Equalization on Graphs: Linear Programming and Message Passing

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***Abstract*— We propose an approximation of maximum-likelihood detection in ISI channels based on linear programming or message passing. We convert the detection problem into a binary decoding problem, which can be easily combined with LDPC decoding. We show that, for a certain class of channels and in the absence of coding, the proposed technique provides the exact ML solution without an exponential complexity in the size of channel memory, while for some other channels, this method has a non-diminishing probability of failure as SNR increases. Some analysis is provided for the error events of the proposed technique under linear programming.**

I. Introduction

Intersymbol interference (ISI) is a characteristic of many data communications and storage channels. Systems operating on these channels employ error-correcting codes in conjunction with some ISI reduction technique, which, in magnetic recording systems, is often a conventional Viterbi detector. It is known that some gain will be obtained if the equalization and decoding blocks are combined at the receiver by exchanging soft information between them. A possible approach to achieving this gain is to use soft-output equalization methods such as the BCJR algorithm [1] or the soft-output Viterbi algorithm (SOVA) [2] along with iterative decoders. However, both BCJR and SOVA suffer from exponential complexity in the length of the channel memory.

Kurkoski *et al.* [3] proposed a bit-based and a state-based graph representation of the ISI channel that can be combined with the Tanner graph of a low-density parity-check (LDPC) code for joint message-passing (MP) decoding. They showed that the bit-based method suffers from a significant performance degradation due to the abundance of 4-cycles, but the state-based method has a performance and overall complexity similar to BCJR, while benefiting from a parallel structure and reduced delay.

Linear programming (LP) has been recently applied by Feldman *et al.* [4] to the decoding of LDPC codes, as an alternative to message-passing techniques. In this method, the binary parity-check constraints of the code are relaxed to a set of linear constraints in the real domain, thus turning the integer problem into an LP problem. While LP decoding performs closely to message-passing algorithms such as the sum-product algorithm (SPA) and the min-sum algorithm (MSA), it is much easier to analyze for finite code lengths.

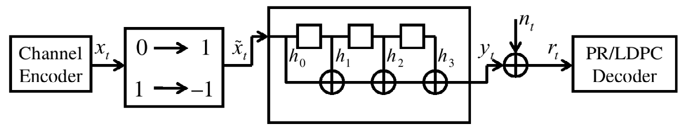

Fig. 1. Binary-input ISI channel.

Motivated by the success of LP decoding, in this work we study the problem of ML detection in the presence of ISI, which can be written as an integer quadratic program (IQP). We convert this problem into a binary decoding problem, which can be used for message-passing decoding, or, after relaxing the binary constraints, LP decoding. Furthermore, decoding an underlying LDPC code can be incorporated into this problem simply by including the parity checks of the code.

By a geometric analysis we show that, in the absence of coding, if the impulse response of the ISI channel satisfies certain conditions, the proposed LP relaxation is guaranteed to produce the ML solution at all SNR values. This means that there are ISI channels, which we call *LP-proper* channels, for which uncoded ML detection can be achieved with a complexity polynomial in the channel memory size. On the other end of the spectrum, some channels are *LP-improper*, i.e. the LP method results in a nonintegral solution with a probability bounded away from zero, even in the absence of noise. Furthermore, we observe some intermediate *asymptotically LP-proper* channels where the performance approaches that of ML detection at high SNR. When message passing is used instead of LP, we observe a similar behavior. Moreover, when LDPC decoding is incorporated in the detector, LP-proper channels achieve very good performance, while some other channels cannot go below a certain word error rate.

The rest of this paper is organized as follows. In Section II, we describe the channel, and introduce the LP relaxation of ML detection. The performance analysis is presented in Section III. Simulation results are given in Section IV, and Section V concludes the paper.

II. Relaxation of the Equalization Problem

A. Channel Model

We consider a partial-response (PR) channel with bipolar (BPSK) inputs, as described in Fig. 1, and use the following notation for the transmitted symbols.

