\left[\sum_{j=0}^{\mu}h_j\tilde{x}_{t-j}\right]^2 \\ &= \sigma^2\underline{\tilde{x}}^T P\underline{\tilde{x}} \\ &= -\sigma^2 n\lambda_0 - 2\sigma^2\sum_{j=1}^{\mu}\lambda_j\rho_j. \end{aligned}\tag{32}$$

Note that $\delta$ and $\zeta^2$ have a similar dependence on the transmitted sequence.

A possible approach for finding the likelihood of occurrence of an all-$\frac{1}{2}$ error event is to maximize $\delta/\zeta$ over all possible transmitted sequences. However, with a long, randomly chosen transmitted sequence, the probability that this ratio will be close to its worst case value may become very low. On the other hand, as we now show, as $n$ grows with $\mu$ remaining fixed, the dependence of both $\delta$ and $\zeta^2$ on the transmitted sequence becomes negligible compared to the constant terms in their respective definitions.

*Lemma 3:* Let $\tilde{x}_1, \ldots, \tilde{x}_n$ be a sequence of i.i.d. $\pm 1$ random variables, each equally likely to be $+1$ or $-1$, and let $\rho_j = \sum_{t=1}^{n-j}\tilde{x}_t\tilde{x}_{t+j}$. Then, for fixed $\mu$, as $n \to \infty$

$$\frac{1}{n}\sum_{j=1}^{\mu}\lambda_j\rho_j \to 0 \quad \text{almost surely.} \tag{33}$$

*Proof:* For each $j = 1, \ldots, \mu$, $\rho_j$ is the sum of $n - j$ terms of the form $\tilde{x}_t\tilde{x}_{t+j}$. Clearly, each of these terms is equally likely to be $+1$ or $-1$. Furthermore, it can be shown that these terms are mutually independent[2]. Hence, using the strong law of large numbers, we have

$$\frac{\rho_j}{n} = \frac{n-j}{n}\left(\frac{1}{n-j}\sum_{t=1}^{n-j}\tilde{x}_t\tilde{x}_{t+j}\right) \to 0 \text{ almost surely,} \tag{34}$$

[2]For a proof of this statement, refer to Proposition 1.1 of [13].

where we have used the fact that $(n-j)/n \to 1$, since $1 \le j \le \mu$. Consequently

$$\frac{1}{n}\sum_{j=1}^{\mu}\lambda_j\rho_j = \sum_{j=1}^{\mu}\lambda_j\frac{\rho_j}{n} \to 0 \text{ almost surely} \quad (35)$$

since the sum is a linear combination of a finite number of variables, each going to zero almost surely. ■

Using Lemma 3 in (31) and (32), with $n$ large, we can write

$$\delta = n\left[|\lambda_0| - \sum_{j=1}^{\mu}|\lambda_j| + o(1)\right], \quad (36)$$

$$\zeta^2 = \sigma^2 n[|\lambda_0| + o(1)] \quad (37)$$

where we have used the fact that $\lambda_0 = -\sum h_i^2 \le 0$. Since the probability of the all-$\frac{1}{2}$ error event is equal to $Q(\delta/\zeta)$, these results motivate us to define the following parameter to characterize the limiting probability as $n \to \infty$.

*Definition 4:* The *LP distance*, $\delta_\infty$, of a partial-response channel is given by

$$\delta_\infty = \frac{1}{|\lambda_0|}\left(|\lambda_0| - \sum_{j=1}^{\mu}|\lambda_j|\right). \quad (38)$$

The LP distance is a dimensionless parameter that can take values between $-\infty$ and 1. The following theorem gives a new sufficient condition for a channel to be improper, expressed in terms of the distance $\delta_\infty$.

*Theorem 3:* The WER of uncoded graph-based detection by LP decoding over an ISI channel with the transmitted sequence generated as a random sequence of i.i.d. Bernouli$(1/2)$ binary symbols goes to 1 as the block length $n$ goes to infinity for any SNR. That is, the channel is improper if the LP distance $\delta_\infty$ of the channel is nonpositive.

