Periodic-Finite-Type Shift Spaces

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Abstract—We study the class of periodic-finite-type (PFT) shift spaces, which can be used to model time-varying constrained codes used in digital magnetic recording systems. A PFT shift is determined by a finite list of periodically forbidden words. We show that the class of PFT shifts properly contains all finite-type (FT) shifts, and the class of almost finite-type (AFT) shifts properly contains all PFT shifts. We establish several basic properties of PFT shift spaces of a given period T, and provide a characterization of such a shift in terms of properties of its Shannon cover (i.e., its unique minimal, deterministic, irreducible graph presentation). We present an algorithm that, given the Shannon cover $\mathcal{G}$ of an irreducible sofic shift X, decides whether or not X is PFT in time that is quadratic in the number of states of $\mathcal{G}$. From any periodic irreducible presentation of a given period, we define a periodic forbidden list, unique up to conjugacy (a circular permutation) for that period, that satisfies certain minimality properties. We show that an irreducible sofic shift is PFT if and only if the list corresponding to its Shannon cover $\mathcal{G}$ and its period is finite. Finally, we discuss methods for computing the capacity of a PFT shift from a periodic forbidden list, either by construction of a corresponding graph or in a combinatorial manner directly from the list itself.

Index Terms—Capacity of constrained system, constrained code, finite-type, periodic constraint, shift spaces, sofic system.

I. INTRODUCTION

DIGITAL data storage systems based upon magnetic and optical recording typically use constrained modulation codes designed to efficiently avoid sequences that are problematic to data recording and retrieval [1]. The family of (d, k)-constrained run-length limited (RLL) codes over the binary alphabet $\{0, 1\}$ is a well known example. The code sequences satisfy the constraint that the number of 0's between consecutive 1's in a sequence is at least d and no more than k. The purpose of these constraints is to aid in timing recovery and

to limit intersymbol interference. The (d, k)-RLL constraint is characterized by a finite list of forbidden words. For example, the $(1, 3)$-RLL sequences are precisely those in which neither of the words $\{11, 0000\}$ appears. Such constraints are called *finite-type (FT)*.

Another widely used family of codes are the c-charge constrained codes over the bipolar alphabet $\{\pm 1\}$. Here, the code sequences limit the running-digital-sums of subsequences to a range of $c > 2$ consecutive integer values. These codes, often called *dc-free*, ensure that the average power spectral density of code sequences vanishes at zero frequency. In contrast to the (d, k)-RLL constraint, the c-charge constraint cannot be characterized by a finite list of forbidden words. However, these constraints can be specified by a countably infinite set of forbidden words. They are representative of constraints called *almost finite-type (AFT)*.

During the past decade, advances in digital recording have led to the introduction of constrained codes that are described by time-varying constraints. An important example is the family of Time-varying Maximum Transition Run codes with parameters $(j, j+1)$, denoted $\mathrm{TMTR}(j, j+1)$. These codes constrain the run-lengths of 1's to be at most j starting at odd time indices and $j + 1$ beginning at even time indices [2]–[5]. These codes were developed for systems employing higher-order partial-response equalization and maximum-likelihood sequence detection. For selected partial-response target channels, they are *distance-enhancing* codes; that is, they eliminate bit patterns occurring in the dominant error events of the target-matched sequence detector [6]–[8]. Recently, generalized TMTR codes, which limit maximum runlengths of 1's beginning at more than two phases, have also been studied [9].

Time-varying constraints also arise in the context of constrained codes with unconstrained positions, introduced in [10] and further studied in [11]–[13]. These codes permit the insertion of parity bits generated by a systematic error-correcting code into specified bit locations in a constrained code sequence, thereby efficiently combining the modulation and error correction functions of the two codes.

In general, these time-varying constraints are not FT, but they all have the property that they can be specified by a finite list of periodically forbidden words. The study of such time-varying constrained systems was initiated in [14], [15], where they were called *periodic-finite-type (PFT)*. The purpose of this paper is to present a detailed analysis of their properties.

Section II reviews necessary concepts, terminology, and notation for use in the rest of the paper.

In Section III, we formulate the definition of PFT constraints in terms of shift spaces, and address their characteristics within the framework of symbolic dynamics. We study basic properties of PFT shifts that are characterized by a finite periodic list of

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forbidden words for a given period T. We refer to such shifts as $\mathrm{PFT}(T)$ shifts, and we say that a shift is PFT if, for some period $T > 0$, it is $\mathrm{PFT}(T)$. We show that PFT shifts are sofic, and we demonstrate that the family of PFT shifts properly contains the family of FT shifts and is properly contained within the family of AFT shifts [16]. We also explore the periods T for which a PFT shift can be $\mathrm{PFT}(T)$.

Section IV gives several characterizations of an irreducible PFT shift in terms of its graph presentations. In particular, we give a necessary and sufficient condition for an irreducible sofic shift to be a $\mathrm{PFT}(T)$ shift, based upon properties of its Shannon cover (i.e., its unique minimal, deterministic, irreducible graph presentation) [17]. This leads to an algorithm that, when presented with the Shannon cover $\mathcal{G}$ of an irreducible sofic shift, decides in time quadratic in the number of states of $\mathcal{G}$ if the shift is PFT.

In Section V, we study periodic forbidden lists that offer a concise description of a PFT shift. From an irreducible presentation with period T, we derive a periodic forbidden list that satisfies a minimality property for the chosen period T. We prove that the list, up to a permutation of the time indices, is unique and independent of the choice of the presentation with period T. The notion of minimality, as well as the definition of the list, are directly inspired by the construction of the set of first offenders of a FT shift [18], [16], so we refer to the periodic forbidden list as the set of periodic first offenders for the period. We then consider the periodic first offenders corresponding to the Shannon cover and the period of its underlying graph. We prove that an irreducible sofic shift is PFT if and only if this list is finite.