Notation 1: The bipolar version of a binary symbol, $b \in \{0,1\}$, is denoted by $\tilde{b} \in \{-1,1\}$, and is given by

$$\tilde{b} = 1 - 2b. \tag{1}$$

The partial-response channel transfer polynomial is $h(D) = \sum_{i=0}^{\mu} h_i D^i$, where μ is the channel memory size. Thus, the output sequence of the PR channel in Fig. 1 before adding the white Gaussian noise can be written as

$$y_t = \sum_{i=0}^{\mu} h_i \tilde{x}_{t-i}. \tag{2}$$

B. Maximum-likelihood (ML) Detection

Having the vector of received samples $\underline{r} = [r_1\ r_2\ \cdots\ r_n]^T$, the ML detector solves the optimization problem

$$\begin{array}{ll} \text{Minimize} & \left\|\underline{r} - \underline{y}\right\|_2 \\ \text{Subject to} & \underline{x} \in \mathscr{C}, \end{array} \tag{3}$$

where $\mathscr{C} \subset \{0,1\}^n$ is the codebook and $\|\cdot\|_2$ denotes the L_2-norm. By expanding the square of the objective function, the problem becomes equivalent to minimizing

$$\begin{aligned} \sum_t (r_t - y_t)^2 &= \sum_t \left[r_t^2 - 2r_t \sum_i h_i \tilde{x}_{t-i} + \left(\sum_i h_i \tilde{x}_{t-i} \right)^2 \right] \\ &= \sum_t \left[r_t^2 - 2r_t \sum_i h_i \tilde{x}_{t-i} + \sum_i h_i^2 \tilde{x}_{t-i}^2 \right. \\ &\quad \left. + \sum_{i \neq j}\sum h_i h_j \tilde{x}_{t-i} \tilde{x}_{t-j} \right], \end{aligned} \tag{4}$$

where, for simplicity, we have dropped the limits of the summations. Equivalently, we can write the problem in a general matrix form

$$\begin{array}{ll} \text{Minimize} & -\underline{q}^T \underline{\tilde{x}} + \frac{1}{2} \underline{\tilde{x}}^T P \underline{\tilde{x}}, \\ \text{Subject to} & \underline{x} \in \mathscr{C}, \end{array} \tag{5}$$

where in this problem $q_t = \sum_i h_i r_{t+i}$, and $P = H^T H$, with H defined as the $n \times n$ Toeplitz matrix

$$H = \begin{bmatrix} h_0 & 0 & \cdots & & & \\ \vdots & \ddots & & & & \\ h_\mu & \cdots & h_0 & 0 & & \\ 0 & h_\mu & & h_0 & 0 & \\ \vdots & & \ddots & & \ddots & \\ 0 & \cdots & 0 & h_\mu & \cdots & h_0 \end{bmatrix}. \tag{6}$$

Here we have assumed that μ zeros are padded at the beginning and the end of the transmitted sequence, so that the trellis diagram corresponding to the ISI channel starts and ends at the zero state. If the signals are uncoded, i.e. $\mathscr{C} = \{0,1\}^n$, and $\underline{q}$ and P are chosen arbitrarily, (5) will represent the general form of an integer quadratic programming (IQP) problem, which is, in general, NP-hard. In the specific case of a PR channel, where we have the Toeplitz structure of (6), the problem can be solved by the Viterbi algorithm with a complexity linear in n, but exponential in μ. However, this model can also be used to describe other problems such as detection in MIMO or two-dimensional ISI channels. Also, when the source symbols have a non-binary alphabet with a regular lattice structure such as the QAM and PAM alphabets, the problem can be reduced to the binary problem of (5) by introducing some new variables.

C. Problem Relaxation

A common approach for solving the IQP problem is to first convert it to an integer LP problem by introducing a new variable for each quadratic term, and then relax the integrality condition; e.g. see [5]. While this relaxed problem does not necessarily have an integer solution, it can be used along with branch-and-cut techniques to solve integer problems of reasonable size. A more recent method is based on dualizing the IQP problem twice to obtain a convex relaxation in the form of a semi-definite program (SDP) [6][7].

