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Linking pairings on singular spaces

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§1. Introduction

In [GM1] [GM2] [GM3] intersection homology groups $IH_i^{\bar{p}}(X)$ were defined for any n dimensional compact oriented pseudomanifold X and any perversity $\bar{p}$ between $\bar{0}=(0,0,\ldots,0)$ and $\bar{t}=(0,1,2,3,\ldots)$. Questions concerning the torsion subgroups of the intersection homology groups have arisen in three contexts:

(A) Is the torsion in $IH_i^{\bar{p}}(X)$ dually paired with the torsion in $IH_j^{\bar{q}}(X)$ when $i+j=n$ and $\bar{p}+\bar{q}=\bar{t}$?

(B) Does the universal coefficient formula hold for $IH_i^{\bar{p}}(X)$?

(C) For a compact $4k$ dimensional pseudomanifold X with even dimensional strata, does the determinant of the intersection pairing on $IH_{2k}^{\bar{m}}(X)$ equal 1?

The answer to all these questions is "yes" if X is a manifold, but "no" if X is a general singular space. However, for singular spaces which are "locally $\bar{p}$-torsion free" the answer is "yes" to each of these questions:

DEFINITION. A pseudomanifold X is locally $\bar{p}$-torsion free if, for each stratum of X,

$$T_{c-2-p(c)}^{\bar{p}}(L)=0$$

where L denotes the link of that stratum, c denotes its codimension, and $T_i^{\bar{p}}(L)$ is the torsion subgroup of $IH_i^{\bar{p}}(L)$.

The answer to question (A) is:

THEOREM 4.4. *Suppose X is a compact oriented n dimensional pseudomanifold. Then there is a canonical torsion pairing*

$$T_i^{\bar{p}}(X)\times T_{n-i-1}^{\bar{q}}(X)\to \mathbf{Q}/\mathbf{Z} \qquad (*)$$

where $\bar{q}=\bar{t}-\bar{p}$. If X is also locally $\bar{p}$-torsion free then this pairing is nondegenerate.

Similarly, the answer to question (B) is:

THEOREM 8.1. *Suppose X is a locally $\bar{p}$-torsion free pseudomanifold. Let G be an abelian group. Then there is a natural exact sequence*

$$0 \to IH_i^{\bar{p}}(X) \otimes G \to IH_i^{\bar{p}}(X; G) \to \mathrm{Tor}\,(IH_{i-1}^{\bar{p}}(X), G) \to 0$$

For spaces which are not locally $\bar{p}$-torsion free there is a new torsion group $R_i^{\bar{p}}(X)$ which in some sense measures the degeneracy of the torsion pairing (*), i.e. there is a sequence

$$\cdots \to T_i^{\bar{p}}(X) \to \mathrm{Hom}\,(T_{n-i-1}^{\bar{q}}(X), \mathbf{Q}/\mathbf{Z}) \to R_i^{\bar{p}}(X) \to T_{i-1}^{\bar{p}}(X) \to \cdots$$

The group $R_i^{\bar{p}}(X)$ is a topological invariant of X and is the hypercohomology of a complex of sheaves which is supported on the singular set of X.

THEOREM 9.3. *For* any *compact oriented n dimensional pseudomanifold X there is a natural nondegenerate pairing*

$$R_i^{\bar{p}}(X) \otimes R_{n-i}^{\bar{q}}(X) \to \mathbf{Q}/\mathbf{Z}$$

where $\bar{q} = \bar{t} - \bar{p}$.

This pairing gives rise to a cobordism invariant characteristic class for certain singular spaces, which was first introduced in [S]:

Suppose X is a compact oriented $4k$ dimensional pseudomanifold with even dimensional strata (or, more generally suppose $IH_{l/2}^{\bar{m}}(L) = 0$ whenever L is the link of a stratum with odd codimensional $c = l + 1$). Then we have a nondegenerate rational pairing

$$I : IH_{2k}^{\bar{m}}(X; \mathbf{Q}) \otimes IH_{2k}^{\bar{m}}(X; \mathbf{Q}) \to \mathbf{Q}$$

and a nondegenerate torsion pairing

$$K : R_{2k}^{\bar{m}}(X) \otimes R_{2k}^{\bar{m}}(X) \to \mathbf{Q}/\mathbf{Z}$$

THEOREM 11.3. *The Witt class (in $W(\mathbf{Q}/\mathbf{Z})$) of the pairing K is equal to the torsion part of the Witt class (in $W(\mathbf{Q})$) of the pairing I. This characteristic class is a cobordism invariant for cobordisms with even dimensional strata.*

This result suggests that the cobordism groups of the spaces (defined in §7.1) which satisfy Poincaré duality over the integers may coïncide with Mishchenko's higher Witt groups of $\mathbf{Z}$ (see [R]).

We are grateful to R. MacPherson for several valuable conversations and in particular for his suggestion that the "peripheral complex" $\underline{\underline{R}}_i^{\bar{p}}$ should be an interesting object to study. Many of the results in this paper have been worked out independently by P. Deligne.

§2. Notation

Our notation follows [GM2] and [GM3]. X will denote an n-dimensional compact oriented piecewise linear pseudomanifold with a P.L. stratification

$$X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_{n-2} = \Sigma = X_{n-1} \subset X_n = X$$

such that each point $x \in X_i - X_{i-1}$ has a neighborhood of the form $U = (i\text{-simplex}) \times \text{cone}\,(L)$ where L is the *link* of the stratum containing x.