We define the size of a periodic forbidden list to be the sum of the lengths of its words. We prove that the minimum size over all periodic forbidden lists for all periods is attained by a periodic forbidden list for a period dividing the period of the graph underlying the Shannon cover.

Finally, in Section VI, we discuss methods for computing the capacity of a PFT shift from a periodic forbidden list description of the shift. The conventional method for computing the capacity of a sofic shift is based upon determining the largest real eigenvalue of the adjacency matrix of a lossless presentation of the system. We review a number of techniques, several of which are formulated in terms of the theory of finite automata, for constructing such a presentation from a finite list of periodically forbidden words.

We then present a quite different method which relies upon the *Inclusion-Exclusion Principle* [19], [20] from enumerative combinatorics. It extends to PFT shifts the technique presented by Pimentel and Uchôa-Filho in [21] for computing the capacity of FT shifts from a finite list of forbidden words. It appears to be quite effective when the size of the periodic forbidden blocks is large compared to the number of blocks in the list, as is the case for some TMTR constraints.

Section VII concludes the paper.

II. BACKGROUND AND TERMINOLOGY

In this section we review terminology and background results to be used in the remainder of the paper. The notation in Sections II-A and II-B follows that found in the text by Lind and Marcus [16], and a thorough presentation may be found there.

Section II-C contains terminology on finite automata relevant to the construction procedures in Section VI-A. A more detailed exposition on automata may be found in [22].

A. Shift Spaces

Let $\Sigma^{\mathbb{Z}}$ denote the set of bi-infinite sequences

$$x = \cdots x_{-3} x_{-2} x_{-1} x_0 x_1 x_2 \ldots$$

whose symbols are drawn from a finite alphabet Σ,

$$\Sigma^{\mathbb{Z}} \stackrel{\text{def}}{=} \{x | x_i \in \Sigma, \quad \forall\, i \in \mathbb{Z}\}.$$

A *word* or *block* $w \in \Sigma^n$, for some integer n, is a finite string of consecutive symbols. We say that w is a *subword*, *subblock*, or *factor* of the sequence x, or equivalently that x *contains* w, if $w = x_i x_{i+1} \ldots x_{i+n-1}$ for some index i. We denote this fact by $w \prec_i x$. To conveniently specify the position of a word within a sequence, we write

$$x_{[i,j]} \stackrel{\text{def}}{=} x_i x_{i+1} \cdots x_j$$

where $i \leqslant j$. We sometimes write $x_{[i]}$ to denote x_i. When the context is clear, we will use similar concepts and notation when x denotes a word.

Let Σ^* be the collection of words over Σ, including the empty word, and let Σ^+ denote the subset of nonempty words in Σ^*. The *length* of a word, $|w|$, is the number of symbols in w, and we refer to a block of length n as an *n-block*. The shift map σ takes a sequence x to the sequence $y = \sigma(x)$ with ith coordinate $y_i = x_{i+1}$. The inverse of the shift map takes a sequence y to $x = \sigma^{-1}(y)$ with ith coordinate $x_i = y_{i-1}$.

When speaking of a finite collection of words $\mathcal{F}$, we say that $\mathcal{F}$ is *antifactorial* or *nonredundant* if no word $u \in \mathcal{F}$ is a factor of any word $w \in \mathcal{F}$ with $u \neq w$.

Let $\mathcal{F}$ be a collection of words over Σ and $\mathsf{X}_{\mathcal{F}}^{\Sigma}$ denote the subset of $\Sigma^{\mathbb{Z}}$ consisting of all bi-infinite sequences that do not contain a word from $\mathcal{F}$. In this context $\mathcal{F}$ is referred to as a *forbidden list*. A *shift space* is a set $X = \mathsf{X}_{\mathcal{F}}^{\Sigma}$. This terminology reflects the fact that X is invariant under the operation of the shift map, i.e., $\sigma(X) = X$. A shift space is a shift of finite type if there exists a finite set $\mathcal{F}$ such that $X = \mathsf{X}_{\mathcal{F}}^{\Sigma}$.

Let $\mathcal{B}_n(X)$ denote the set of all length-n words that occur as factors of sequences in X. The *language* of X is the collection

$$\mathcal{B}(X) \stackrel{\text{def}}{=} \bigcup_{n=0}^{\infty} \mathcal{B}_n(X),$$

where $\mathcal{B}_0(X) = \{\epsilon\}$, and ϵ denotes the empty word. The language of a shift space determines the space [16, Prop. 1.3.4]. That is, a bi-infinite sequence x belongs to the shift space X if and only if all of its subblocks belong to $\mathcal{B}(X)$. Considering $\mathcal{B}_N(X)$ as an alphabet, the Nth higher power code $\gamma_N : X \to (\mathcal{B}_N(X))^{\mathbb{Z}}$ is the mapping

$$(\gamma_N(x))_{[i]} = x_{[iN, iN+N-1]}$$

which takes a sequence from X and breaks it into a sequence of nonoverlapping N-blocks. The image of X under γ_N, $X^N \stackrel{\text{def}}{=} \gamma_N(X)$, is called the Nth higher power shift of X.