In this work, we use the linear relaxation due to the lower complexity of solving LPs compared to SDPs. Unlike in [5], where the auxiliary variables are each defined as the product of two 0-1 variables, we define them as the product of ± 1 variables, which, as we will see, translates into the modulo-2 addition of two bits when we move to the 0-1 domain. This relaxation is more suitable for our purpose, since modulo-2 additive constraints are similar to parity-check constraints; thus, message-passing decoders designed for linear codes can be applied without any modification. However, it can be shown that this relaxation gives the exact same solution as in [5].

To linearize (4), we define

$$\tilde{z}_{t,j} = \tilde{x}_t \cdot \tilde{x}_{t-j}, \quad j = 1, \ldots, \mu,\ t = j+1, \ldots, n. \tag{7}$$

In the binary domain, this will be equivalent to

$$z_{t,j} = x_t \oplus x_{t-j}, \tag{8}$$

where $\oplus$ stands for modulo-2 addition. Hence, the right-hand side of (4) is a linear combination of $\{x_t\}$ and $\{z_{t,j}\}$, plus a constant, given that ${\tilde{x}_i}^2 = 1$ is a constant. With some simplifications, the IQP in (5) can be rewritten as

$$\begin{array}{ll} \underset{x,z}{\text{Minimize}} & \sum_t q_t x_t + \sum_t \sum_j \lambda_{t,j} z_{t,j}, \\ \text{Subject to} & \underline{x} \in \mathscr{C}, \\ & z_{t,j} = x_t \oplus x_{t-j},\ j = 1, \ldots, \mu, \\ & \qquad t = j+1, \ldots, n, \end{array} \tag{9}$$

where, in the equalization problem,

$$\lambda_{t,j} = -P_{t,t-j} = -\sum_{i=0}^{\min(\mu-j, n-t)} h_i h_{i+j}. \tag{10}$$

In this optimization problem, we call $\{x_i\}$ the information bits, and $\{z_{t,j}\}$ the state bits. It can be seen from (10) that $\lambda_{t,j}$ is independent of t, except for indices near the two ends of the block; i.e. $1 \leq t \leq \mu$ and $n - \mu + 1 \leq t \leq n$. In practice, this "edge effect" can be neglected due to the zero padding at the transmitter. For clarity, we sometimes drop the first subscript in $\lambda_{t,j}$, when the analysis is specific to the PR detection problem.

The combined equalization and decoding problem (9) has the form of a single decoding problem, which can be represented by a low-density Tanner graph. Fig. 2 shows an example of the combination of a PR channel of memory size 2 with an LDPC code. We call the upper and lower layers of this Tanner graph the code layer and the PR layer (or the PR graph), respectively. The PR layer of the graph consists of μn check nodes $c_{t,j}$ of degree 3, each connected to two information bit nodes x_t, x_{t-j}, and one distinct state bit node, $z_{t,j}$. Also, the PR layer can contain cycles of length 6 and higher. If a coefficient, $\lambda_{t,j}$, is zero, its corresponding state bit node, $z_{t,j}$, and the check node it is connected to can be eliminated from the graph, as they have no effect on the decoding process.

It follows from (10) that the coefficients of the state bits in the objective function, $\{\lambda_{t,j}\}$, are only a function of the PR channel impulse response, while the coefficients of the information bits are the results of matched filtering the noisy received signal by the channel impulse response, and therefore dependent on the noise realization. Once the variable coefficients in the objective function are determined, LP decoding can be applied to solve a linear relaxation of decoding on this Tanner graph. We call this method *LP detection*. In the relaxation of [4], the binary parity-check constraint corresponding to each check node c is relaxed as follows. Let N_c be the index set of neighbors of check node c, i.e. the variable nodes it is directly connected to in the Tanner graph. Then, we include the following constraints

$$\sum_{i\in V} x_i - \sum_{i\in N_c\setminus V} x_i \le |V|-1,\ \ \forall\, V\subset N_c \text{ s.t. } |V| \text{ is odd.} \quad (11)$$

In addition, the integrality constraints $x_i \in \{0,1\}$ are relaxed to box constraints $0 \le x_i \le 1$. This relaxation has the "ML certificate property," i.e. if the solution of the relaxed LP is integral, it will also be the solution of (9).

