Efficient Two-Write WOM-Codes

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***Abstract*—A *Write Once Memory (WOM)* is a storage medium with binary memory elements, called *cells*, that can change from the zero state to the one state only once. Examples of WOMs are punch cards, optical disks, and more recently flash memories. A t-write WOM-code is a coding scheme for storing t messages in n cells in such a way that each cell can change its value only from the zero state to the one state. The WOM-rate of a t-write WOM-code is the ratio of the total amount of information written to the WOM in t writes to the number of cells.**

In this paper we present a family of 2-write WOM-codes. It is shown how to construct from each linear code $\mathcal{C}$ a 2-write WOM-code. Then, we find 2-write WOM-codes that improve the best known WOM-rate with two writes. This scheme is proved to be capacity achieving when the parity check matrix of the linear code $\mathcal{C}$ is chosen uniformly at random. Finally, we show how to take advantage of 2-write WOM-codes in order to construct codes for the Blackwell channel.

I. Introduction

Rivest and Shamir presented in [16] the Write Once Memory (WOM) model, suggested for memories like punch cards and optical disks. A WOM consists of binary memory elements that can only be changed from a zero state to a one state. WOM-codes, designed for such memories, address the problem of multiple writes to the memory. The work of Rivest and Shamir paved the way for more research [4], [6], [7], [20] and recently, such codes have been suggested for application to flash memories [12]–[14].

The atomic memory element in flash memories is a floating gate cell. The memory consists of arrays of these cells (typically, blocks contain 2^{20} cells). Each cell can have multiple levels, where the cell level is a function of the number of electrons trapped within it [8]. It is possible to increase an individual cell level in the block by charging it with electrons. However, it is impossible to reduce its level, unless the entire block is erased and then reprogrammed [8]. This model is a generalization of the WOM model [13], [14]. In fact, it was already described before in [6], [7], however without mentioning the connection to flash memories. In this work, we are only concerned with cells that take on two levels.

The problem in the WOM model which has attracted the most attention is to maximize the total amount of information that can be written into n memory cells in t writes, while preserving the constraint that on each write one can only change cells in the zero state to the one state. A code that is designed for this problem is called a ***t*-write WOM-Code** $\mathcal{C}_{\mathcal{W}}$. If M_i messages can be written on the i-th write, $1 \leqslant i \leqslant t$, then the ***WOM-rate*** $\mathcal{R}_t(\mathcal{C}_{\mathcal{W}})$ of the t-write WOM-code $\mathcal{C}_{\mathcal{W}}$ is the ratio of the total amount of information written to the memory, $\sum_{i=1}^{t} \log_2 M_i$, to the number of cells n,

$$\mathcal{R}_t(\mathcal{C}_{\mathcal{W}}) = \frac{\sum_{i=1}^{t} \log_2 M_i}{n}.$$

The first example of a WOM-code was presented by Rivest and Shamir for storing two bits twice using only three cells [16]. Since then, several more WOM-code constructions were presented, including tabular WOM-codes and "linear" WOM-codes [16]. WOM-codes based on projective geometries were presented in [15]. In [4] and [10], a "coset-coding" technique based upon binary linear codes is used to construct WOM-codes. An information-theoretical viewpoint of WOM-codes was discussed by Wolf, et al. [19]. The capacity of a WOM and a noisy WOM was studied by Heegard [11]. The WOM model has been generalized for the multi-level case in [7] and was later discussed again in [6]. Error-correcting WOM-codes were studied in [20], [21] and recently, in [12], Jiang discussed the generalization of error-correcting WOM-codes for the flash/floating codes model [13], [14]. Recently, position modulation codes were introduced by Wu and Jiang in order to construct WOM-codes with multiple writes [23].

The capacity region of a WOM with t writes was proved in [11], [6] to be

$$\begin{aligned} C_t = \Big\{ &(R_1, \ldots, R_t) \mid R_1 \leqslant h(p_1), R_2 \leqslant (1-p_1)h(p_2), \ldots, \\ &R_{t-1} \leqslant \Big(\prod_{i=1}^{t-2} (1-p_i) \Big) h(p_{t-1}), R_t \leqslant \prod_{i=1}^{t-1} (1-p_i), \\ &\text{where } 0 \leqslant p_1, \ldots, p_{t-1} \leqslant 1/2 \Big\} \end{aligned}$$

and $\log_2(t+1)$ is the maximum achievable WOM-rate, $\sum_{i=1}^{t} R_i$. In particular, for two writes the capacity region is

$$C_2 = \{(R_1, R_2) \mid \exists p \in [0, 0.5], R_1 \leqslant h(p), R_2 \leqslant 1-p\}. \quad (1)$$

The WOM-rate is maximized for $p = 1/3$ and its value is $\log_2 3 \approx 1.58$.