*Proof:* As mentioned above, the probability of the all-$\frac{1}{2}$ event is equal to $Q(\delta/\zeta)$. From (36) and (37), we have, for large $n$

$$\frac{\delta}{\zeta} = \sqrt{n}\left(\delta_\infty\frac{\sqrt{|\lambda_0|}}{\sigma} + o(1)\right). \quad (39)$$

If $\delta_\infty < 0$, the RHS will approach $-\infty$ as $n$ increases, hence, $Q(\delta/\zeta)$ will go to 1. ■

The theorem shows that $\delta_\infty$ can provide evidence that a channel is improper. It is, therefore, natural to ask how it behaves for proper channels; i.e., channels satisfying the CC in Theorem 1. The following lemma provides an answer.

*Lemma 4:* For proper channels that satisfy the NC, $\delta_\infty \ge \frac{1}{2}$.

*Proof:* For any ISI channel, we can write

$$\left[\sum_{i=0}^{\mu}h_i\right]^2 = \sum_i h_i^2 + \sum_{i,j;i\ne j}\sum h_ih_j$$
$$= |\lambda_0| - 2\sum_{j=1}^{\mu}\lambda_j \ge 0. \quad (40)$$

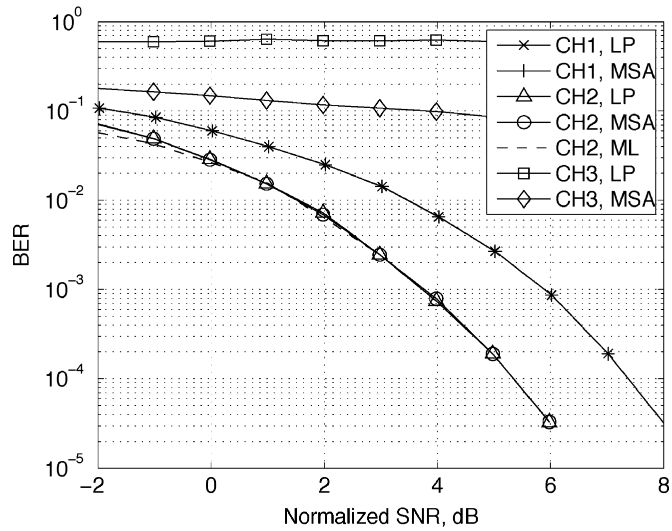


Fig. 4. BER for CH1-CH3. SNR is defined as the ratio of the transmitted signal power to the received noise variance.

Hence

$$\sum_{j=1}^{\mu}\lambda_j \le \frac{1}{2}|\lambda_0|. \quad (41)$$

Since the NC is satisfied, $\lambda_j \ge 0,\ \forall j \ge 1$. Therefore, we have

$$\delta_\infty = \frac{1}{|\lambda_0|}\left(|\lambda_0| - \sum_{j=1}^{\mu}\lambda_j\right)$$
$$\ge \frac{1}{2}. \quad (42)$$

■

### E. Simulation Results

We have simulated graph-based detection on the PR Tanner graph using LP decoding and the min-sum algorithm (MSA) for three PR channels of memory size 3:

1) **CH1:** $h(D) = 1 - D - 0.5D^2 - 0.5D^3$ (with $\delta_\infty = \frac{1}{2}$, satisfies CC; proper);
2) **CH2:** $h(D) = 1 + D - D^2 + D^3$ (with $\delta_\infty = \frac{1}{2}$);
3) **CH3:** $h(D) = 1 + D - D^2 - D^3$ (EPR4 channel with $\delta_\infty = 0$; improper).

Bit error rates (BER) obtained on these uncoded channels using LP and MSA are shown in Fig. 4. Since CH1 satisfies the CC, LP will achieve ML performance on this channel. For CH2, the BER curve can be compared to that of the ML detector, which is also shown. Except at very low SNR where there is a small difference, the performance of LP and ML are nearly equal, which means that CH2 is an asymptotically proper channel. For both CH1 and CH2, the MSA detector converges in at most three iterations and has a BER very close to that of LP detection. In contrast, CH3 (i.e., the EPR4 channel) is an improper channel, and we observe that the BERs of the LP and MSA detectors are almost constant. It is interesting to note that Kurkoski *et al.* also observed a similar constant error-floor behavior in their message-passing detection of the uncoded EPR4 channel [4]. The floor was eliminated by the introduction of a suitably chosen