The coefficients in the linear objective function, after some normalization, can also be treated as log-likelihood ratios (LLR) of the corresponding bits, which can be used for iterative message-passing decoding. In this work, we have used the Min-Sum Algorithm (MSA), since, similar to LP decoding, it is not affected by the uniform normalization of the variable coefficients in (9).

III. Performance Analysis

In this section, we study the performance of LP detection in the absence of coding, i.e. solving (5) with $\mathscr{C} = \{0,1\}^n$. It is known that if the off-diagonal elements of P are all nonpositive; i.e. $\lambda_{t,j} \ge 0,\ \forall j \ne 0, t$, the 0-1 problem is solvable in polynomial time by reducing it to the MIN-CUT problem; e.g. see [8]. As an example, Sankaran and Ephremides [9] argued using this fact that when the spreading sequences in a synchronous CDMA system have nonpositive cross correlations, optimal multiuser detection can be done in polynomial time. In this section, we derive a slightly weaker

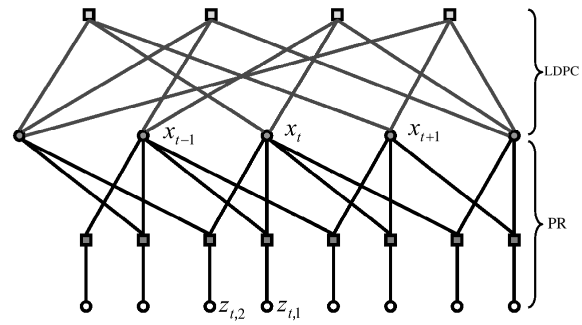

Fig. 2. PR channel and LDPC code represented by a Tanner graph.

condition than the nonnegativity of $\lambda_{t,j}$, as the necessary and sufficient condition for the success of the LP relaxation to result in an integer solution for any value of $\underline{q}$ in (9). This analysis also sheds some light on the question of how the algorithm behaves when this condition is not satisfied.

For a check node in the Tanner graph connecting information bit nodes x_t and x_{t-j} and state bit node $z_{t,j}$, the constraints (11) can be summarized as

$$\begin{aligned} z_{t,j} &\ge \max[x_t - x_{t-j}, x_{t-j} - x_t] \\ z_{t,j} &\le \min[x_t + x_{t-j}, 2 - x_t - x_{t-j}], \end{aligned} \quad (12)$$

which can be further simplified as

$$|x_t - x_{t-j}| \le z_{t,j} \le 1 - |x_t + x_{t-j} - 1|. \quad (13)$$

Since there is exactly one such pair of upper and lower bounds for each state bit, in the solution vector, $z_{t,j}$ will be equal to either the lower or upper bound, depending on the sign of its coefficient in the linear objective function, $\lambda_{t,j}$. Hence, having the coefficients, the cost of $z_{t,j}$ in the objective function can be written as

$$\lambda_{t,j} z_{t,j} = \begin{cases} \lambda_{t,j}|x_t - x_{t-j}| & \text{if } \lambda_{t,j} \ge 0, \\ \lambda_{t,j} - \lambda_{t,j}|x_t + x_{t-j} - 1| & \text{if } \lambda_{t,j} < 0, \end{cases} \quad (14)$$

where the first term in the second line is constant and does not affect the solution. Consequently, by substituting (14) in the objective function, the LP problem will be projected into the original n-dimensional space, giving the equivalent minimization problem

$$\begin{aligned} \text{Minimize} \quad & f(\underline{x}) = \sum_t q_t x_t + \sum_{t,j:\lambda_{t,j}>0}\sum |\lambda_{t,j}||x_t - x_{t-j}| \\ & \qquad + \sum_{t,j:\lambda_{t,j}<0}\sum |\lambda_{t,j}||x_t + x_{t-j} - 1|, \\ \text{Subject to} \quad & 0 \le x_t \le 1,\ \forall t = 1,\dots,n, \end{aligned} \quad (15)$$

which has a convex and piecewise-linear objective function. Each absolute value term in this expression corresponds to a check node in the PR layer of the Tanner graph representation.