The symbol $IH_i^{\bar{p}}(X)$ denotes the ith intersection homology group of X, with perversity $\bar{p} = (p_2, p_3, p_4, \ldots)$ where $p_c \le p_{c+1} \le p_c + 1$ and $p_2 = 0$. This group is canonically isomorphic to the hypercohomology group $\mathcal{H}^{-i}(\underline{\underline{IC}}^{\cdot}_{\bar{p}})$ of the complex of sheaves $\underline{\underline{IC}}^{\cdot}_{\bar{p}}$ which was constructed by Deligne [GM3]. It does not depend on the choice of P.L. structure or on the choice of stratification of X.

§3. Linking products in intersection homology

3.1. Let X be a compact n dimensional piecewise linear stratified pseudomanifold and suppose $\bar{p} + \bar{q} = \bar{r}$ are perversities as in [GM2] §1. We shall define a product

$$L : T_i^{\bar{p}}(X) \times T_j^{\bar{q}}(X) \to IH_{i+j-n-1}^{\bar{r}}(X; \mathbf{Q}/\mathbf{Z}) \tag{1}$$

where $T_i^{\bar{p}}(X)$ denotes the torsion subgroup of $IH_i^{\bar{p}}(X)$. Let $\xi \in IC_i^{\bar{p}}(X)$ and $\eta \in IC_j^{\bar{q}}(X)$ be cycles which represent torsion classes $[\xi] \in T_i^{\bar{p}}(X)$ and $[\eta] \in T_j^{\bar{q}}(X)$. Then there are integers m_1 and m_2 and chains $\tilde{\xi} \in IC_{i+1}^{\bar{p}}(X)$, $\tilde{\eta} \in IC_{j+1}^{\bar{q}}(X)$ such that $\partial\tilde{\xi} = m_1\xi$ and $\partial\tilde{\eta} = m_2\eta$. We may choose ξ and $\tilde{\xi}$ so as to be dimensionally transverse to η and $\tilde{\eta}$ by [Mc].

Define $L([\xi], [\eta])$ to be the homology class of the intersection cycle $(1/m_1)\tilde{\xi} \cap \eta \in IC_{i+j-n-1}^{\bar{r}}(X; \mathbf{Q}/\mathbf{Z})$ (see [GM2] §2.1). It is easy to check that $L([\xi], [\eta])$ is well defined and is equal to the homology class of the cycle $(-1)^i(1/m_2)\xi \cap \tilde{\eta}$. Furthermore $L([\xi][\eta]) = (-1)^{(n-i)(n-j)}L([\eta], [\xi])$.

3.2. The torsion product for complementary dimensions $(i + j = n - 1)$ and perversities $(\bar{p} + \bar{q} = \bar{t} = (0, 1, 2, 3, \ldots))$ may also be constructed by the sheaf theoretic techniques of [GM3] from the intersection product, as follows: If $\underline{\underline{\mathbf{D}}}^{\cdot}_X$ denotes the dualizing complex on X, we have the product morphism

$$\underline{\underline{IC}}^{\cdot}_{\bar{p}} \otimes \underline{\underline{IC}}^{\cdot}_{\bar{q}} \to \underline{\underline{\mathbf{D}}}^{\cdot}_X[n]$$

and its adjoint

$$\underline{\underline{IC}}^{\cdot}_{\bar{p}} \to R\,\underline{\underline{\mathrm{Hom}}}^{\cdot}(\underline{\underline{IC}}^{\cdot}_{\bar{q}}, \underline{\underline{\mathbf{D}}}^{\cdot}_X)[n] \tag{2}$$

Applying the hypercohomology functor $\mathcal{H}^{-i}$ and the universal coefficient theorem [B], we obtain a commuting diagram with exact columns:

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
T^{\bar{p}}_i & \longrightarrow & \mathrm{Ext}\,(IH^{\bar{t}-\bar{p}}_{n-i-1}(X), \mathbf{Z}) \cong \mathrm{Hom}\,(T^{\bar{t}-\bar{p}}_{n-i-1}, \mathbf{Q}/\mathbf{Z}) \\
\downarrow & & \downarrow \\
IH^{\bar{p}}_i(X) & \longrightarrow & \mathcal{H}^{-i}(X; R\,\underline{\underline{\mathrm{Hom}}}^{\cdot}\,(\underline{\underline{IC}}^{\cdot}_{\bar{t}-\bar{p}}, \underline{\underline{\mathbf{D}}}^{\cdot}_X)[n]) \\
\downarrow & & \downarrow \\
IH^{\bar{p}}_i/T^{\bar{p}}_i & \longrightarrow & \mathrm{Hom}\,(IH^{\bar{t}-\bar{p}}_{n-i}(X), \mathbf{Z}) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

The adjoint of the homomorphism on the top line is the desired product

$$L : T^{\bar{p}}_i \times T^{\bar{t}-\bar{p}}_{n-i-1} \to \mathbf{Q}/\mathbf{Z} \tag{3}$$

3.3. PROPOSITION. *The linking product* (3) *coincides with the augmented product* (1):

$$T^{\bar{p}}_i \times T^{\bar{q}}_j \to IH^{\bar{r}}_{i+j-n-1}(X; \mathbf{Q}/\mathbf{Z}) \to H_0(\text{point}, \mathbf{Q}/\mathbf{Z}) = \mathbf{Q}/\mathbf{Z} \tag{*}$$

when $j = n - i - 1$ *and* $\bar{q} = \bar{t} - \bar{p}$.