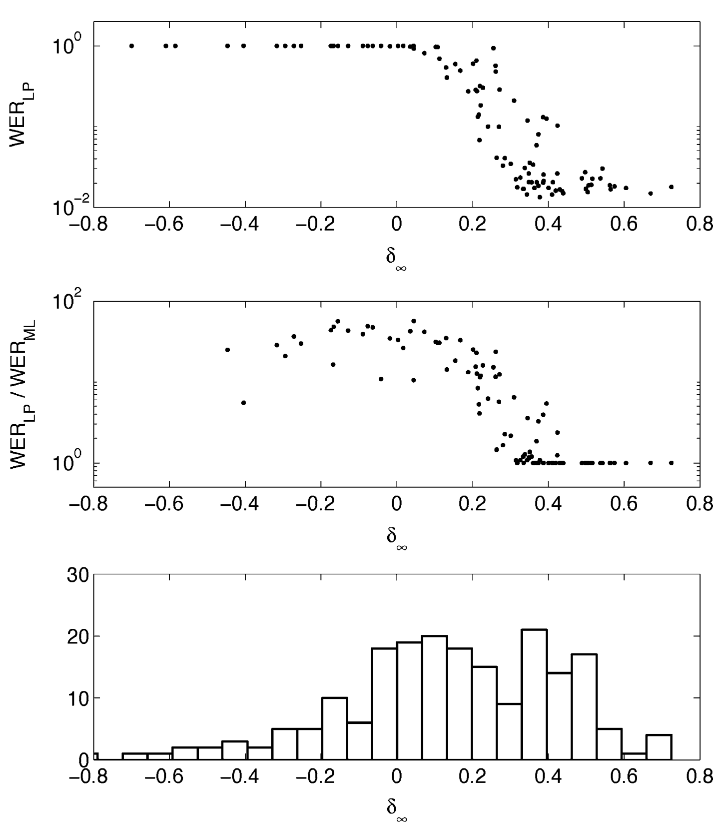


Fig. 5. Upper and middle plots: Performance of graph-based LP detection versus $\delta_\infty$ for random ISI channels of memory 4, with $\mathrm{SNR} = 11$ dB. Lower plot: Histogram of $\delta_\infty$ values for the same set of channels.

precoder. In an LDPC-coded system, on the other hand, their simulation results showed no evidence of a minimum BER. In the next section, we will similarly observe that, in the presence of coding, the error floor observed in Fig. 4 does not appear for message-passing algorithms on the proposed graph representation of the EPR4 channel.

We also studied the relationship between the ISI channel parameter $\delta_\infty$ and the performance of graph-based detection. We randomly generated 200 ISI channels with memory 4, choosing the taps of the impulse response to be i.i.d. samples from a zero-mean, unity-variance Gaussian distribution. The total energy of the impulse response was also normalized to one, i.e., $|\lambda_0| = 1$. For each channel, we then simulated the uncoded LP detector at an SNR of 11 dB with randomly chosen channel input sequences of length 100. In the upper and middle plots of Fig. 5, respectively, we show the WER of LP detection as well as the ratio of that WER to the WER of ML detection as a function of $\delta_\infty$ for all 200 of the ISI channels.

The results in Fig. 5 demonstrate a strong correlation between the performance of the LP detection algorithm and the value of $\delta_\infty$. In particular, almost every channel with $\delta_\infty < 0.1$ has a WER close to 1, while for almost every channel with $\delta_\infty > 0.4$, the WER of LP detection is very close to that of ML detection. Simulation results for the MSA detector showed a similar behavior, except for some channels with $0.05 < \delta_\infty < 0.3$, for which MSA performance was significantly better than that of the LP detector. This suggests that, for MSA, the transition from "improper" to "proper" error-rate characteristics begins at smaller values of $\delta_\infty$ than for LP.

Finally, to give a sense of the probability density function of $\delta_\infty$ corresponding to this random construction of ISI channels, we show in the lower plot of Fig. 5 a histogram of the $\delta_\infty$ values of the 200 ISI channels whose performance was simulated.

## IV. Combined Equalization and LDPC Decoding

One of the main advantages of the graph-based detection proposed in Section II is that it lends itself to the combining of the channel equalization with the decoding of an underlying error-correcting code. In this section, we study this joint detection scheme using both linear progamming and iterative, message-passing algorithms.

### A. Coded LP Detection

Joint LP equalization and decoding is represented by a linear relaxation of the IQP problem in (9), incorporating the constraints that apply to the uncoded graph-based detection, as well as the linear inequalities corresponding to the relaxation of parity-check constraints of the code. The latter constraints cut from the feasible polytope some of the fractional vertices that can cause the uncoded LP detector to fail, but they also add new fractional vertices to the polytope.