A. LP-Proper Channels: Guaranteed ML Performance

For a class of channels, which we call *LP-proper channels*, the proposed LP relaxation of uncoded ML detection always gives the ML solution. The following theorem provides a criterion for recognizing LP-proper channels.

Theorem 1: The LP relaxation of the integer optimization problem (9), in the absence of coding, is exact for every transmitted sequence and every noise configuration if and only if the following condition is satisfied for $\{\lambda_{t,j}\}$:

Weak Nonnegativity Condition (WNC): Every check node $c_{t,j}$, connected to variable nodes x_t and x_{t-j}, which lies on a cycle in the PR Tanner graph corresponds to a nonnegative coefficient; i.e. $\lambda_{t,j} \geq 0$.

Proof: We first prove that WNC is sufficient for guaranteed convergence of LP to the ML sequence, and then show that if this condition is not satisfied, there are cases where the LP algorithm fails. Some of the details in the proof are omitted, and the reader is referred to a future longer paper [10]. In the proof, we make use of the following definition.

Definition 1: Consider a piecewise-linear function $f : \mathbb{R}^n \mapsto \mathbb{R}$. We call $\underline{a}$ a *breakpoint* of f if the derivative of $f(\underline{a} + s\underline{v})$ with respect to s changes at $s = 0$, for any nonzero vector $\underline{v} \in \mathbb{R}^n$.

1) Sufficiency: It is sufficient to show that under WNC, the solution of (15) is always at one of the vertices of the unit cube, $[0, 1]^n$. In the absence of the box constraints, the minimum of the objective function has to occur either at infinity or at a breakpoint of this piecewise-linear function. Each breakpoint, $\underline{a}$, is determined by making n of the absolute value terms *active*, i.e. setting their arguments equal to zero. When the feasible region is restricted to the unit cube $[0, 1]^n$, the optimum can also occur at the boundaries of this region. We can assume that the optimum lies on a number, k, of hyperplanes corresponding to the box constraints, making exactly k variables equal to either 0 or 1. In order to determine the remaining $n - k$ fractional variables, at least $n - k$ other equations are needed, resulting from making a number of absolute value terms active. Each of these equations will have one of the two forms $x_t = x_{t-j}$ or $x_t + x_{t+j} = 1$, depending on whether $\lambda_{t,j} > 0$ or $\lambda_{t,j} < 0$, respectively. In each of these equations, either both, or neither of its variables can be integer. Since the former case does not provide an equation in terms of the fractional variables, we can assume that all these equations only involve fractional variables.

Now the question becomes under what condition such equations can have a unique and nonintegral solution. We can illustrate this system of equations by a *dependence graph*, where the vertices correspond to the $n - k$ fractional variable nodes, and between vertices x_s and x_{s-i} there is a *positive* edge if $\lambda_{s,i} > 0$ and a *negative* edge if $\lambda_{s,i} < 0$. An example of a dependence graph satisfying WNC is shown in Fig. 3. In general, this graph contains L clusters, $C_1, C_2, \ldots, C_L$, of cycles, and also J additional edges which do not lie on any cycle. If WNC is satisfied, each cluster will contain only positive edges. This means that if we separate the equations corresponding only to vertices within a cluster C_i of cycles, these equations will all have the form $x_t = x_{t-j}$, where vertices t and $t - j$ belong to C_i. Consequently, all vertices in C_i have the same value, β_i. The values of $\{\beta_i\}$ should be determined by the edges (equations) between the clusters.