Let X be a shift space over Σ, and let $\Psi : \mathcal{B}_{m+a+1}(X) \to \Gamma$ be a mapping from allowed $(m+a+1)$-blocks in X to symbols in an alphabet Γ. The sliding block code with *memory* m and anticipation a induced by Ψ is the mapping $\psi : X \to \Gamma^{\mathbb{Z}}$ defined by

$$y = \psi(x),$$

where, for $x \in X$,

$$y_i = \Psi(x_{[i-m, i+a]}).$$

A sliding block code $\psi : X \to Y$ is a *conjugacy* from X to Y if it is invertible. The shifts X and Y are *conjugate* if $Y = \psi(X)$ and ψ is a conjugacy.

B. Sofic Shifts

A labeled directed graph $\mathcal{G} = (G, \mathcal{L})$ consists of a directed graph $G = (\mathcal{V}, E)$ with a finite set of states $\mathcal{V} = \mathcal{V}(G)$, a finite set of directed edges $E = E(G)$ connecting the states, and a *labeling* $\mathcal{L} : E \to \Sigma$ that assigns a label to each edge. Each directed edge e has an initial state, $\mathsf{i}(e)$ and a terminal state $\mathsf{t}(e)$. A *path* in the graph is a finite sequence of edges $\pi = e_1 e_2 \cdots e_N$ such that $\mathsf{t}(e_j) = \mathsf{i}(e_{j+1})$. The initial state of a path $\pi = e_1 e_2 \cdots e_N$ is defined as $\mathsf{i}(\pi) = \mathsf{i}(e_1)$, and the terminal state is defined as $\mathsf{t}(\pi) = \mathsf{t}(e_N)$. A path is a *cycle* if $\mathsf{i}(\pi) = \mathsf{t}(\pi)$. The label of π is the word $\mathcal{L}(\pi) = \mathcal{L}(e_1)\mathcal{L}(e_2)\ldots\mathcal{L}(e_N)$.

Whereas a path is finite, a *walk* on G is a bi-infinite sequence of edges $\xi = \cdots e_{-1} e_0 e_1 \cdots$ such that $\mathsf{t}(e_j) = \mathsf{i}(e_{j+1})$. The label of a walk is the sequence

$$\mathcal{L}_\infty(\xi) \stackrel{\text{def}}{=} \cdots \mathcal{L}(e_{-1})\mathcal{L}(e_0)\mathcal{L}(e_1)\ldots.$$

A graph G is *irreducible* if for any pair of states $I, J \in \mathcal{V}$ there exists a path with $\mathsf{i}(\pi) = I$ and $\mathsf{t}(\pi) = J$. An *irreducible component* of a graph G is a maximal (with respect to inclusion of vertices) irreducible subgraph of G.

A vertex $I \in \mathcal{V}$ is *stranded* if either no edges start at I or no edges terminate at I. A graph is *essential* if no vertex is stranded.

A graph has *local anticipation* a if a is the smallest nonnegative integer such that, for each $I \in \mathcal{V}$, all paths of length $a+1$ that start at I and have the same label start with the same edge. Similarly, a graph has *local memory* m if m is the smallest nonnegative integer such that, for each $I \in \mathcal{V}$, all paths of length $m+1$ that end at I and have the same label end with the same edge. A graph is *deterministic* if it has local anticipation 0, i.e., if edges with the same initial state have distinct labels.

A graph is (m, a)-*definite* if, given any word $w = w_{[-m, a]}$, the set of paths $\pi = e_{-m}\ldots e_0 \ldots e_a$ that generate w all agree in the edge e_0. If a graph is (m, a)-definite for some integers m and a, it is said to be *definite*. An irreducible graph is definite if and only if no two distinct cycles generate the same word (see, for instance, [23, Prop. 2.4]). An $(m, 0)$-definite graph is said to be *finite-memory*.

A *sofic shift* $\mathsf{X}_{\mathcal{G}}$ is the set of bi-infinite sequences obtained by reading the labels of walks on G:

$$\mathsf{X}_{\mathcal{G}} \stackrel{\text{def}}{=} \{x | \mathcal{L}_\infty(\xi) = x \text{ for some } \xi, \text{ a walk on } G\}.$$

We say that $\mathcal{G}$ is a *presentation* or *cover* of $\mathsf{X}_{\mathcal{G}}$, or $\mathcal{G}$ *presents* $\mathsf{X}_{\mathcal{G}}$. A sofic shift is *irreducible* if it has an irreducible presentation. The set of finite words generated by paths in $\mathcal{G}$, denoted $\mathcal{S}(\mathcal{G})$, is called a *constrained system*, and similar terminology is used in that context.

Let $\mathcal{G}$ be a deterministic graph. For any word $u \in \mathcal{B}(\mathsf{X}_{\mathcal{G}})$, we denote by $\tau(u)$ the set of terminal states of all paths with label u. If the cardinality of $\tau(u)$ is 1, then u is called a *synchronizing word*, and it is said to *focus* to the single state in $\tau(u)$.

An irreducible sofic shift is *almost-finite-type (AFT)* if it has a presentation with finite local anticipation and finite local memory. Since every sofic shift has a deterministic presentation [16, Th. 3.3.2], a sofic shift is AFT if and only if it has an irreducible, deterministic presentation with finite local memory.

Sofic shifts are shift spaces [16, Th. 3.1.4]. Hence, for every $\mathsf{X}_{\mathcal{G}}$ there exists a forbidden list, $\mathcal{F}$, of words over Σ such that $\mathsf{X}_{\mathcal{G}} = \mathsf{X}_{\mathcal{F}}^{\Sigma}$.