Unfortunately, except for certain structured codes, no general systematic method has been reported in the literature for computing the performance of LP decoding of explicitly given codes even on a memoryless channel. Performance analysis becomes even harder in the case of LP detection on ISI channels. Hence, we have not been able to fully generalize to the coded case the conditions derived for the uncoded LP detection in the previous section. However, we can generalize Theorem 3, which formulated in terms of $\delta_\infty$ a condition that implies the channel is improper, with nondecreasing WER asymptotically in the block length.

*Corollary 3:* Consider a linear code with no "trivial" (i.e., degree-1) parity checks, used on a channel satisfying $\delta_\infty < 0$. Then, coded LP detection on this system has a nondiminishing WER for large block lengths.

*Proof:* We have shown in Section III-C that if this condition is satisfied, the all-$\frac{1}{2}$ vector will with high probability have a lower cost than the transmitted vector.[3] It is now enough to show that the all-$\frac{1}{2}$ vector will not be cut from the polytope by any error-correcting code. To see this, consider a relaxed parity-check inequality of the form

$$\sum_{i\in V} x_i - \sum_{i\in N_c\setminus V} x_i \leq |V| - 1 \quad \forall\, V \subset N_c \text{ s.t. } |V| \text{ is odd} \tag{43}$$

where $N_c \geq 2$. To prove that this constraint is satisfied by the all-$\frac{1}{2}$ vector, we consider two cases: $|V| = 1$, and $|V| \geq 2$. If $|V| = 1$, the first sum in (43) will be equal to $1/2$, and the second sum will be greater than or equal to $1/2$, since $N_c\setminus V$ has at least one element. Hence, the left-hand side (LHS) of (43) will be less than or equal to $|V| - 1 = 0$. Also, if $|V| \geq 2$, the first sum will be equal to $|V|/2 \leq |V| - 1$, while the second sum is nonnegative. Therefore, the LHS of the inequality will be less than or equal to its RHS. Consequently, in both cases (43) will be satisfied. ∎

### B. Coded Message-Passing Detection

Similar to LP detection, graph-based IMP detection can be extended to coded systems by adding the parity-check constraints of the code to the PR Tanner graph, as shown in Fig. 2, and treating it as a single Tanner graph defining a linear code. Despite many similarities, LP and IMP decoding schemes behave differently when used for joint detection and decoding. For example, there is no immediately evident analog to Corollary 3 for IMP detection. On the contrary, we have observed in simulation studies that there are improper channels for which coded IMP detection does not exhibit the poor performance seen in the uncoded setting.

In this paper, we used both the min-sum algorithm (MSA) and the sum-product algorithm (SPA) for the implementation of coded IMP detection. As in the uncoded case, we use as the objective coefficients of MSA the same coefficients as those used for LP detection, i.e., $\{q_t\}$ and $\{\lambda_{t,j}\}$, since MSA is invariant under coefficient scaling. For SPA, note that each $q_t$ contains a Gaussian noise term with variance proportional to $\sigma^2$. Hence, one can argue that a suitable normalization of the objective coefficients to estimate the "equivalent LLRs" is achieved by multiplying all the objective coefficients by $2/\sigma^2$. An advantage of this normalization is that, in the absence of ISI, the normalized coefficients become the true LLR values of the received samples. Consequently, we have used for SPA detection the equivalent LLRs obtained by this normalization.

[3]The derivation of the limit of $\delta$ was based on the assumption that the transmitted sequence is an i.i.d. sequence, so that (33) holds. While the transmitted sequence is no longer i.i.d. in the presence of coding, we implicitly assume that (33) still holds. This is a sufficiently accurate assumption for all codes of practical interest. In particular, (33) can be proved for a random ensemble of LDPC codes.

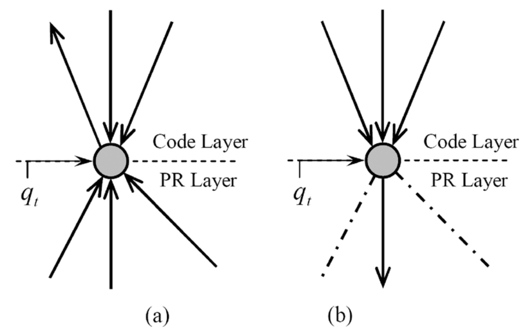


Fig. 6. Selective message passing at the information bit nodes. (a) Calculating a message outgoing to the code layer. (b) Calculating a message outgoing to the PR layer.