COROLLARY. *If the morphism* (2) *is a quasi-isomorphism, then the linking pairing* (*) *is nondegenerate.*

The proof of Prop. 3.3 is similar to the proof of Corollary 3.6 in [GM3] and will be omitted.

§4. Spaces for which the linking pairing is nondegenerate

4.1. DEFINITION. *A stratified pseudomanifold X is locally $\bar{p}$-torsion free if for each stratum of X we have*

$$T^{\bar{p}}_{q(c)}(L)=0 \tag{4}$$

where L is the link of the stratum, c is the codimension and $q(c)=c-2-p(c)$.

4.2. *Remark.* If X is a locally $\bar{p}$-torsion free space and L is the link of any stratum of X, then L is also a locally $\bar{p}$-torsion free space.

4.3. PROPOSITION. *X is locally $\bar{p}$-torsion free with respect to one stratification iff the same is true with respect to any refinement of that stratification.*

Proof. The link L' of a stratum in the refinement has the form of a join, $L'=S^k * L$ where L is the link of a stratum in the original stratification. We must verify that

$$T^{\bar{p}}_r(L')=0 \tag{4'}$$

where $r=l+k-1-p(l+k+1)$ and $l=\dim(L)$. For $k=0$, L' is the suspension of L and

$$IH^{\bar{p}}_i(\Sigma L)=\begin{cases} IH^{\bar{p}}_{i-1}(L) & \text{if } i>l-p(l)-1 \\ 0 & \text{if } i=l-p(l)-1 \\ IH^{\bar{p}}_i(L) & \text{if } i<l-p(l)-1 \end{cases}$$

There are three possibilities: $p(l+2)=p(l)$, $p(l+2)=p(l)+1$, or $p(l+2)=p(l)+2$. In each case one calculates $T^{\bar{p}}_r(L)=0$ assuming (4) holds.

For $k>0$, equation (4′) may be verified by repeated application of the case $k=0$. Q.E.D.

4.4. THEOREM. *Suppose X is a compact n dimensional piecewise linear stratified pseudomanifold which is locally $\bar{p}$-torsion free. Then the morphism* (2) *is a quasi-isomorphism, so the linking pairing* (*) *is non-singular.*

Theorem 1 depends on a result from homological algebra which we now describe.

§5. Truncation of complexes

5.1. If $C^\cdot$ is a (cochain) complex of free abelian groups,

$$\cdots \xrightarrow{d} C^a \xrightarrow{d} C^{a+1} \xrightarrow{d} C^{a+2} \xrightarrow{d} \cdots$$

we denote by $\mathrm{Hom}^\cdot(C^\cdot, \mathbf{Z})$ the dual complex,

$$\mathrm{Hom}^b(C^\cdot, \mathbf{Z}) = \mathrm{Hom}(C^{-b}, \mathbf{Z}).$$

Deligne has defined truncation functors ([D1] [D2] [GM3])

$$(\tau_{\le a} C^\cdot)^n = \begin{cases} 0 & \text{if } n > a \\ \ker d & \text{if } n = a \\ C^n & \text{if } n < a \end{cases}$$

$$(\tau^{\ge a} C^\cdot)^n = \begin{cases} C^n & \text{if } n > a \\ \operatorname{coker} d^{a-1} & \text{if } n = a \\ 0 & \text{if } n < a \end{cases}$$

It is easy to verify the following facts from homological algebra:

5.2. PROPOSITION. *Let $C^\cdot$ be a complex of free abelian groups. Then the following natural sequence is split exact*

$$0 \to \mathrm{Ext}(H^{-m+1}(C^\cdot), \mathbf{Z}) \to H^m(\mathrm{Hom}^\cdot(C^\cdot, \mathbf{Z})) \to \mathrm{Hom}(H^{-m}(C^\cdot), \mathbf{Z}) \to 0$$

Consequently,

$$H^a\,\mathrm{Hom}(\tau^{\ge b} C^\cdot, \mathbf{Z}) = \begin{cases} 0 & \text{if } a \ge -b+2 \\ \mathrm{Ext}(H^b(C^\cdot), \mathbf{Z}) & \text{if } a = -b+1 \\ H^a\,\mathrm{Hom}^\cdot(C^\cdot, \mathbf{Z}) & \text{if } a \le -b \end{cases}$$

$$H^a\,\mathrm{Hom}(\tau_{\le b} C^\cdot, \mathbf{Z}) = \begin{cases} H^a(\mathrm{Hom}^\cdot(C^\cdot, \mathbf{Z})) & \text{if } a \ge -b+1 \\ \mathrm{Hom}(H^b(C^\cdot), \mathbf{Z}) & \text{if } a = -b \\ 0 & \text{if } a \le -b-1 \end{cases}$$

(*The same result holds if we drop the hypothesis that $C^\cdot$ is free, and replace* $\mathbf{Z}$ *by its injective resolution* $\mathbf{Q} \to \mathbf{Q}/\mathbf{Z}$).