There is a unique, up to labeled graph isomorphism, deterministic graph presenting an irreducible sofic shift with the minimal number of states [16, Th. 3.3.18]. This graph is referred to as the *Shannon cover* of the shift. It is also called the *Fischer cover*. One can obtain the Shannon cover from any presentation via determinizing and state-minimizing algorithms, e.g., [16, pp. 92], [22, p. 68]. A Shannon cover always has at least one synchronizing word [17]. An irreducible sofic shift is FT (resp. AFT) if and only if the Shannon cover is definite (resp. has finite local memory) [17].

The *follower set* $\mathsf{F}_{\mathcal{G}}(I)$ of state I in $\mathcal{V}$ is the collection of labels of paths starting at I:

$$\mathsf{F}_{\mathcal{G}}(I) \stackrel{\text{def}}{=} \{\mathcal{L}(\pi) | \mathcal{L}(\pi) \in \mathcal{B}(\mathsf{X}_{\mathcal{G}}) \text{ and } \mathsf{i}(\pi) = I\}.$$

Note that for a graph, $\mathcal{G}$:

$$\bigcup_{I \in \mathcal{V}(\mathcal{G})} \mathsf{F}_{\mathcal{G}}(I) = \mathcal{B}(\mathsf{X}_{\mathcal{G}}).$$

The follower set of a collection of states is simply the union of their respective follower sets. The Nth higher power graph $\mathcal{G}^N = (G^N, \mathcal{L}^N)$ of $\mathcal{G}$ is the labeled graph with underlying graph G^N and the naturally induced labeling $\mathcal{L}^N$. Specifically, the vertex set is $\mathcal{V}(G^N) = \mathcal{V}(G)$, and there is one edge e_π in $E(G^N)$ from I to J with label $\mathcal{L}^N(e_\pi) = \mathcal{L}(\pi)$ for each path π of length N from I to J in G. The Nth higher power graph presents the Nth higher power shift, $\mathsf{X}_{\mathcal{G}^N} = (\mathsf{X}_{\mathcal{G}})^N$.

For $I, J \in \mathcal{V}$, let A_{IJ} denote the number of edges from I to J in G. The *adjacency matrix* of G is the $|\mathcal{V}| \times |\mathcal{V}|$ matrix $A_G = [A_{IJ}]$.

Given a nonnegative matrix A, the period of state I, $\mathrm{per}(I)$, is the greatest common divisor of those integers $n \geqslant 1$ for which $(A^n)_{II} > 0$, if such integers exist. Otherwise, we define $\mathrm{per}(I) = \infty$. The *period* $\mathrm{per}(A)$ of A is defined as the greatest common divisor of the finite periods $\mathrm{per}(I)$, or as ∞ if none of the state periods $\mathrm{per}(I)$ is finite. The period of a graph, $\mathrm{per}(G)$, is the period of its adjacency matrix. It is the same as the greatest common divisor of the lengths of cycles in G. The periods of the states in an irreducible graph are equal. For a labeled graph $\mathcal{G} = (G, \mathcal{L})$, the period of $\mathcal{G}$ is defined as $\mathrm{per}(G)$.

Let $\mathcal{G}$ be a labeled graph. If p is a positive integer, a *coloring* of $\mathcal{G}$ in p colors, or a *p-coloring* for short, is a function c from $\mathcal{V}(\mathcal{G})$ to $\{0, 1, \ldots, p-1\}$ such that, whenever there is an edge from a state I to a state J, $c(J) = c(I) + 1 \bmod p$. Note that an irreducible presentation has a coloring in p colors if and only if its period is a multiple of p.

We say that a graph G is *T-partite* if the vertices of G may be divided into T disjoint subsets $D_0, D_1, \ldots, D_{T-1}$ such that any edge that begins in D_i terminates in $D_{(i+1) \bmod T}$. If G is irreducible and $\mathrm{per}(G) = T$ then G is T-partite, and the sets $D_0, D_1, \ldots, D_{T-1}$ are referred to as the *period classes* of the graph.

The *T-cascade* of a graph G is the T-partite graph with vertex set given by T copies $\mathcal{V}_0, \mathcal{V}_1, \ldots, \mathcal{V}_{T-1}$ of the vertex set $\mathcal{V}(G)$ and exactly one edge e from $I \in \mathcal{V}_i$ to $J \in \mathcal{V}_{(i+1) \bmod T}$ for each edge e from I to J in G. For a sofic shift $\mathsf{X}_{\mathcal{G}}$ presented by $\mathcal{G} = (G, \mathcal{L})$, the T-cascade of $\mathsf{X}_{\mathcal{G}}$ is the shift presented by the T-cascade of G with the natural labeling induced by $\mathcal{L}$.

If $\mathcal{G} = (G, \mathcal{L})$ is irreducible with $\mathrm{per}(G) = p$, then $\mathcal{G}^T = (G^T, \mathcal{L}^T)$ decomposes into $q = \gcd(p, T)$ irreducible components. Moreover, it is easy to verify that each component has period p/q.

C. Finite Automata

A *language* over Σ is a subset $L \subseteq \Sigma^*$. A *finite automaton* M is defined by a quadruple $M = (\mathcal{G}, \Sigma, I_0, F)$, where Σ is the *input alphabet*, $\mathcal{G} = (V, E, \mathcal{L})$ is a finite-state labeled graph, $I_0 \in V$ is the *initial* state, and $F \subseteq V$ is the set of *final* states. Elements of F are *accepting* states of the automaton; any other state is a *nonaccepting* state.