Decoding with MSA (respectively, SPA) on the Tanner graph defining the code represents an approximation of ML decoding (respectively, *a posteriori* probability decoding). The approximation becomes exact if the messages incoming to any node are statistically independent. This happens if the Tanner graph is cycle-free and the channel observations (i.e., the *a priori* LLRs) are independent. In the graph-based detection that we propose, neither of these two conditions is satisfied. In particular, the PR layer of the graph may contain many cycles of length 6, while the channel observations $\{q_t\}$ result from matched filtering of the received signal and, therefore, contain colored noise.

*1) Selective Message-Passing Algorithm:* In order to mitigate the positive feedback resulting from the cycles of the PR layer, we propose an alternative IMP approach, which we call *selective message passing (SMP)*, for coded detection. In SMP, we use a modified combining rule for messages at the information bit nodes, illustrated in Fig. 6. The outgoing message from an information bit node along an edge $e$ in the code layer is computed from the channel observation and the messages coming into the bit node along all edges, excluding edge $e$. On the other hand, the outgoing message along an edge in the PR layer is computed from the channel observation and incoming messages along the edges in the code layer only. Since there are no 4-cycles in the PR layer of the graph, this modification prevents any "closed-loop" circulation of messages within the PR layer. In other words, message passing inside the PR layer becomes effectively loop-free. However, there still remain cycles in the code layer, as well as cycles that have edges in both the code and PR layers. Simulation studies, discussed in more detail in Section IV-C, showed that SMP improved coded IMP detection on some channels that would otherwise perform poorly, such as the EPR4 channel.

*2) Performance Analysis:* Density evolution provides a powerful tool for analyzing the average performance of message-passing decoding of LDPC code ensembles on memoryless channels in the limit as the code length goes to infinity. The calculation of the ensemble average performance inherently assumes that the messages entering each node of the Tanner graph are statistically independent, an assumption that becomes accurate as the code length goes to infinity and the

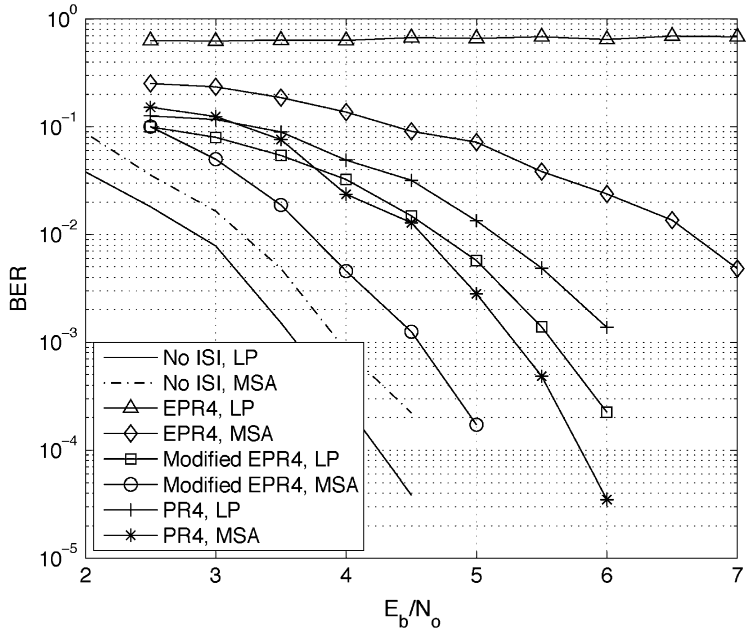


Fig. 7. BER versus $E_b/N_o$ for coded LP detection and coded MSA detection in four channels.

topology of the Tanner graph of the code approaches that of a cycle-free graph.