For years, the best known WOM-rate for two writes, achieved by Rivest and Shamir, was $2 \cdot \log_2(26)/7 \approx 1.34$. No other better construction was reported until very recently when Wu discussed the problem of 2-write WOM-codes [22]. Several constructions of low-complexity WOM-codes were introduced and by computer search some more codes were found, improving the best known WOM-rate to be $(\log_2(176) + \log_2(76))/10 \approx 1.37$. Inspired by the strong connection between WOM-code and coding for memory with defective cells, Wu reported another construction for ϵ-error 2-write WOM-codes which achieves the capacity region. In ϵ-error 2-write WOM-codes the second write is not guaranteed in the worst case but is allowed with high probability.

In this work, we consider two problems related to 2-write WOM-codes:

1) The number of messages written to the memory on each write is the same.

978-1-4244-8264-1/10/$26.00

2) Different number of messages can be written on each write.

For the case of 2-write WOM-codes, the theoretical bound on the WOM-rate for the first problem is approximately 1.5458 [16] and in the second problem it is $\log_2 3$ [11], [6]. Since the best known WOM-rate for the first problem is approximately 1.34 and 1.37 for the second problem, there is still room for improvement in closing these gaps.

In this paper, we present a 2-write WOM-code construction which reduces the gaps between the upper bound and lower bound on the WOM-rates. In Section III, we present our 2-write WOM-codes construction. The construction generates from every linear code a 2-write WOM-code. Two examples of such codes having better WOM-rates than the best known ones are presented. Then, in Section IV, it is shown that by choosing uniformly at random the parity check matrix of the linear code in our construction, it is possible to achieve the capacity region C_2. The random coding scheme enables us to run a computer search in order to find more 2-write WOM-codes and further improve the best known WOM-rate. Finally, we discuss the connection between the Blackwell channel and 2-write WOM-codes and show how to take advantage of 2-write WOM-codes in order to construct codes for the Blackwell channel.

II. Preliminaries

In this work, the memory elements, called ***cells***, have two states: zero and one. At the beginning, all the cells are in their zero state. A cell can change its state from zero to one. This operation is irreversible in the sense that a cell cannot change its state from one to zero unless the entire memory is erased. The *memory-state vectors* are all the binary vectors of length n, $\{0,1\}^n$. For two memory-state vectors $\boldsymbol{c}, \boldsymbol{c}' \in \{0,1\}^n$, we say $\boldsymbol{c} \geqslant \boldsymbol{c}'$ if and only if $\boldsymbol{c}_i \geqslant \boldsymbol{c}'_i$ for all $1 \leqslant i \leqslant n$.

Definition. *An $[n, M_1, \ldots, M_t, t]$ **t-write WOM-Code** $\mathcal{C}_\mathcal{W}$ is a coding scheme which consists of n cells and t pairs of encoding and decoding maps, denoted by $\mathcal{E}_i$ and $\mathcal{D}_i$ for $1 \leqslant i \leqslant t$. The t-write WOM-code $\mathcal{C}_\mathcal{W}$ satisfies the following properties:*

1) $\mathcal{E}_1 : \{1, \ldots, M_1\} \to \{0,1\}^n$,
2) *For* $2 \leqslant i \leqslant t$,
$$\mathcal{E}_i : \{1, \ldots, M_i\} \times \{0,1\}^n \to \{0,1\}^n,$$
such that, for all $(m, \boldsymbol{c}) \in \{1, \ldots, M_i\} \times \{0,1\}^n$,
$$\mathcal{E}_i(m, \boldsymbol{c}) \geqslant \boldsymbol{c}.$$
3) *For* $1 \leqslant i \leqslant t$,
$$\mathcal{D}_i : \{0,1\}^n \to \{1, \ldots, M_i\},$$
such that $\mathcal{D}_1(\mathcal{E}_1(m)) = m$ *for all* $m \in \{1, \ldots, M_1\}$, *and for* $2 \leqslant i \leqslant t$, $\mathcal{D}_i(\mathcal{E}_i(m, \boldsymbol{c})) = m$ *for all* $(m, \boldsymbol{c}) \in \{1, \ldots, M_i\} \times \{0,1\}^n$.