§6. Proof of Theorem 4.4

Let us assume by induction that the theorem has been proven for pseudomanifolds of dimension $\leq n-1$. Let X be a pseudomanifold of dimension n. The morphism (2) is clearly a quasi-isomorphism over the nonsingular part of X, so it suffices to verify that the complex of sheaves

$$\underline{\underline{S}}^{\cdot} = R\,\underline{\underline{\mathrm{Hom}}}^{\cdot}(\underline{\underline{IC}}^{\cdot}_{\bar{q}}, \underline{\underline{\mathbf{D}}}^{\cdot}_X)[n]$$

satisfies the axioms [AX1] of [GM3] §3.3, since these axioms uniquely determine $\underline{\underline{IC}}^{\cdot}_{\bar{p}}$. Specifically, we will verify the support axioms

$$[\mathrm{AX1}](c)\ \underline{\underline{H}}^m(\underline{\underline{S}}^{\cdot} \mid U_{k+1}) = 0 \qquad \text{for all } m > p(k) - n$$

$$[\mathrm{AX1}](d')\ \underline{\underline{H}}^m(j_k^!\underline{\underline{S}}^{\cdot} \mid U_{k+1}) = 0 \qquad \text{for all } m \leq p(k) - n + 1$$

where $j_k : Y = X_{n-k} - X_{n-k-1} \to U_{k+1} = X - X_{n-k-1}$ is the closed inclusion of the stratum with codimension k into U_{k+1}.

Verification of $[AX1](d')$. Let $\underline{\underline{\mathbf{D}}}^{\cdot}_Y$ and $\underline{\underline{\mathbf{D}}}^{\cdot}_{k+1}$ denote the dualizing complexes of Y and U_{k+1} respectively. Then

$$\begin{aligned} j_k^!\underline{\underline{S}}^{\cdot} &\cong \mathrm{dual}\ j_k^*\ \mathrm{dual}\ \underline{\underline{S}}^{\cdot} \\ &\cong R\,\underline{\underline{\mathrm{Hom}}}^{\cdot}(j_k^*\underline{\underline{IC}}^{\cdot}_{\bar{q}}, \underline{\underline{\mathbf{D}}}^{\cdot}_Y)[n] \\ &\cong R\,\underline{\underline{\mathrm{Hom}}}^{\cdot}(j_k^*\underline{\underline{IC}}^{\cdot}_{\bar{q}}[k-2n], \underline{\underline{\mathbf{Z}}}_Y) \end{aligned}$$

These complexes are cohomologically locally constant on Y, so the stalk of the sheaf $R\,\underline{\underline{\mathrm{Hom}}}$ is the $R\,\mathrm{Hom}$ of the stalks. Let j_y denote the inclusion of a point $y \in Y$. Then the stalk cohomology at y is

$$\begin{aligned} \underline{\underline{H}}^m(j_k^!\underline{\underline{S}}^{\cdot})_y &= \mathrm{Hom}(j_y^*\underline{\underline{IC}}^{\cdot}_{\bar{q}}[k-2n], \mathbf{Z}) \\ &\cong \mathrm{Ext}(H^{-m+1}(j_y^*\underline{\underline{IC}}_{\bar{q}}[k-2n], \mathbf{Z})) \oplus \mathrm{Hom}(H^{-m}(j_y^*\underline{\underline{IC}}^{\cdot}_{\bar{q}}[k-2n], \mathbf{Z}) \\ &= 0 \qquad \text{whenever } -m+k-2n > q(k)-n \text{ by } [\mathrm{AX1}](c) \text{ for } \underline{\underline{IC}}^{\cdot}_{\bar{q}}. \end{aligned}$$

This holds if $m \leq p(k) - n + 1$. (This verification did *not* use the assumption $T^{\bar{p}}_{q(k)}(L) = 0$).

Verification of $[AX1](c)$. We shall show the stalk cohomology over points $y \in Y$ satisfies the required vanishing condition

$$\begin{aligned} j_k^*\underline{\underline{S}}^{\cdot} \mid U_{k+1} &\cong R\,\underline{\underline{\mathrm{Hom}}}^{\cdot}(j_k^!\underline{\underline{IC}}^{\cdot}_{\bar{q}}, \underline{\underline{\mathbf{D}}}^{\cdot}_Y)[n] \\ &\cong R\,\underline{\underline{\mathrm{Hom}}}^{\cdot}(j_k^!\underline{\underline{IC}}^{\cdot}_{\bar{q}}, \underline{\underline{\mathbf{Z}}}_Y)[2n-c] \\ &\cong R\,\underline{\underline{\mathrm{Hom}}}^{\cdot}(\tau^{\geq q(k-n)+1} j_k^* R i_* i^* \underline{\underline{IC}}_{\bar{q}}[-1], \mathbf{Z})[2n-c] \end{aligned}$$

because of the distinguished triangle

$$\begin{array}{ccc} Rj_{k*}j_k^!\underline{\underline{IC}}^{\cdot}_{\bar q}\mid U_{k+1} & \longrightarrow & \underline{\underline{IC}}^{\cdot}_{\bar q}\mid U_{k+1}=\tau_{\le q(k)-n}Ri_*i^*\underline{\underline{IC}}^{\cdot}_{\bar q} \\ {\scriptstyle[1]}\nwarrow & & \swarrow \\ & Ri_*i^*\underline{\underline{IC}}^{\cdot}_{\bar q}\mid U_{k+1} & \end{array}$$

(Here, $i: U_k \to U_{k+1}$ is the inclusion.)