Given the success of density evolution in predicting the waterfall behavior of IMP decoding of LDPC codes, it is natural to try to extend this approach to the message-passing decoding of coded ISI channels. Unfortunately, we face some challenges in applying density evolution to the proposed message-passing detector in the presence of ISI. A key requirement in the application of density evolution is that the incoming messages at each node be statistically independent, or at least have a controlled statistical dependence which enables us to estimate the statistics of the outgoing messages. As is done in the setting of memoryless channels, we can consider the limiting case of an infinite-length, random LDPC code with corresponding cycle-free graph structure in the code layer of the Tanner graph. In the PR layer, however, there will generally be many cycles, with cycle length as short as 6, irrespective of the code length, as well as an overall lack of randomness in the graph interconnections. These properties notwithstanding, use of the SMP algorithm presented above may allow a meaningful application of density evolution. Specifically, a message passed from an information bit node to a check node in the PR layer of the Tanner graph is only derived from the channel observations, $\{q_t\}$, and a number of statistically independent messages coming from the code layer, as illustrated in Fig. 6. Furthermore, since each check node in the PR layer is connected to only two information bit nodes and one state bit node with a fixed and deterministic message, one can approximate the statistics of the messages going from these check nodes to the information bit nodes. In particular, the known pattern of statistical dependence between channel observations, $\{q_t\}$, makes it possible to approximate the dependence among the messages coming into an information bit node from nearby check nodes in the PR layer and, hence, to estimate the statistics of messages entering the code layer of the Tanner graph.

Another challenge in applying density evolution to the analysis of detectors for channels with memory is that, unlike the case of memoryless channels, the performance of the detector depends on the transmitted codeword. Hence, the simplifying assumption that the all-zeros codeword is transmitted cannot be used, and one needs to instead assume that a random (but valid) codeword is transmitted. In [15], Kavčić and Mitzenmacher proposed a method to tackle this problem based on *message flow neighborhoods* in the channel factor graph. A similar technique may be applicable to the performance analysis of the SMP detector.

In summary, with careful consideration of the structure of the PR layer, and the assumption of a random tree-like structure in the code layer, it should be possible to use suitably modified density evolution techniques to analyze the performance of the proposed SMP detection technique. We leave the pursuit of this idea to future research.

### C. Simulation Results

In this subsection, we present simulation results for coded detection in the presence of ISI using the schemes proposed above. In all cases, we have used a rate-$1/4$, length-200, (3,4)-regular LDPC code. The following PR channels were considered:

1) **Memoryless Channel:** $h(D) = 1$;
2) **PR4 Channel:** $h(D) = 1 - D^2$ ($\delta_\infty = \frac{1}{2}$; proper),
3) **EPR4 Channel:** $h(D) = 1 + D - D^2 - D^3$ ($\delta_\infty = 0$, improper);
4) **Modified EPR4:** $h(D) = 1 + D - D^2 + D^3$ ($\delta_\infty = \frac{1}{2}$, asymptotically proper).

In Fig. 7, the BER of coded LP and MSA detection on these channels has been plotted as a function of $E_b/N_o$. For all but the memoryless channel, coded MSA detection outperforms coded LP detection. In particular, for the EPR4 channel, coded LP detection has a BER of about $1/2$ for all SNR values, as predicted

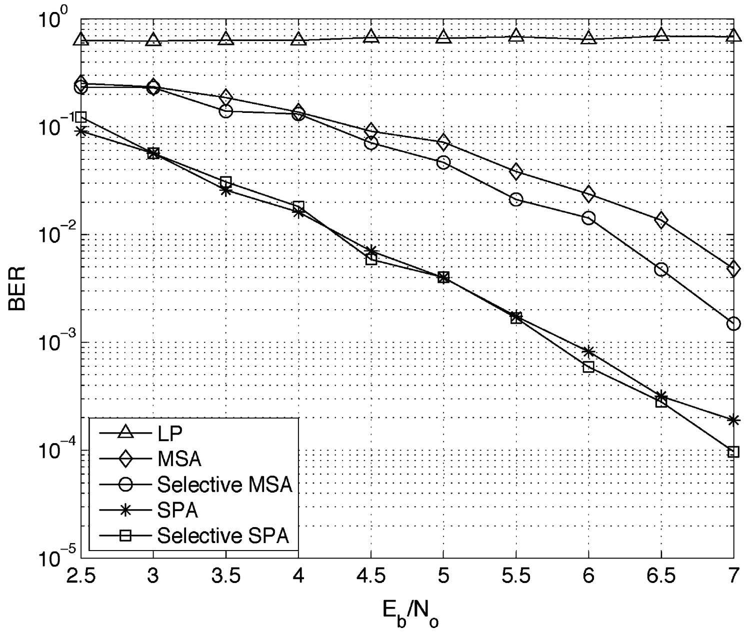


Fig. 8. BER versus $E_b/N_o$ for various coded detection schemes in the EPR4 channel.

by Corollary 3, while coded MSA detection has a monotonically decreasing BER.