*The **WOM-Rate** of a t-write WOM-code $\mathcal{C}_\mathcal{W}$ is defined to be*
$$\mathcal{R}_t(\mathcal{C}_\mathcal{W}) = \frac{\sum_{i=1}^t \log_2 M_i}{n}.$$

Remark 1. We assume that the write number on each write is known. This knowledge does not affect the WOM-rate. Indeed, assume that there exsits an $[n, M_1, \ldots, M_t, t]$ t-write WOM-code $\mathcal{C}_\mathcal{W}$ where the write number is known. Assume also that the WOM-rate of $\mathcal{C}_\mathcal{W}$ is $\mathcal{R}_t(\mathcal{C}_\mathcal{W}) = \frac{\sum_{i=1}^t \log_2 M_i}{n}$. It is possible to change this code to an $[Nn+t, M_1^N, \ldots, M_t^N, t]$ t-write WOM-code $\mathcal{C}'_\mathcal{W}$ by having N blocks of the t-write WOM-code $\mathcal{C}_\mathcal{W}$ and t more cells indicating the write number. Then, the WOM-rate of $\mathcal{C}'_\mathcal{W}$ is
$$\mathcal{R}_t(\mathcal{C}'_\mathcal{W}) = \frac{\sum_{i=1}^t \log_2 M_i^N}{Nn+t} = \frac{N\sum_{i=1}^t \log_2 M_i}{Nn+t}$$
$$= \frac{N\sum_{i=1}^t \log_2 M_i}{Nn} \cdot \frac{Nn}{Nn+t} = \frac{\mathcal{R}_t(\mathcal{C}_\mathcal{W})}{1+\frac{t}{Nn}}.$$
Therefore, for N large enough it is possible to achieve the WOM-rate of the t-write WOM-code $\mathcal{C}_\mathcal{W}$. For simplicity, we will assume in the rest of this paper that the write number is known in the encoding process.

III. Two-Write WOM-Codes Construction

In this section we present our construction of 2-write WOM-codes. Let $\mathcal{C}[n,k]$ be a linear code with parity check matrix $\mathcal{H}$. For each $\boldsymbol{v} \in \{0,1\}^n$ we define the matrix $\mathcal{H}_{\boldsymbol{v}}$ as follows. The i-th column of $\mathcal{H}_{\boldsymbol{v}}$, $1 \leqslant i \leqslant n$, is the i-th column of $\mathcal{H}$ if $\boldsymbol{v}_i = 0$ and otherwise it is the zeros column. The set $V_\mathcal{C}$ is defined to be
$$V_\mathcal{C} = \{\boldsymbol{v} \in \{0,1\}^n \mid \text{rank}(\mathcal{H}_{\boldsymbol{v}}) = n-k\}. \quad (2)$$
We first note the following claim.

Claim 1. *If a vector $\boldsymbol{v}$ belongs to $V_\mathcal{C}$, its weight is at most k.*

The support of a binary vector $\boldsymbol{v}$, denoted by supp($\boldsymbol{v}$), is the set $\{i \mid \boldsymbol{v}_i = 1\}$. We say that a binary vector $\boldsymbol{v}$ covers a binary vector $\boldsymbol{u}$ if and only if $\{i \mid \boldsymbol{u}_i = 1\} \subseteq \{i \mid \boldsymbol{v}_i = 1\}$. The dual of the code $\mathcal{C}$ is denoted by $\mathcal{C}^\perp$. The next lemma is a variation of a well known lemma (see e.g. [4]).

Lemma 2. *Let $\mathcal{C}[n,k]$ be a linear code with parity check matrix $\mathcal{H}$. For each vector $\boldsymbol{v} \in \{0,1\}^n$, rank$(\mathcal{H}_{\boldsymbol{v}}) = n-k$ if and only if $\boldsymbol{v}$ does not cover any non-zero codeword in $\mathcal{C}^\perp$.*

Lemma 2 implies that for each linear code $\mathcal{C}$, the set $V_\mathcal{C}$ does not depend on the structure of the parity check matrix of the code $\mathcal{C}$ and so we can define the set $V_\mathcal{C}$ to be
$$V_\mathcal{C} = \{\boldsymbol{v} \in \{0,1\}^n \mid \boldsymbol{v} \text{ does not cover any non-zero } c \in \mathcal{C}^\perp\}.$$

The next theorem presents our 2-write WOM-codes. This coding scheme is very similar to the one recently presented in [22] for the ϵ-error case.