An automaton is *deterministic* if $\mathcal{G}$ is deterministic. A word w is *accepted* by automaton $M = (\mathcal{G}, \Sigma, I_0, F)$ if there exists a path π on $\mathcal{G}$ with $\mathrm{i}(\pi) = I_0$, $\mathrm{t}(\pi) \in F$, and $\mathcal{L}(\pi) = w$. The language accepted by the automaton, $L(M)$, is the set of words accepted by the automaton. A *regular* language (or set) is a language accepted by a finite automaton. In a deterministic automaton, there exists a unique path from the initial state to an accepting state that generates each $w \in L(M)$.

There is a natural correspondence between languages of sofic shifts and regular languages. The language of a sofic shift is a regular language [16], [18, eq. A.12]. However, not all regular languages are languages of sofic shifts. In particular, if $M = (\mathcal{G}, \Sigma, I_0, F)$, then $L(M)$ does not necessarily equal $\mathcal{B}(\mathsf{X}_{\mathcal{G}})$. Simple counter-examples may be constructed from graphs with initial or final states that are stranded.

III. PERIODIC-FINITE-TYPE (PFT) SHIFT SPACES

In this section, we formally introduce the class of periodic-finite-type (PFT) shift spaces and study their relationship to FT shifts and AFT shifts.

A. Periodic Forbidden Words

In Section II-A, we defined a shift space in terms of a forbidden list $\mathcal{F}$. Here, we will define a sequence space in terms of a set of periodically forbidden words. A subtlety is required in the definition to ensure shift invariance.

The notion of periodically forbidden words [14] generalizes the notion of minimal forbidden words (or minimal forbidden factors) of a bi-infinite sequence (see for instance [24]–[26]).

Let Σ be a finite alphabet. Let T be a positive integer (the *period*), and let $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_{T-1})$ be a list of T possibly empty sets of finite-length words. The list is said to be *regular* (resp. *finite*) if all its sets are regular (resp. finite) sets.

Let X_0 be the set of bi-infinite sequences x over Σ such that, for each integer i, one has

$$u \prec_i x \Rightarrow u \notin \mathcal{F}_{i \bmod T}.$$

Hence, at position i, the bi-infinite sequence x avoids the words in $\mathcal{F}_{i \bmod T}$, for all i. A word $f \in \mathcal{F}_i$ is said to have *phase* equal to i, and we sometimes denote such a word together with its phase by (f, i). The set of all bi-infinite sequences obtained by all integer shifts of the bi-infinite sequences in X_0 defines a subshift X. The list $\mathcal{F}$ is called a periodic forbidden list of the shift X for the period T. Note that the definition of X depends on the choice of the alphabet Σ.

More formally, we have the following definition.

Definition 1: Given a period T and a periodic forbidden list $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_{T-1})$, the shift $X = \mathsf{X}_{\{\mathcal{F}, T\}}^{\Sigma}$ is defined as the set of all bi-infinite sequences x over the alphabet Σ such that there exists some integer $k \in [0, T-1]$ with the property that the k-shifted sequence $\sigma^k(x)$ satisfies

$$u \prec_i \sigma^k(x) \Rightarrow u \notin \mathcal{F}_{i \bmod T}$$

for every integer i. Note that k may depend upon x.

Shift invariance of $X = \mathsf{X}_{\{\mathcal{F}, T\}}^{\Sigma}$ is an immediate consequence of the definition. Sometimes we will use the simpler notation $\mathsf{X}_{\{\mathcal{F}, T\}}$ or $\mathsf{X}_{\mathcal{F}}$ to denote the shift X when the context prevents any confusion.

Proposition 1: A shift is a sofic shift if and only if it has a regular periodic forbidden list for any period.

Proof: Let X be a sofic shift over a finite alphabet Σ. Hence $\mathcal{B}(X)$ is a regular language. For any positive integer T, the list $\mathcal{F}$ defined by $\mathcal{F}_i = \Sigma^* - \Sigma^* \mathcal{B}(X) \Sigma^*$, for any $0 \leqslant i \leqslant T-1$, is a regular periodic forbidden list of X for the period T.

Conversely, suppose $X = \mathsf{X}_{\{\mathcal{F}, T\}}$ for a period T where $\mathcal{F}_i$ is a regular language for any $0 \leqslant i \leqslant T-1$. Let $\mathcal{G}$ be a finite-state automaton accepting the regular language $W = \Sigma^* - \cup_{i=0}^{T-1} (\Sigma^T)^* \Sigma^i \mathcal{F}_i \Sigma^*$. The finite-state labeled graph obtained from this automaton by removing the nonfinal states of $\mathcal{G}$ and by keeping its essential part (i.e., the states belonging to a bi-infinite path) is a presentation of the shift X. Note that if the essential part is empty, then the shift is empty and thus sofic. ∎

It follows from the definition that the list

$$\mathcal{F}' = (\mathcal{F}_{T-1}, \mathcal{F}_0, \ldots, \mathcal{F}_{T-2})$$

formed by adding one, modulo T, to the phase of each (f, i) pair in $\mathcal{F}$, satisfies $\mathsf{X}_{\{\mathcal{F}, T\}} = \mathsf{X}_{\{\mathcal{F}', T\}}$. We refer to the periodic forbidden lists obtained by repeated application of this procedure as the *conjugates* of the list $\mathcal{F}$.
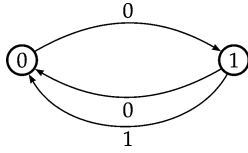
Fig. 1. The periodic-finite-type shift $\mathsf{X}_{\mathcal{F}}$ for the period 2 over $\{0,1\}$ with $\mathcal{F}_0 = \{1\}$, $\mathcal{F}_1 = \emptyset$.