Thus the stalk at a point $y \in Y$ of $H^m(S^{\cdot})$ is

$$\underline{\underline{H}}^m(\underline{\underline{S}}^{\cdot}_y)=\begin{cases} H^{m+2n-c-1}(\operatorname{Hom}(j_y^*i_*i^*\underline{\underline{IC}}^{\cdot}_{\bar q},\mathbf{Z})) & \text{if } m\le p(k)-n \\ \operatorname{Ext}(H^{q-n+1}(j_y^*i_*i^*\underline{\underline{IC}}^{\cdot}_{\bar q}),\mathbf{Z}) & \text{if } m=p(k)-n+1 \\ 0 & \text{if } m>p(k)-n+1 \end{cases}$$

by Proposition 5.2.

Therefore axiom (c) will be satisfied iff the following group vanishes:

$$\operatorname{Ext}(H^{q-n+1}(j_y^*i_*i^*\underline{\underline{IC}}^{\cdot}_{q}),\mathbf{Z}).$$

This is isomorphic to $\operatorname{Ext}(IH^{\bar q}_{k-2-q(k)}(L),\mathbf{Z})$ by [GM3] §2.2. However the link L of the stratum Y is a $k-1$ dimensional pseudomanifold which is locally $\bar q$-torsion free (see Remark 4.2) so the theorem applies to L by induction and

$$T^{\bar q}_{c-2-q(c)}(L) \quad \text{is} \quad \mathbf{Q/Z}\text{-dual to } T^{\bar p}_{q(c)}(L)$$

which is 0 by assumption.

§7. Spaces which satisfy Poincaré duality

In this section we describe a class of spaces such that the intersection homology group with middle perversity $\bar m=(0,0,1,1,2,2,\ldots)$ satisfies Poincaré duality over the integers.

7.1. THEOREM. *Suppose the compact oriented n dimensional pseudomanifold X satisfies the following two conditions*:

(a) *For each stratum of odd codimension c,*

$$IH^{\bar m}_{(c-1)/2}(L;\mathbf{Z})=0$$

(b) *For each stratum of even codimension c,*

$$T^{\bar{m}}_{c/2-1}(L) = 0$$

where L is the link of the stratum in question. Then there is a canonical split exact sequence

$$0 \to \operatorname{Hom}(T^{\bar{m}}_{i-1}(X), \mathbf{Q}/\mathbf{Z}) \to IH^{\bar{m}}_{n-i}(X; \mathbf{Z}) \to \operatorname{Hom}(IH^{\bar{m}}_{i}(X; \mathbf{Z}), \mathbf{Z}) \to 0$$

which is compatible with the intersection pairing and the linking pairing.

Proof. Conditions (a) and (b) guarantee (by Theorem 4.4) that the morphism induced by the product

$$\underline{\underline{IC}}^{\bullet}_{\bar{m}} \to R\,\underline{\underline{\operatorname{Hom}}}^{\bullet}(\underline{\underline{IC}}^{\bullet}_{\bar{n}}, \underline{\underline{\mathbf{D}}}^{\bullet}_{X})[n]$$

is a quasi-isomorphism, and condition (a) guarantees that the natural map $\underline{\underline{IC}}^{\bullet}_{\bar{m}} \to \underline{\underline{IC}}^{\bullet}_{\bar{n}}$ is a quasi-isomorphism (see the obstruction sequence argument in [S] or [GM3] §5.6). Together they imply that $\underline{\underline{IC}}_{\bar{m}}$ is self dual. The exact sequence is the universal coefficient theorem of [8].

7.2. *Remarks.* If X satisfies properties (a) or (b) above, with respect to one stratification then it satisfies the same properties with respect to any refinement of the stratification.

§8. Change of coefficients in intersection homology

In this section we will assume G is an abelian group. Recall that $IH^{\bar{p}}_{i}(X; G)$ is defined to be the ith homology group of the complex of chains $IC^{\bar{p}}_{i}(X; C)$ which consists of those $\xi \in C_i(X) \otimes G$ such that $|\xi|$ is $(\bar{p}, i)$-allowable and $|\partial\xi|$ is $(\bar{p}, i-1)$-allowable ([GM2] §6.3).

8.1. THEOREM. *Suppose X is a* P.L. *stratified pseudomanifold and for each stratum of X,*

$$\operatorname{Tor}(IH^{\bar{p}}_{q(c)}(L), G) = 0$$

where L is the link of the stratum, c is its codimension, and $q(c) = c - 2 - p(c)$. Then *there is a canonical exact sequence*

$$0 \to IH_i^{\bar{p}}(X) \otimes G \to IH_i^{\bar{p}}(X; G) \to \operatorname{Tor}(IH_{i-1}^{\bar{p}}(X), G) \to 0$$

which is split.

Remark. If X is locally $\bar{p}$-torsion free then the hypothesis holds for any abelian group G.

8.2. *Proof.* $IH_i^{\bar{p}}(X; G)$ is the hypercohomology group $\mathcal{H}^{-i}(\underline{\underline{IC}}_{\bar{p}}^{\cdot}(\underline{\underline{G}}))$ of the complex of sheaves which is obtained by applying Deligne's construction to the constant sheaf G on $X - \Sigma$. We shall show that $\underline{\underline{IC}}_{\bar{p}}^{\cdot}(G)$ and $\underline{\underline{IC}}_{\bar{p}}^{\cdot} \otimes G$ are quasi-isomorphic under the hypotheses of the theorem. (The short exact sequence is then the statement of the universal coefficient theorem for the complex $\underline{\underline{IC}}_{\bar{p}}^{\cdot} \otimes G$).