Theorem 3. *Let $\mathcal{C}[n,k]$ be a linear code with parity check matrix $\mathcal{H}$ and let $V_\mathcal{C}$ be the set defined in* (2). *Then there exists an $[n, |V_\mathcal{C}|, 2^{n-k}, 2]$ 2-write WOM-code of WOM-rate*
$$\frac{\log_2 |V_\mathcal{C}| + (n-k)}{n}.$$

Proof: We need to show the existence of the encoding and decoding maps on the first and second writes. First, let $\{v_1, v_2, \ldots, v_{|V_\mathcal{C}|}\}$ be an ordering of the set $V_\mathcal{C}$. The first and the second writes are implemented as follows.

1) On the first write, a symbol over an alphabet of size $|V_\mathcal{C}|$ is written.

The encoding and decoding maps $\mathcal{E}_1, \mathcal{D}_1$ are defined as follows. For each $m \in \{1 \ldots, |V_\mathcal{C}|\}, \mathcal{E}_1(m) = v_m$ and $\mathcal{D}_1(v_m) = m$.

2) On the second write, we write a vector $\boldsymbol{s}_2$ of $n-k$ bits. Let $\boldsymbol{v}_1$ be the programmed vector on the first write and $\boldsymbol{s}_1 = \mathcal{H} \cdot \boldsymbol{v}_1$, then

$$\mathcal{E}_2(\boldsymbol{s}_2, \boldsymbol{v}_1) = \boldsymbol{v}_1 + \boldsymbol{v}_2,$$

where $\boldsymbol{v}_2$ is a solution of the equation $\mathcal{H}_{\boldsymbol{v}_1} \cdot \boldsymbol{v}_2 = \boldsymbol{s}_1 + \boldsymbol{s}_2$. For the decoding map $\mathcal{D}_2$, if $\boldsymbol{c}$ is the vector of programmed cells, then the decoded value of the $n-k$ bits is given by

$$\mathcal{D}_2(\boldsymbol{c}) = \mathcal{H} \cdot \boldsymbol{c} = \mathcal{H} \cdot \boldsymbol{v}_1 + \mathcal{H} \cdot \boldsymbol{v}_2 = \boldsymbol{s}_1 + \boldsymbol{s}_1 + \boldsymbol{s}_2 = \boldsymbol{s}_2.$$

The success of the second write results from the condition that for every vector $\boldsymbol{v} \in V_\mathcal{C}$, $\text{rank}(\mathcal{H}_{\boldsymbol{v}}) = n - k$. ∎

There is no condition on the code $\mathcal{C}$ and therefore we can use any linear code in this construction, though we seek to find codes that maximize the WOM-rate $\frac{\log_2(|V_\mathcal{C}|)+n-k}{n}$. Next, we show two examples of 2-write WOM-codes that achieve better WOM-rates than the best known ones.

Example 1. Let us demonstrate how Theorem 3 works for the first order Reed-Muller code $\mathcal{C}[16, 5, 8]$. Its dual code is the second order Reed-Muller $\mathcal{C}[16, 11, 4]$, which is the extended Hamming code of length 16. Hence, we are interested in the size of the set

$$V_1 = \left\{\boldsymbol{v} \in \{0,1\}^{16} \mid \boldsymbol{v} \text{ does not cover any } c \in \mathcal{C}[16, 11, 4]\right\}.$$

According to Claim 1, the set V_1 does not contain vectors of weight greater than five. This extended Hamming code has 140 codewords of weight four and no codewords of weight five. The set V_1 consists of the following vector sets.

1) All vectors of weight at most three. There are $\sum_{i=0}^{3} \binom{16}{i} = 697$ such vectors.
2) All vectors of weight four that are not codewords. There are $\binom{16}{4} - 140 = 1680$ such vectors.
3) All vectors of weight five that do no cover a codeword of weight four. There are $\binom{16}{5} - 12 \cdot 140 = 2688$ such vectors. Since the minimum distance of the code is four, a vector of weight five can cover at most one codeword of weight four.