B. PFT Shifts

A shift space X is *periodic-finite-type (PFT)* for a positive integer period T if it can be described as $X = \mathsf{X}_{\{\mathcal{F},T\}}^{\Sigma}$, where $\mathcal{F}$ is a *finite* periodic forbidden list $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_{T-1})$. We say that such a shift X is $\mathrm{PFT}(T)$. Note that a shift is finite-type if and only if it is $\mathrm{PFT}(1)$.

Example 1: Consider the PFT sofic shift X over the alphabet $\{0,1\}$ presented by the graph shown in Fig. 1. For $T = 2$, the shift X has the periodic forbidden list $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1)$, with $\mathcal{F}_0 = \{1\}$, $\mathcal{F}_1 = \emptyset$.

It is easy to see that, for a $\mathrm{PFT}(T)$ shift $\mathsf{X}_{\mathcal{F}}$ over the alphabet Σ, one can construct a periodic forbidden list $\mathcal{F}'$ in which all words have the same phase, the same length, or both. A common phase is obtained by taking each word $f \in \mathcal{F}_i$, prepending each of the $|\Sigma|^i$ prefixes of length i to f, and associating phase 0 with each of the resulting words. The sets corresponding to the other phases are defined to be empty sets. A common word length is achieved by replacing each f in $\mathcal{F}_i$ with the words obtained by appending each of the $|\Sigma|^{\ell-|f|}$ suffixes to f, where $\ell \geqslant \max_{f \in \mathcal{F}} |f|$, so that each word has length ℓ. Finally, a list that satisfies both properties may be constructed by applying the first transformation followed by the second.

C. PFT Sofic Shifts

The following theorem, an analog to [16, Th. 3.1.5] for shifts of finite type, establishes that PFT shift spaces are sofic shifts by explicitly constructing a presentation.

Theorem 2: Every periodic-finite-type shift space is sofic.

Proof: Let $\mathsf{X}_{\mathcal{F}}$ be a $\mathrm{PFT}(T)$ shift space. Assume, without loss of generality, that $\mathcal{F}_i = \emptyset$ for $i = 1, \ldots, T-1$, and that each word $w \in \mathcal{F}_0$ has length $|w| = \ell$.

For $\ell \geqslant 1$, let $\mathcal{U}(\ell)$ be the graph with vertex set $\mathcal{V}(\mathcal{U}(\ell)) = \Sigma^{\ell}$, the set of all ℓ-blocks of letters from Σ. For each pair of vertices $I = a_1 a_2 \ldots a_{\ell}$ and $J = b_1 b_2 \ldots b_{\ell}$ in $\mathcal{V}(\mathcal{U}(\ell))$ with $a_2 a_3 \ldots a_{\ell} = b_1 b_2 \ldots b_{\ell-1}$, draw an edge from I to J with label b_{ℓ}.

Let $\mathcal{U}(\ell, T)$ be the T-cascade of $\mathcal{U}(\ell)$ with vertex sets $\mathcal{V}_0, \mathcal{V}_1, \ldots \mathcal{V}_{T-1}$. Let $\mathcal{U}(\ell, T, \mathcal{F})$ be the graph formed from $\mathcal{U}(\ell, T)$ by deleting the edges starting and ending at each vertex $I = a_1 a_2 \ldots a_{\ell} \in \mathcal{V}_{\ell \bmod T}$ such that $I = w$ where $w \in \mathcal{F}_0$, as well as the vertex itself.

Let $\mathcal{G}$ be the essential subgraph of $\mathcal{U}(\ell, T, \mathcal{F})$. We will show that $\mathsf{X}_{\mathcal{F}} = \mathsf{X}_{\mathcal{G}}$. Choose $x = \mathcal{L}_{\infty}(\cdots e_{-1} e_0 e_1 \cdots) \in \mathsf{X}_{\mathcal{G}}$. Suppose that $\mathsf{i}(e_0) \in \mathcal{V}_k \cap \mathcal{V}(\mathcal{G})$.

Let $y = \sigma^k(x)$. Then $y_{[m,m+\ell-1]} \neq w$ for each $w \in \mathcal{F}_0$ and $m \in \mathbb{Z}$ with $m \bmod T = 0$. Therefore $y \in \mathsf{X}_{\mathcal{F}}$ and we conclude that $\mathsf{X}_{\mathcal{G}} \subseteq \mathsf{X}_{\mathcal{F}}$.

To show the reverse inclusion, choose $x \in \mathsf{X}_{\mathcal{F}}$, and let k be an integer such that $y = \sigma^k(x)$ satisfies $y_{[m,m+\ell-1]} \neq w$

for each $w \in \mathcal{F}_0$ and $m \in \mathbb{Z}$ with $m \bmod T = 0$. Since $\mathcal{U}(\ell, T)$ presents $\Sigma^{\mathbb{Z}}$, y is the label of a walk on $\mathcal{U}(\ell, T)$. Let $\xi = (\ldots e_{-1} e_0 e_1 \ldots)$ be the walk on $\mathcal{U}(\ell, T)$ such that $\mathcal{L}_{\infty}(\xi) = y$ and $\mathsf{i}(e_0) \in \mathcal{V}_0$. Suppose an edge in ξ is deleted when constructing $\mathcal{G}$ (so that $y \notin \mathsf{X}_{\mathcal{G}}$). This occurs only if $y_{[m,m+\ell-1]} = w$ for some $w \in \mathcal{F}_0$ and $m \in \mathbb{Z}$ with $m \bmod T = 0$, contradicting the properties of y. Therefore $x \in \mathsf{X}_{\mathcal{G}}$ and $\mathsf{X}_{\mathcal{F}} \subseteq \mathsf{X}_{\mathcal{G}}$. ∎

The constructive proof of Theorem 2 provides a method to obtain a presentation of a PFT shift. However, the complexity grows exponentially with the length of the longest element in $\mathcal{F}$. In Section VI, we discuss alternative algorithms for generating presentations of PFT shifts.