The quasi-isomorphism is obtained by verifying the axioms [AX1] for the complex $\underline{\underline{IC}}_{\bar{p}}^{\cdot} \otimes G$. Since the verification is analogous to that in §6, we omit it here but remark that the relevant lemma from homological algebra is the following:

LEMMA. *Let $C^{\cdot}$ be a chain complex of free abelian groups. Then*

$$H^n \tau_{\leq a}(C^{\cdot} \otimes G) = \begin{cases} 0 & \textit{for} \quad n > a \\ H^n[(\tau_{\leq a} C^{\cdot}) \otimes G] \oplus \operatorname{Tor}(H^{n+1}(C^{\cdot}), G) & \textit{for} \quad n = a \\ H^n[(\tau_{\leq a} C^{\cdot}) \otimes G] & \textit{for} \quad n < a \end{cases}$$

§9. The peripheral complex $\underline{\underline{\mathbf{R}}}_{\bar{p}}^{\cdot}$

9.1. Let X be an n dimensional compact oriented pseudomanifold.

DEFINITION. $\underline{\underline{R}}_{\bar{p}}^{\cdot}$ is the (algebraic) mapping cone of the morphism (2). $R_i^{\bar{p}}(X)$ is the hypercohomology group $\mathcal{H}^{-i}(\underline{\underline{R}}_{\bar{p}}^{\cdot})$.

9.2. *Remarks.* (1) We have a distinguished triangle in $D^b(X)$,

$$\begin{array}{ccc} \underline{\underline{IC}}_{\bar{p}}^{\cdot} & \longrightarrow & R\, \underline{\underline{\operatorname{Hom}}}^{\cdot}(\underline{\underline{IC}}_{\bar{q}}^{\cdot}, \underline{\underline{\mathbf{D}}}_X^{\cdot})[n] \\ & {\scriptstyle [1]}\nwarrow \quad \swarrow & \\ & \underline{\underline{\mathbf{R}}}_{\bar{p}}^{\cdot} & \end{array}$$

Thus, X is locally $\bar{p}$-torsion free if and only if $\underline{\underline{R}}_{\bar{p}}^{\cdot} \cong 0$.

(2) The cohomology sheaves associated to $\underline{\underline{R}}^{\cdot}_{\bar{p}}$ are supported on the singular set of X since the morphism (2) is a quasi-isomorphism over the nonsingular part of X.

(3) The hypercohomology groups $R^{\bar{p}}_i(X) = \mathcal{H}^{-i}(\underline{\underline{R}}^{\cdot}_{\bar{p}})$ are torsion groups since the morphism (2) becomes a quasi-isomorphism when both sides are tensored with the rationals (see [GM3] §5.3). Thus, the torsion sub-groups of the hypercohomology groups of the complexes in the above triangle can be identified as follows:

$$\cdots \longrightarrow T^{\bar{p}}_i \longrightarrow \operatorname{Hom}(T^{\bar{q}}_{n-i-1}, \mathbf{Q}/\mathbf{Z}) \xrightarrow{\alpha_i} R^{\bar{p}}_i(X) \xrightarrow{\beta_i} T^{\bar{p}}_{i-1} \longrightarrow \cdots$$

This sequence is exact except at $R^{\bar{p}}_i(X)$ (see diag. 11.3).

9.3. PROPOSITION. *There is a canonical nondegenerate pairing*

$$K : R^{\bar{p}}_i(X) \times R^{\bar{q}}_{n-i}(X) \to \mathbf{Q}/\mathbf{Z}$$

such that $K(\alpha_i(a), b) = a \cdot \beta_{n-i}(b)$ *for all* $a \in \operatorname{Hom}(T^{\bar{q}}_{n-i-1}, \mathbf{Q}/\mathbf{Z})$ *and all* $b \in R^{\bar{q}}_{n-i}$.

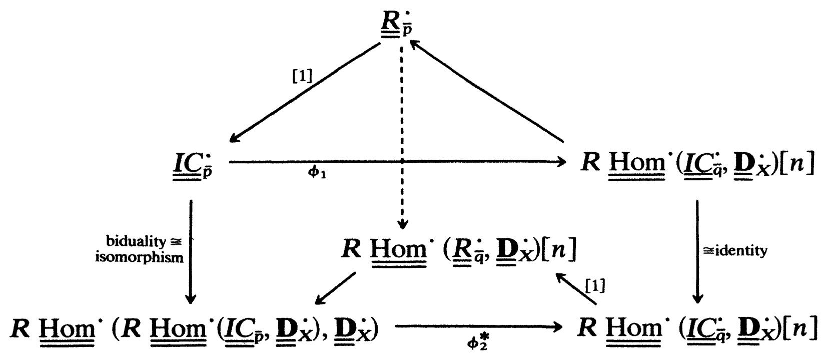
Proof. First we define K. The intersection product ([GM3] §5.2)

$$\underline{\underline{IC}}^{\cdot}_{\bar{p}} \otimes \underline{\underline{IC}}^{\cdot}_{\bar{q}} \to \underline{\underline{\mathbf{D}}}^{\cdot}_X[n]$$

induces a pair of adjoint morphisms

$$\phi_1 : \underline{\underline{IC}}^{\cdot}_{\bar{p}} \to R\,\underline{\underline{\operatorname{Hom}}}^{\cdot}(\underline{\underline{IC}}^{\cdot}_{\bar{p}}, \underline{\underline{\mathbf{D}}}^{\cdot}_X)[n]$$
$$\phi_2 : \underline{\underline{IC}}^{\cdot}_{\bar{q}} \to R\,\underline{\underline{\operatorname{Hom}}}^{\cdot}(\underline{\underline{IC}}^{\cdot}_{\bar{p}}, \underline{\underline{\mathbf{D}}}^{\cdot}_X)[n]$$

and $\underline{\underline{R}}^{\cdot}_{\bar{p}} =$ mapping cone (ϕ_1); $\underline{\underline{R}}^{\cdot}_{\bar{q}} =$ mapping cone (ϕ_2). Dualizing ϕ_2 gives rise to a pair of distinguished triangles