For the EPR4 channel, the BER for LP, MSA, selective MSA, SPA, and selective SPA are compared in Fig. 8. SPA shows a gain of about 2 dB over MSA. (Although not shown in the figure, the gap between MSA and SPA for the other three PR channels was between 0.3 and 0.7 dB.) Selective message passing provides an additional gain of approximately 0.5 dB for MSA, but appears to offer little benefit when applied to SPA.

## V. Conclusion

In this paper, we introduced a new graph representation of ML detection in ISI channels, which can be used for combined equalization and decoding using LP relaxation or iterative message-passing methods. By a geometric study of the problem, we derived a necessary and sufficient condition for the equalization problem to give the ML solution for all transmitted sequences and all noise configurations under LP relaxation. Moreover, for certain other channels violating this condition, the performance of LP is very close to that of ML at high SNR. For a third class of channels, graph-based has a probability of failure bounded away from zero at all SNR, even in the absence of noise. In a step toward the analysis of the performance in the general case, we derived a distance $\delta_\infty$ for ISI channels, which can be used as a tool to estimate the asymptotic behavior of the proposed detection method. Simulation results show that message-passing techniques have a similar performance to that of LP detection for most channels. In addition, we studied graph-based joint detection and decoding of channels with LDPC-coded inputs. Simulation results indicate that, in contrast to the uncoded case, message-passing detection significantly outperforms LP detection for some channels.

## Appendix
## Equivalence of the LP Relaxation of Binary and Bipolar Forms of the IQP Problem

In this Appendix we show that if we first rewrite the objective function of the IQP problem (5) in terms of 0-1 variables, and then relax the resulting problem in to an LP, as done in [8], the resulting problem will be equivalent to the LP detection proposed in II.C. Clearly, since the parity-check constraints of an underlying code only involve the information variables, their relaxation will not be affected by the way the auxiliary variable nodes are defined. Hence, for simplicity of the expressions, here we only consider an uncoded scenario.

We prove the claim by showing that the relaxation of 0-1 form of the IQP problem results in an equivalent minimization problem identical to (15). We first start by rewriting the objective function of the IQP problem in (5) as a function of the 0-1 vector $\underline{x}$. Replacing $\underline{\tilde{x}}$ by $1-2\underline{x}$, the objective function becomes

$$\begin{aligned}&-\sum_{t=1}^{n} q_t + 2\sum_{t=1}^{n} q_t x_t + \frac{1}{2}\sum_{t=1}^{n}\sum_{s=1}^{n} P_{t,s}\\&-\sum_{t=1}^{n}\sum_{s=1}^{n} P_{t,s}x_s\\&-\sum_{t=1}^{n}\sum_{s=1}^{n} x_t P_{t,s} + 2\sum_{t=1}^{n}\sum_{s=1}^{n} x_t P_{t,s} x_s. \qquad (44)\end{aligned}$$

After removing the constant terms and factors and using (10), we obtain the equivalent objective function

$$\sum_t q_t x_t + 2\sum_t\sum_j \lambda_{t,j} x_t - 2\sum_t\sum_j \lambda_{t,j} w_{t,j} \qquad (45)$$

where we have defined the auxiliary variables $\{w_{t,j}\}$ as

$$w_{t,j} = x_t \cdot x_{t-j}, \quad j = 1, \ldots, \mu \quad t = j+1, \ldots, n. \tag{46}$$

As the next step, we need to relax the constraints above into linear constraints. Similar to [8], as well as the relaxation of Feldman *et al.* [6], we use linear constraints that define the convex hull of all binary 3-tuples, $(x_t, x_{t-j}, w_{t,j}) \in \{0,1\}^3$ that satisfy (46). This results in the four linear inequalities, which can be summarized as

$$\max(0, x_t + x_{t-j} - 1) \leq w_{t,j} \leq \min(x_t, x_{t-j}) \tag{47}$$

or equivalently

$$\frac{x_t + x_{t-j} - 1}{2} + \frac{|x_t + x_{t-j} - 1|}{2} \leq w_{t,j} \leq \frac{x_t + x_{t-j}}{2} - \frac{|x_t - x_{t-j}|}{2}. \tag{48}$$