Therefore, we get $|V_1| = 697 + 1680 + 2688 = 5065$ and the WOM-rate is

$$(\log_2(5065) + 11)/16 = 1.4566.$$

It is possible to modify this code such that on the first write only 11 bits are written. Thus, we achieve a 2-write WOM-code with the same rate on the first and second write. The WOM-rate in this case is $22/16 = 1.375$, which is the best known in case that the rate in the two writes is the same.

Example 2. In this example we will use the $\mathcal{C}[23, 11, 8]$ Golay code. Its dual code is the $\mathcal{C}[23, 12, 7]$ Golay code so we are interested in the size of the set

$$V_2 = \left\{\boldsymbol{v} \in \{0,1\}^{23} \mid \boldsymbol{v} \text{ does not cover any } c \in \mathcal{C}[23, 12, 7]\right\}.$$

According to Claim 1, there are no vectors of weight greater than 11 in the set V_2. The Golay code $\mathcal{C}[23, 12, 7]$ has $A_7 = 253$ codewords of weight seven, $A_8 = 506$ codewords of weight eight, and $A_{11} = 1288$ codewords of weight 11. The set V_2 consists of the following vector sets.

1) All vectors of weight at most 6. This number of vectors is $\sum_{i=0}^{6} \binom{23}{i} = 145499$.
2) All vectors of weight between 7 and 10 besides those that cover a codeword of weight 7 or 8. Since the minimum distance of the code is 7 every vector can cover at most one codeword. Hence, this number of vectors is

$$\sum_{i=7}^{10} \binom{23}{i} - A_7 \cdot \sum_{i=7}^{10} \binom{16}{i-7} - A_8 \cdot \sum_{i=8}^{10} \binom{15}{i-8}$$
$$= 2459160$$

3) All vectors of weight 11 that are not codewords and do not cover a codeword of weight either 7 or 8. This number was shown in [5] to be 695520.

Therefore, for the $[23, 11, 8]$ Golay code we get

$$|V_2| = 145499 + 2459160 + 695520 = 3300179,$$

and thus the WOM-rate is

$$(\log_2(3300179) + 12)/23 = 1.4632.$$

Remark 2. The encoding and decoding maps of the second write are implemented by the parity check matrix of the linear code $\mathcal{C}$ as described in the proof of Theorem 3. A naive scheme to implement the encoding and decoding maps of the first write is simply by a lookup table of the set $V_\mathcal{C}$. However, this can be done more efficiently using algorithms to encode and decode permutations, e.g. [2]. We omit the details due to the lack of space and leave them to an extended version of this work.

IV. Random Coding

The scheme we described in Section III can work for any linear code $\mathcal{C}$. Given a linear code $\mathcal{C}[n, k]$ with parity check matrix $H_\mathcal{C}$, we denote $R_1(\mathcal{C}) = \frac{\log_2|V_\mathcal{C}|}{n}, R_2(\mathcal{C}) = \frac{n-k}{n}$ so the WOM-rate of the generated WOM-codes is $R_1(\mathcal{C}) + R_2(\mathcal{C})$. Our goal in this section is to show that it is possible to achieve the capacity region C_2 defined in (1), by choosing uniformly at random the parity check matrix of the linear code $\mathcal{C}$. We prove that in the following theorem.

Theorem 4. *For any $(R_1, R_2) \in C_2$ and $\epsilon > 0$ there exists a linear code $\mathcal{C}$ satisfying $R_1(\mathcal{C}) \geqslant R_1 - \epsilon, R_2(\mathcal{C}) \geqslant R_2 - \epsilon$.*

Proof: Let $p \in [0, 0.5]$ be such that $R_1 \leqslant h(p)$ and $R_2 \leqslant 1 - p$. Let $k = \lceil np \rceil$ for n large enough and let us choose uniformly at random an $(n-k) \times n$ matrix H. The matrix H will be the parity check matrix of the linear code $\mathcal{C}$ that will be used to construct the 2-write WOM-code. For each vector $\boldsymbol{v} \in \{0,1\}^n$, let us define the indicator random variable $X_{\boldsymbol{v}}$ as follows

$$X_{\boldsymbol{v}} = \begin{cases} 1 & \text{if } \boldsymbol{v} \in V_\mathcal{C} \\ 0 & \text{otherwise} \end{cases}$$

where $V_\mathcal{C}$ is the set defined in (2). Then the number of vectors in $V_\mathcal{C}$ is $X = \sum_{\boldsymbol{v} \in \{0,1\}^n} X_{\boldsymbol{v}}$, and

$$E[X] = \sum_{\boldsymbol{v} \in \{0,1\}^n} E[X_{\boldsymbol{v}}] = \sum_{\boldsymbol{v} \in \{0,1\}^n} \Pr\{X_{\boldsymbol{v}} = 1\}. \quad (3)$$