From this diagram we obtain a quasi-isomorphism,

$$\underline{\underline{R}}^{\cdot}_{\bar{p}} \to R\ \underline{\underline{\mathrm{Hom}}}^{\cdot}\,(\underline{\underline{R}}^{\cdot}_{\bar{q}}, \underline{\underline{\mathbf{D}}}^{\cdot}_X)[n+1]$$

Applying hypercohomology and the universal coefficient theorem we obtain an isomorphism

$$\tilde{K}: R^{\bar{p}}_i(X) \to \mathrm{Hom}\,(R^{\bar{q}}_{n-i}(X), \mathbf{Q}/\mathbf{Z})$$

whose adjoint is the desired $K: R^{\bar{p}}_i(X) \otimes R^{\bar{q}}_{n-i}(X) \to \mathbf{Q}/\mathbf{Z}$.

The compatibility of K with α and β is equivalent to the statement that the following diagram commutes:

$$\begin{array}{ccc} R^{\bar{p}}_i(X) & \xleftarrow{\ \alpha\ } & \mathrm{Hom}\,(T^{\bar{q}}_{n-i-1}, \mathbf{Q}/\mathbf{Z}) \\ {\scriptstyle \tilde{K}}\downarrow & & \downarrow{\scriptstyle \text{identity}} \\ \mathrm{Hom}\,(R^{\bar{q}}_{n-i}(X), \mathbf{Q}/\mathbf{Z}) & \xleftarrow{\ \beta^*\ } & \mathrm{Hom}\,(T^{\bar{q}}_{n-i-1}, \mathbf{Q}/\mathbf{Z}) \end{array}$$

However this diagram is simply the torsion in the hypercohomology of the right hand face of the preceding diagram.

9.4. EXAMPLE. If the singular set of X consists of a single stratum Σ of codimension c, then the stalk homology of $\underline{\underline{R}}^{\cdot}_{\bar{p}}$ is, for any $x \in \Sigma$,

$$\mathscr{H}^{-i}(\underline{\underline{R}}^{\cdot}_{\bar{p}})_x = \begin{cases} T_{c-2-p(c)}(L) & \text{if } i = n - p(c) - 1 \\ 0 & \text{if } i \neq n - p(c) - 1 \end{cases}$$

where L is the link of the stratum.

If X is obtained from an n-dimensional manifold M by attaching the cone on its boundary ∂M, then

$$R^{\bar{m}}_i(X) = \begin{cases} T_{[(n-1)/2]}(\partial M) & \text{if } i = [(n+1)/2] \\ 0 & \text{if } i \neq [(n+1)/2] \end{cases}$$

where [ ] denotes the integer part. If $\dim(M) = 4k$ then the equivalence class in the Witt ring $W(\mathbf{Q}/\mathbf{Z})$ of the torsion pairing

$$T^{\bar{m}}_{2k-1}(\partial M) \times T^{\bar{m}}_{2k-1}(\partial M) \to \mathbf{Q}/\mathbf{Z}$$

is called the *peripheral invariant* in [AHV].

§10. Spaces for which the peripheral complex is self dual

The canonical morphism $\underline{\underline{IC}}^{\cdot}_{\bar{p}} \to \underline{\underline{IC}}^{\cdot}_{\bar{q}}$ (where $\bar{p} \le \bar{q}$) induces canonical morphisms on the peripheral complexes, $\underline{\underline{R}}^{\cdot}_{\bar{p}} \to \underline{\underline{R}}^{\cdot}_{\bar{q}}$. The spaces for which $\underline{\underline{R}}^{\cdot}_{\bar{m}}$ is self dual over $\mathbf{Q}/\mathbf{Z}$ are the spaces such that $\underline{\underline{R}}^{\cdot}_{m} \to \underline{\underline{R}}^{\cdot}_{n}$ is a quasi-isomorphism.

10.1. THEOREM. *Suppose X is a compact oriented n dimensional pseudomanifold such that, for each stratum with* odd *codimension c,*

$$IH^{\bar{m}}_{(c-1)/2}(L; \mathbf{Z}) = 0$$

where L is the link of the stratum in question. Then $\underline{\underline{R}}^{\cdot}_{\bar{m}} \to \underline{\underline{R}}^{\cdot}_{\bar{n}}$ is a quasi-isormorphism so K induces a nondegenerate product

$$K: R^{\bar{m}}_{i}(X) \times R^{\bar{m}}_{n-i}(X) \to \mathbf{Q}/\mathbf{Z}.$$

Proof. The assumption implies $\underline{\underline{IC}}^{\cdot}_{\bar{m}} \to \underline{\underline{IC}}^{\cdot}_{\bar{n}}$ and $R\,\underline{\underline{\mathrm{Hom}}}^{\cdot}(\underline{\underline{IC}}^{\cdot}_{\bar{n}}, \underline{\underline{\mathbf{D}}}^{\cdot}_{X}) \to R\,\underline{\underline{\mathrm{Hom}}}^{\cdot}(\underline{\underline{IC}}^{\cdot}_{\bar{n}}, \underline{\underline{\mathbf{D}}}^{\cdot}_{X})$ are quasi-isomorphisms (see [S] or [GM3] §5.6). Therefore the induced map $\underline{\underline{R}}^{\cdot}_{\bar{m}} \to \underline{\underline{R}}^{\cdot}_{\bar{n}}$ is also a quasi-isomorphism. Q.E.D.