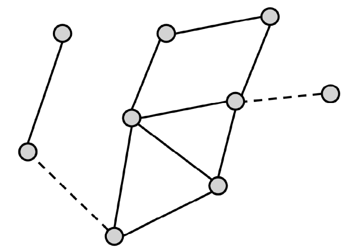

Fig. 3. The dependence graph of the system of linear equations with one cluster of cycles. Solid lines represent positive edges and dashed lines represent negative edges.

If we modify the dependence graph by concentrating each cluster to a vertex of value β_i, the graph will not contain any cycle. Therefore, the system of equations for determining the variables is under-determined, and none of the nodes in this graph will have a unique solution. Hence, the only case that will have a unique solution for all the variables is $k = n$, which means that all of the variables $\{x_i\}$ are integral.

2) Necessity: (Outline) We prove the necessity by a counter-example. Consider a case where the realization of the noise sequence is such that the received sequence is zero. Since this makes the $\{q_t\}$ equal to zero, we will be left with the positive-weighted sum of a number of absolute value terms. The objective function will become zero if and only if all these terms are zero, which is satisfied if $\underline{x} = [\frac{1}{2}, \ldots, \frac{1}{2}]^T$. By using similar ideas to those in the sufficiency proof, we can show that if WNC is not satisfied, this is the unique vector that makes all the terms equal to zero, and therefore, this fractional vector will be the solution of LP detection. ■

Corollary 1: The solutions of the LP relaxation of uncoded ML equalization are in the space $\{0, \frac{1}{2}, 1\}^n$. (Proof omitted)■

In a PR channel, due to the periodic structure of the Tanner graph, WNC implies that either the graph is acyclic, or all the coefficients $\{\lambda_{t,j}\}$ are nonnegative. An interesting application of this result is in 2-D channels, for which there is no feasible extension of the Viterbi algorithm. In a 2-D channel, the received signal $r_{t,s}$ at coordinate (t, s) in terms of the transmitted symbol array $\{\tilde{x}_{t,s}\}$ has the form

$$r_{t,s} = \sum_{i=1}^{\mu} \sum_{j=1}^{\nu} h_{i,j} \tilde{x}_{t-i,s-j} + n_{t,s}. \tag{16}$$

Hence, the state variable defined as $z_{(t,s),(k,l)} = x_{t,s} \oplus x_{t-k,s-l}$ will have the coefficient

$$\gamma_{k,l} = -\sum_{i=1}^{\mu} \sum_{j=1}^{\nu} h_{i,j} h_{i+l,j+l}. \tag{17}$$

Theorem 1 guarantees that ML detection can be achieved by linear relaxation if $\gamma_{k,l} \geq 0, \ \forall k, l > 0$. An example of a 2-D channel satisfying this condition is given by the matrix

$$[h_{i,j}] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \tag{18}$$

B. High SNR Analysis: asymptotically LP-Proper Channels

Here we outline a method to characterize the event that LP detection fails to find the ML solution at high SNR. This analysis is motivated by the observation that for some channels not satisfying WNC, this event at high SNR is dominated by the failure of the ML detector to find the transmitted sequence. We call these channels *asymptotically LP-proper*, since their high-SNR performance is similar to that of ML.

Given the solution $\hat{\underline{x}}$ of LP detection, let the *fractional set*, $F \subset \{1, \ldots, n\}$, be the set of indices of elements of $\hat{\underline{x}}$ that have fractional values in the solution. We know from Corollary 1 that these fractional values are all equal to $\frac{1}{2}$. A reasonable assumption supported by our simulations at high SNR is that if the ML solution, $\underline{x}$, is correct, the integer elements of $\hat{\underline{x}}$ are correct, as well. For the objective function f of (15) we have