We claim that $\Pr\{X_{\boldsymbol{v}} = 1\}$ depends on $\boldsymbol{v}$ only through its weight, $wt(\boldsymbol{v})$. If so, then (3) simplifies to

$$E[X] = \sum_{i=0}^{n} \binom{n}{i} \Pr\{X_{\boldsymbol{v}:wt(\boldsymbol{v})=i} = 1\}$$
$$= \sum_{i=0}^{k} \binom{n}{i} \Pr\{X_{\boldsymbol{v}:wt(\boldsymbol{v})=i} = 1\},$$

because if $wt(\boldsymbol{v}) \geqslant k+1$ then $X_{\boldsymbol{v}} = 0$ (Claim 1).

Now, let us determine the value of $\Pr\{X_{\boldsymbol{v}} = 1\}$ for a vector $\boldsymbol{v}$ of weight $0 \leqslant i \leqslant k$. Note that $\boldsymbol{v} \in V_{\mathcal{C}}$ if and only if the submatrix of size $(n-k) \times (n - wt(\boldsymbol{v}))$ induced by the zero entries of the vector $\boldsymbol{v}$ is full rank. It is well known, e.g. [3], that if we choose an $m \times n$ matrix, where $m \leqslant n$, uniformly at random then the probability that it is full rank is $\prod_{j=n-m+1}^{n}(1-2^{-j})$. Therefore, if we choose an $(n-k) \times (n-i)$ matrix uniformly at random then the probability that it is full rank is $\prod_{j=k-i+1}^{n-i}\left(1-2^{-j}\right)$. Note that

$$\prod_{j=k-i+1}^{n-i}\left(1-2^{-j}\right) > \prod_{j=1}^{\infty}\left(1-2^{-j}\right)$$
$$> \left(1-\frac{1}{2}\right)\left(1-\sum_{j=2}^{\infty}2^{-j}\right) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4},$$

and hence, $\Pr\{X_{\boldsymbol{v}} = 1\} = \prod_{j=k-i+1}^{n-i}\left(1-2^{-j}\right) > 1/4$. According to Lemma 4.8 in [18],

$$\sum_{i=0}^{k}\binom{n}{i} \geqslant \frac{1}{n+1}2^{nh\left(\frac{k}{n}\right)}$$

and therefore, we get

$$E[X] = \sum_{i=0}^{k}\binom{n}{i}\prod_{j=k-i+1}^{n-i}\left(1-2^{-j}\right) > 2^{nh\left(\frac{k}{n}\right)-2-\log_2(n+1)}.$$

It follows that there exists a parity check matrix H of a linear code $\mathcal{C}$, such that the size of the set $V_{\mathcal{C}}$ is at least $2^{nh\left(\frac{k}{n}\right)-2-\log_2(n+1)}$ and

$$R_1(\mathcal{C}) \geqslant h\left(\frac{k}{n}\right) - \frac{2+\log_2(n+1)}{n}$$
$$\geqslant h(p) - \frac{2+\log_2(n+1)}{n} \geqslant R_1 - \epsilon$$
$$R_2(\mathcal{C}) = \frac{n-k}{n} \geqslant (1-p) - \frac{1}{n} \geqslant R_2 - \epsilon$$

for n large enough. ■

Random coding was proved to be capacity-achieving by constructing a partition code [11], [6]. However, our random coding scheme has more structure that enables us to look for codes with a relatively large block length. We ran a computer search to look for such codes. The parity check matrix of the linear code $\mathcal{C}$ was chosen uniformly at random and then the size of the set $V_{\mathcal{C}}$ was computed. The results are shown in Figure 1. Note that if (R_1, R_2) and (R_3, R_4) are two achievable rate points then for each $t \in \mathbb{Q}$ the point $(tR_1+(1-t)R_2, tR_3+(1-t)R_4)$ is an achievable WOM-rate point, too. This can simply be done by block sharing of large number of blocks. Therefore, the achievable region is convex.