10.2 DEFINITION. If X is a $4k$ dimensional space which satisfies the hypotheses of Theorem 9.1 then the equivalence class in $W(\mathbf{Q}/\mathbf{Z})$ of the pairing

$$K: R^{\bar{m}}_{2k}(X) \times R^{\bar{m}}_{2k}(X) \to \mathbf{Q}/\mathbf{Z}$$

is called the *peripheral invariant* of X. (Here $W(\mathbf{Q}/\mathbf{Z})$ is the Witt ring of $\mathbf{Q}/\mathbf{Z}$ and it consists of certain equivalence classes of symmetric $\mathbf{Q}/\mathbf{Z}$-valued pairings on finite abelian groups [MH]).

§11. Relation between the Witt class and the peripheral invariant

11.1. DEFINITION. An oriented pseudomanifold X is a rational Witt space if, for each stratum of X with odd codimension c,

$$IH^{\bar{m}}_{(c-1)/2}(L; \mathbf{Q}) = 0$$

where L is the link of that stratum. If $\dim(X) = 4k$ define $w(X) \in W(\mathbf{Q})$ to be the equivalence class (in the Witt ring of $\mathbf{Q}$) of the nondegenerate symmetric intersection pairing

$$IH^{\bar m}_{2k}(X;\,\mathbf{Q})\times IH^{\bar m}_{2k}(X;\mathbf{Q})\to \mathbf{Q}.$$

We recall the following fact from [S]:

THEOREM. *If X is a rational Witt space then $w(X)$ is a cobordism invariant (for cobordisms which are also rational Witt spaces). The association $X\mapsto w(x)$ determines an isomorphism.*

$$\Omega^{\text{Witt}}_{4k}\cong W(\mathbf{Q})$$
$$\Omega^{\text{Witt}}_{j}=0 \quad \textit{if} \quad j\not\equiv 0 \pmod 4.$$

11.2. The structure of $W(\mathbf{Q})$ is given by the following split exact sequence [MH]:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & W(\mathbf{Z}) & \longrightarrow & W(\mathbf{Q}) & \xrightarrow{\delta} & W(\mathbf{Q}/\mathbf{Z}) & \longrightarrow & 0\\
 & & \wr\| & & & & \wr\| & & \\
 & & \mathbf{Z} & & & & \bigoplus_{p \text{ prime}} W(\mathbf{Z}/p\mathbf{Z}) & &
\end{array}$$

11.3. THEOREM. *Suppose X is a $4k$ dimensional oriented pseudomanifold which satisfies the hypothesis of* §9.1, *i.e.,*

$$IH^{\bar m}_{(c-1)/2}(L;\mathbf{Z})=0$$

whenever L is the link of a stratum with odd codimension, c. Then X is a rational Witt space, and $\delta w(X)$ is equal to the peripheral invariant of X.

Proof. Consider the exact sequence on hypercohomology which is associated to the distinguished triangle of §9.2:

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & T^{\bar m}_{2k}(X) & \longrightarrow & \mathrm{Hom}(T^{\bar n}_{2k-1},\mathbf{Q}/\mathbf{Z}) & \xrightarrow{\alpha} & R^{\bar m}_{2k} & \xrightarrow{\beta} & T^{\bar m}_{2k-1}(X) & \longrightarrow & \\
 & & \downarrow & & \downarrow & & \downarrow{\cong} & & \downarrow & & \\
\cdots & \longrightarrow & IH^{\bar m}_{2k}(X) & \longrightarrow & IH^{2k}_{\bar n}(X) & \longrightarrow & R^{\bar m}_{2k} & \longrightarrow & IH^{\bar m}_{2k-1}(X) & \longrightarrow & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & IH^{\bar m}_{2k}/T^{\bar m}_{2k} & \xrightarrow{\theta} & \mathrm{Hom}(IH^{\bar n}_{2k},\mathbf{Z}) & \longrightarrow & 0 & \longrightarrow & IH^{\bar m}_{2k-1}/T^{\bar m}_{2k-1} & \longrightarrow & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0 & &
\end{array}$$

By [AHV] (Lemma 1.4), $\delta w(X)$ coincides with the Witt class of the induced pairing on $K = \operatorname{coker}(\theta)$. However, $K \cong \ker \beta / \operatorname{Im} \alpha$ since we may view the diagram above as a short exact sequence of chain complexes with the middle complex acyclic. But we have already shown (§9.4) that $(\ker \beta) = (\operatorname{Im} \alpha)^{\perp}$ so by [AHV] (Lemma 1.3), the Witt class of the pairing on $R^{\bar{m}}_{2k}(X)$ also coincides with the Witt class of the pairing on $\ker \beta / \operatorname{Im} \alpha$ Q.E.D.

Remark. The diagram and preceding argument may be found in [BM] in the case that X has isolated singularities.

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