The construction in Theorem 2 actually implies a stronger result, namely, that any PFT shift is AFT.

Theorem 3: Irreducible PFT shifts are AFT.

Proof: Let $\mathsf{X}_{\{\mathcal{F},T\}}$ be a $\mathrm{PFT}(T)$ shift over the alphabet Σ. It is easy to see that the graph $\mathcal{G}$ constructed in Theorem 2 is deterministic. Therefore, to prove that $\mathsf{X}_{\{\mathcal{F},T\}}$ is AFT, it suffices to show that $\mathcal{G}$ has finite local memory. In fact, since $\mathcal{G} \subseteq \mathcal{U}(\ell, T)$, and the operation of passing to a subgraph preserves the property of finite local memory, it suffices to verify that $\mathcal{U}(\ell, T)$ has this property. Without loss of generality, consider a vertex $I \in \mathcal{V}_0$, with $I = (a_1 a_2 \ldots a_{\ell})$. Let $\pi = e_0 e_1 \ldots e_{\ell}$ and $\pi' = e_0' e_1' \ldots e_{\ell}'$ be two paths of length $\ell+1$ that terminate in I and generate the word $b_0 b_1 \ldots b_{\ell}$. Let $J = \mathsf{i}(e_{\ell})$ and $J' = \mathsf{i}(e_{\ell}')$. From the definition of $\mathcal{U}(\ell, T)$, it follows that $J \in \mathcal{V}_{T-1}$ and $J' \in \mathcal{V}_{T-1}$, and, moreover, both J and J' correspond to the state $b_0 b_1 \ldots b_{\ell-1} = b_0 a_1 a_2 \ldots a_{\ell-1}$. The edge from this state to state I with label a_{ℓ} is unique, implying that $e_{\ell} = e_{\ell}'$. Thus $\mathcal{U}(\ell, T)$ has finite local memory. ∎

The sliding block coding theorem [16, Th. 5.5.6] holds for AFT systems [27]. Therefore there exist sliding-block-decodable finite-state codes into irreducible PFT shifts at rational rates less than or equal to the Shannon capacity of the shift. (In Section VI, we address the computation of the capacity of PFT shifts.)

D. Proper PFT Shifts

We further distinguish a PFT shift as *proper* if it is not FT. For any proper PFT shift, there exists a word that is allowed in some, but not all, phases. Hence proper PFT shifts are $\mathrm{PFT}(p)$ only for $p > 1$. The PFT(2) shift of Example 1 is proper. Here are two further examples of proper PFT constraints that have found practical application in magnetic recording systems. Historically, these constraints provided the motivation for the definition and study of PFT shifts.

Example 2: The well-known biphase shift is a PFT(2) shift over the binary alphabet with $\mathcal{F}_0 = \{00, 11\}$ and $\mathcal{F}_1 = \emptyset$. Fig. 2 illustrates $\mathcal{U}(\ell, T, \mathcal{F})$, as described in the proof of Theorem 2, where the cyclic nature of the cascade is represented by redrawing $\mathcal{V}_1$. Deleted edges and states are drawn with dashed lines. The Shannon cover is illustrated in Fig. 3. It is easily shown and well known that the biphase shift is not FT (see, for example, [16, Th. 3.4.17], [17, p. 1657]) and hence is proper PFT. ∎
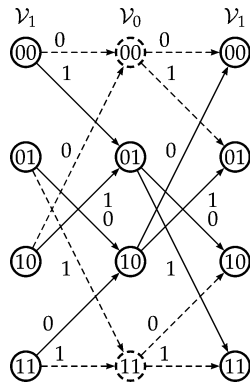
Fig. 2. $\mathcal{U}(\ell, T, \mathcal{F})$ presenting the biphase shift.
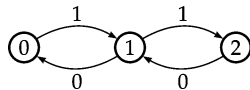

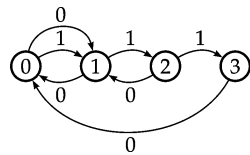Fig. 3. Shannon cover of the biphase constraint.

Fig. 4. Shannon cover of the TMTR shift.

Example 3: The time-varying maximum-transition-run (TMTR) shift [2]–[4] is a binary PFT(2) shift with $\mathcal{F}_0 = \{111\}$ and $\mathcal{F}_1 = \emptyset$. The Shannon cover is shown in Fig. 4. It is easy to verify the TMTR shift is not FT; for example, note that the Shannon cover contains the cover for the biphase shift, Fig. 3, as a subgraph. Therefore it cannot be definite, implying that the TMTR shift is a proper PFT shift. ∎

E. Periods of PFT Shifts

We now explore the periods T with which a PFT shift can be associated.

Lemma 4: If X is an irreducible PFT(T) shift, then X is PFT(nT) for any positive integer n.

Proof: If $X = \mathsf{X}_{\mathcal{F}}$ with $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_{T-1})$, then we have trivially also $X = \mathsf{X}_{\mathcal{E}}$ with $\mathcal{E} = (\mathcal{E}_i)_{0 \leqslant i \leqslant nT-1}$ and $\mathcal{E}_i = \mathcal{F}_{i \bmod T}$. ∎

Proposition 5: If X is an irreducible PFT(T) shift which has an irreducible presentation of period q, then X is $PFT(\gcd(T,q))$.