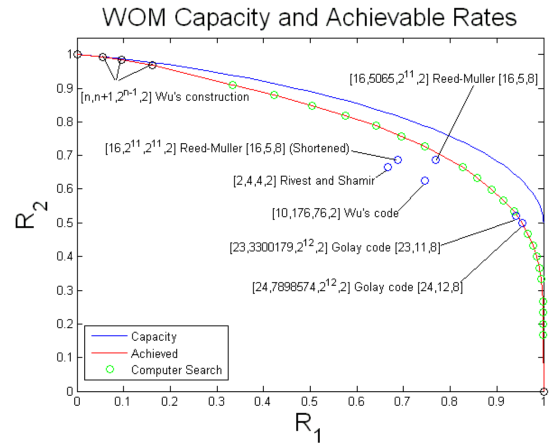

Fig. 1. The capacity region and achievable rates 2-write WOM-codes.

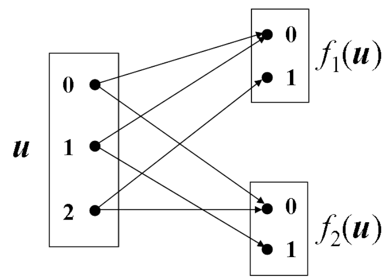

Fig. 2. The Blackwell Channel.

V. Application to the Blackwell Channel

The Blackwell channel, introduced first by Blackwell [1], is one example of a deterministic broadcast channel. The channel is composed of one transmitter and two receivers. The input to the transmitter is ternary and the channel output to each receiver is a binary symbol. Let $\boldsymbol{u}$ be the ternary input vector to the transmitter of length n. For $1 \leqslant i \leqslant n$, $f(\boldsymbol{u}_i) = (f(\boldsymbol{u}_i)_1, f(\boldsymbol{u}_i)_2)$ is a binary vector of length two defined as follows (see Figure 2):

$$f(0) = (0,0),\ f(1) = (0,1),\ f(2) = (1,0).$$

The binary vectors $f_1(\boldsymbol{u}), f_2(\boldsymbol{u})$ are defined to be

$$f_1(\boldsymbol{u}) = (f(\boldsymbol{u}_1)_1, f(\boldsymbol{u}_2)_1, \ldots, f(\boldsymbol{u}_n)_1),$$
$$f_2(\boldsymbol{u}) = (f(\boldsymbol{u}_1)_2, f(\boldsymbol{u}_2)_2, \ldots, f(\boldsymbol{u}_n)_2),$$

and are the output vectors to the two receivers.

The capacity region of the Blackwell channel was found by Gel'fand [9] and consists of five regions, given by their boundaries:

1) $\{(R_1, R_2) \mid 0 \leqslant R_1 \leqslant 1/2, R_2 = 1\}$,
2) $\{(R_1, R_2) \mid R_1 = 1-p, R_2 = h(p), 1/3 \leqslant p \leqslant 1/2\}$,
3) $\{(R_1, R_2) \mid R_1 + R_2 = \log_2 3, \frac{2}{3} \leqslant R_1 \leqslant \log_2 3 - \frac{2}{3}\}$,
4) $\{(R_1, R_2) \mid R_1 = h(p), R_2 = 1-p, 1/3 \leqslant p \leqslant 1/2\}$,
5) $\{(R_1, R_2) \mid R_1 = 1, 0 \leqslant R_2 \leqslant 1/2\}$.

The connection between the Blackwell channel and 2-write WOM-codes was suggested by Roth [17]. The next theorem shows that from every 2-write WOM-code of rate (R_1, R_2) it is possible to construct codes for the Blackwell channel of rates (R_1, R_2) and (R_2, R_1).

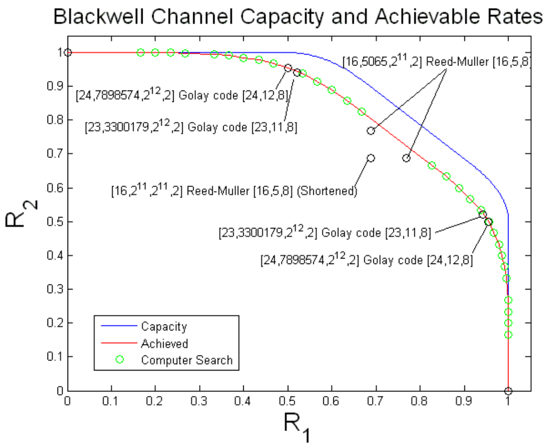

Fig. 3. The capacity region and the achievable rates of the Blackwell channel.