$$f(\hat{\underline{x}}) < f(\underline{x}). \tag{19}$$

By expanding f, this inequality can be written in terms of $\{\lambda_{t,j}\}$, $\underline{x}$, and $\underline{q}$. Moreover, each q_t is a function of the channel coefficients, the transmitted sequence $\underline{x}$, and the additive noise. Hence, (19) will provide a condition in terms of $\underline{x}$ and the noise. In the absence of noise, and for a given fractional set, F, we can set up an integer optimization problem to find the transmitted sequence that minimizes $f(\hat{\underline{x}}) - f(\underline{x})$. We have studied the special case where $F = \{1, \ldots, n\}$. In this case, $f(\hat{\underline{x}}) - f(\underline{x}) = \delta$, where δ, which we call the all-$\frac{1}{2}$ distance of the channel, becomes independent of $\underline{x}$ for large block lengths. This distance can be used to estimate the high-SNR probability of failure. In particular, if for a channel $\delta \leq 0$, LP detection fails with a non-diminishing probability. We refer to these channels as *LP-improper* channels.

IV. Simulation Results

We have simulated graph-based detection using LP decoding and MSA for three PR channels of memory size 3:

1) **CH1:** $h(D) = 1 - D - 0.5D^2 - 0.5D^3$ (satisfies WNC),
2) **CH2:** $h(D) = 1 + D - D^2 + D^3$,
3) **CH3:** $h(D) = 1 + D - D^2 - D^3$ (EPR4 channel).

Uncoded bit error rates (BER) of detection on these channels using LP and MSA are shown Fig. 4. Since CH1 satisfies WNC, LP will be equivalent to ML on this channel. For CH2, we have also provided the BER of ML. Except at very low SNR where we see a small difference, the performance of LP and ML are nearly equal, which means that CH2 is an asymptotically LP-proper channel. For both CH1 and CH2, MSA converges in at most 3 iterations and has a BER very close to that of LP. On the other hand, for CH3, we observe that the BERs of LP and MSA are almost constant. Hence, CH3 is an LP-improper channel. The results for detection with LDPC coding are omitted due to space limitations [10]. However, for the cases that we have simulated, all the LP-proper and asymptotically LP-proper channels also showed a good performance in the presence of coding.

V. Conclusion

We introduced a new graph representation of ML detection in ISI channels, which can be used for combined equalization and decoding using LP relaxation or iterative message-passing methods. By a geometric study of the problem, we derived

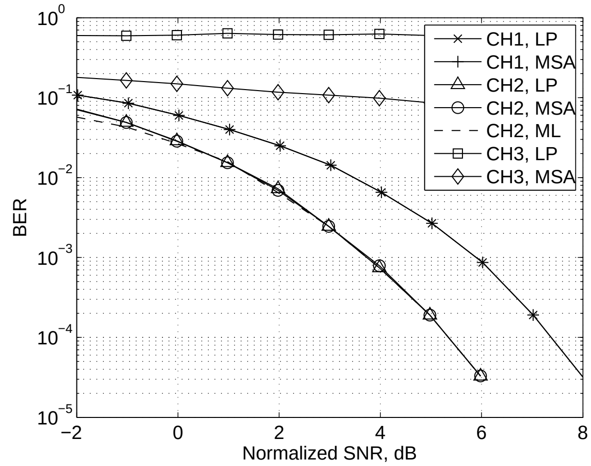

Fig. 4. BER for CH1-CH3. SNR is defined as the transmitted signal power to the received noise variance.

necessary and sufficient conditions for the LP relaxation of the channel equalization problem to give the ML solution for all transmitted sequences and all noise configurations. In addition, for certain other channels, the performance of LP approaches that of ML at high SNRs. For a third class of channels, LP detection has a probability of failure bounded away from zero even in the absence of noise. In a step to characterize these channels, we showed how a condition can be derived for the failure of LP detection, which can be used to estimate the asymptotic behavior of this detection method. Simulation results show that message-passing decoding techniques have a similar performance to that of LP decoding for most channels.

Acknowledgment

This work is supported in part by NSF Grant CCF-0514859.

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