Following a line of reasoning similar to that in Section III, we know that the value of $w_{t,j}$ at the solution of the relaxed LP problem will be equivalent to either the upper or lower bound given by (47), depending on the sign of its coefficient $-2\lambda_{t,j}$ in the objective function. Hence, similar to (14), the cost of $w_{i,j}$ in the optimum objective function will be

$$-2\lambda_{t,j} w_{t,j} = -\lambda_{t,j}(x_t + x_{t-j}) + \begin{cases} \lambda_{t,j}|x_t - x_{t-j}| & \text{if } \lambda_{t,j} \geq 0 \\ \lambda_{t,j} - \lambda_{t,j}|x_t + x_{t-j} - 1| & \text{if } \lambda_{t,j} < 0. \end{cases} \tag{49}$$

Substituting the expression above in (45), and neglecting the constant terms, yields the equivalent optimization problem

$$\begin{aligned} \text{Minimize } f(\underline{x}) = & \sum_t q_t x_t + 2\sum_t \sum_j \lambda_{t,j} x_t \\ & - \sum_t \sum_j \lambda_{t,j}(x_t + x_{t-j}) \\ & + \sum_{t,j:\lambda_{t,j}>0} \sum |\lambda_{t,j}||x_t - x_{t-j}| \\ & + \sum_{t,j:\lambda_{t,j}<0} \sum |\lambda_{t,j}||x_t + x_{t-j} - 1| \\ \text{Subject to } \quad & 0 \leq x_t \leq 1, \quad \forall t = 1, \ldots, n. \end{aligned} \tag{50}$$

Since the second and third sums in the objective function cancel each other, the equivalent problem becomes identical to that in (15). This proves that the information bits have equal optimal values in the two LP relaxations obtained from the 0-1 and bipolar forms of the IQP problem.


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**Mohammad H. Taghavi** (M'09) received the B.Sc. degree from the Sharif University of Technology, Tehran, Iran, in 2003 and the M.Sc. and Ph.D. degrees from the University of California at San Diego, La Jolla, in 2005 and 2008, respectively, all in electrical engineering.

Since 2008, he has been a Senior Engineer with Qualcomm, Inc., San Diego, CA, where he conducts research and development on wireless cellular and local area networks and contributes to the IEEE 802.11 standards. His research interests include error-correcting codes, wireless communications, information theory, and optimization theory.

Dr. Taghavi was a recipient of the Gold Medal of the Iranian National Physics Olympiad and the Gold Medal of the 30th International Physics Olympiad, Padova, Italy, 1999.

**Paul H. Siegel** (M'82–SM'90–F'97) received the S.B. and Ph.D. degrees in mathematics from the Massachusetts Institute of Technology (MIT), Cambridge, in 1975 and 1979, respectively.

He held a Chaim Weizmann Postdoctoral Fellowship at the Courant Institute, New York University. He was with the IBM Research Division in San Jose, CA, from 1980 to 1995. He joined the faculty at the University of California, San Diego, in July 1995, where he is currently a Professor of electrical and computer engineering in the Jacobs School of Engineering. He is affiliated with the California Institute of Telecommunications and Information Technology, the Center for Wireless Communications, and the Center for Magnetic Recording Research, where he holds an endowed chair and serves as Director. His primary research interests lie in the areas of information theory and communications, particularly coding and modulation techniques, with applications to digital data storage and transmission.

Prof. Siegel was a member of the Board of Governors of the IEEE Information Theory Society from 1991 to 1996 and was re-elected for a 3-year term in 2009. He served as Co-Guest Editor of the May 1991 Special Issue on "Coding for Storage Devices" of the IEEE TRANSACTIONS ON INFORMATION THEORY. He served the same TRANSACTIONS as Associate Editor for Coding Techniques from 1992 to 1995, and as Editor-in-Chief from July 2001 to July 2004. He was also Co-Guest Editor of the May/September 2001 two-part issue on "The Turbo Principle: From Theory to Practice" of the IEEE JOURNAL ON SELECTED AREAS IN COMMUNICATIONS. He was co-recipient, with R. Karabed, of the 1992 IEEE Information Theory Society Paper Award and shared the 1993 IEEE Communications Society Leonard G. Abraham Prize Paper Award with B. Marcus and J. K. Wolf. With J. B. Soriaga and H. D. Pfister, he received the 2007 Best Paper Award in Signal Processing and Coding for Data Storage from the Data Storage Technical Committee of the IEEE Communications Society. He holds several patents in the area of coding and detection and was named a Master Inventor at IBM Research in 1994. He is a member of Phi Beta Kappa and the National Academy of Engineering.