Theorem 5. *If (R_1, R_2) is an achievable rate of a 2-write WOM-code, then (R_1, R_2) and (R_2, R_1) are achievable rates on the Blackwell channel.*

Proof:

Assume that there exists a $[n, 2^{nR_1}, 2^{nR_2}, 2]$ 2-write WOM-code and let $\mathcal{E}_1, \mathcal{E}_2$ and $\mathcal{D}_1, \mathcal{D}_2$ be its encoding and decoding maps. We claim that there exists a coding scheme for the Blackwell channel of rate (R_1, R_2). Let $(m_1, m_2) \in \{1, \ldots, 2^{nR_1}\} \times \{1, \ldots, 2^{nR_2}\}$ be two messages and let $\boldsymbol{v}_1 = \mathcal{E}_1(m_1)$ and $\boldsymbol{v}_2 = \mathcal{E}_2(m_2, \boldsymbol{v}_1)$. Let $\boldsymbol{u}$ be a ternary vector of length n defined as follows. For $1 \leqslant i \leqslant n$, $\boldsymbol{u}_i = f^{-1}(\boldsymbol{v}_{1,i}, \overline{\boldsymbol{v}_{2,i}})$. The vector $\boldsymbol{u}$ is well-defined since for all $1 \leqslant i \leqslant n$, $(\boldsymbol{v}_{1,i}, \boldsymbol{v}_{2,i}) \neq (1, 0)$ and hence $(\boldsymbol{v}_{1,i}, \overline{\boldsymbol{v}_{2,i}}) \neq (1, 1)$. The vector $\boldsymbol{u}$ is the input to the transmitter. Then, the vector $f_1(\boldsymbol{u})$ is transmitted to the first receiver and the vector $f_2(\boldsymbol{u})$ to the second receiver. Note that $f_1(\boldsymbol{u}) = \boldsymbol{v}_1$ and $f_2(\boldsymbol{u}) = \overline{\boldsymbol{v}_2}$. Therefore, the first receiver decodes its message according to $\mathcal{D}_1(f_1(\boldsymbol{u})) = \mathcal{D}_1(\boldsymbol{v}_1) = m_1$ and the second receiver decodes its message according to $\mathcal{D}_2(\overline{f_2(\boldsymbol{u})}) = \mathcal{D}_2(\boldsymbol{v}_2) = m_2$.

Similarly, it is possible to achieve the rate (R_2, R_1). Now we let $\boldsymbol{v}_2 = \mathcal{E}_2(m_2)$ and $\boldsymbol{v}_1 = \mathcal{E}_1(m_1, \boldsymbol{v}_2)$. The vector $\boldsymbol{u}$ is defined as $\boldsymbol{u}_i = f^{-1}(\overline{\boldsymbol{v}_{1,i}}, \boldsymbol{v}_{2,i})$ for $1 \leqslant i \leqslant n$. The decoded message by the first receiver is $\mathcal{D}_1(\overline{f_1(\boldsymbol{u})})$ and $\mathcal{D}_2(f_2(\boldsymbol{u}))$ is the decoded message by the second receiver. ■

Remark 3. It is possible to define the Blackwell channel differently such that the forbidden pair of bits is not $(1, 1)$ but another combination. Our construction of the codes can be adjusted accordingly.

Now, we can use our 2-write WOM-codes in order to define codes for the Blackwell channel. By time sharing the achievable region is convex and hence we get in Figure 3 the capacity and achievable regions for the Blackwell channel.

VI. Conclusion

The 2-write WOM-codes problem was discussed. We showed a construction providing from each linear code $\mathcal{C}$ a 2-write WOM-code. We showed that if the parity check matrix of the linear code $\mathcal{C}$ is chosen uniformly at random then it is possible to achieve the capacity region of the WOM problem with two writes. Then, we ran a computer search to find more 2-write WOM-codes with high WOM-rates. When the same number of messages is written on each write our best construction achieves WOM-rate 1.375 and by computer search we found a 2-write WOM-code of WOM-rate 1.4546. When a different number of messages can be written on each write, the best construction has WOM-rate 1.4632 and by computer search the best WOM-rate we found is 1.4928. Finally, we showed how WOM-codes with two writes provide codes for the Blackwell channel.

VII. Acknowledgement

This work was supported in part by the University of California Lab Fees Research Program, Award No. 09-LR-06-118620-SIEP and the Center for Magnetic Recording Research at the University of California, San Diego.

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