Proof: Let $X = \mathsf{X}_{\mathcal{F}}$ with $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_{T-1})$. Let $d = \gcd(T,q)$ and $k = T/d$. Let $Y = \mathsf{X}_{\mathcal{E}}$ with $\mathcal{E} = (\mathcal{E}_0, \mathcal{E}_1, \ldots, \mathcal{E}_{d-1})$ and $\mathcal{E}_i = \cup_{j=0}^{k-1} \mathcal{F}_{i+jd}$. It is straightforward to see that $Y \subseteq X$. We now show that $X \subseteq Y$ which implies $X = Y$ and the conclusion.

Let us assume that there is a bi-infinite sequence x in $X - Y$. Suppose x is periodic. Since $x \notin Y$, for each integer $0 \leqslant \ell \leqslant d-1$, there are integers $0 \leqslant i \leqslant d-1$, $0 \leqslant j \leqslant k-1$, a positive integer n, and a finite factor u of x at position $\ell+nd+i$ such that $u \in \mathcal{F}_{i+jd}$. Moreover, since x is periodic, one may assume without loss of generality that the distance between two positions $\ell + nd + i$ is greater than the maximal length of the
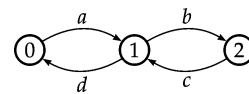

Fig. 5. The Shannon cover of the $abcd$ shift.

words in the list $\mathcal{F}$. Let π be a path labeled by x in the irreducible presentation of X of period q. Let I be the state in π at position $\ell+nd+i$. Since the presentation is irreducible and of period q, there is a finite number m of cycling paths of lengths $(\ell_i)_{1 \leqslant i \leqslant m}$ around the state I such that $\gcd((\ell_i)_{1 \leqslant i \leqslant m}) = q$.

It is well known that for any integer N greater than the Frobenius number [28] of the $(\ell_i/q)_{1 \leqslant i \leqslant m}$, N can be expressed as a nonnegative integer linear combination of the $(\ell_i/q)_{1 \leqslant i \leqslant m}$. As a consequence, there is a positive integer N such that for any nonnegative integer r, $(NT+r)q$ can be expressed as a nonnegative integer linear combination of the $(\ell_i)_{1 \leqslant i \leqslant m}$ and there is thus a cycle around I of length $NTq+rq$. Since $\gcd(T,q) = d$, there are integers a, b such that $aT = -bq + d$. One can moreover choose $b > 0$. Let M be a positive integer such that $b(j-n) + MT > 0$. We choose $r = b(j-n) + MT$. Hence there is a cycle around I of size $Z = NTq+b(j-n)q+MTq$. Its length is thus equal to $jd - nd \bmod T$.

The bi-infinite sequence labeling a path obtained from x by inserting this cycle at position $\ell + nd + i$ belongs to X. At the position $\ell+nd+i+Z$, equal to $\ell+i+jd \bmod T$, this sequence contains a factor in $\mathcal{F}_{i+jd \bmod T}$. By inserting such cycles simultaneously into x at all positions $\ell + nd + i$, we get a sequence y which belongs to X since it labels a path in $\mathcal{G}$. However, every shift by ℓ positions of the sequence y has a factor at a position equal to $i + jd \bmod T$ which belongs to $\mathcal{F}_{i+jd}$. Hence $y \notin X$, a contradiction.

This shows that any periodic $x \in X$ must in fact belong to Y. Since X is irreducible, any word in the language $\mathcal{B}(X)$ is a factor of a periodic sequence in X and, therefore, $\mathcal{B}(X) \subseteq \mathcal{B}(Y)$. Since a shift is determined by its language, we conclude that $X \subseteq Y$, as desired. ∎

Let $\mathcal{G}$ be a presentation of a PFT(T) shift $\mathsf{X}_{\mathcal{F}}$. The following proposition gives a condition that can be used to determine if $\mathsf{X}_{\mathcal{F}}$ is *not* a proper PFT shift, namely, the period of $\mathcal{G}$ and the period T associated with the forbidden list must share a nontrivial common factor if $\mathsf{X}_{\mathcal{F}}$ is proper.

Corollary 6: If $\mathcal{G}$ is an irreducible presentation of a proper PFT(T) shift $\mathsf{X}_{\mathcal{F}}$ over an alphabet Σ, then $\gcd(\mathrm{per}(\mathcal{G}), T) \neq 1$.

Proof: Indeed, if $\gcd(\mathrm{per}(\mathcal{G}), T) = 1$, then by Proposition 5, $\mathsf{X}_{\mathcal{F}}$ is PFT(1), and therefore FT. ∎

Note that the PFT shifts in Examples 2 and 3 above—the biphase and TMTR shifts—are not FT. The period associated with each of their respective forbidden lists is $T = 2$, and the graph period of each of their respective Shannon covers is also 2. Hence, $\gcd(\mathrm{per}(\mathcal{G}), 2) = 2 \neq 1$, in accordance with Corollary 6.

Example 4: The graph $\mathcal{G}$ in Fig. 5 is the Shannon cover of a shift that we will refer to as the *abcd shift*. The *abcd* shift is clearly FT, and therefore not proper PFT. Since any FT shift may be described as a PFT(T) shift for arbitrary period T by assigning all phases $0, 1, \ldots, T-1$ to each word in a finite
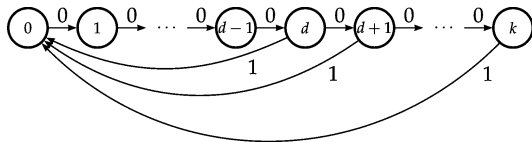
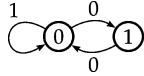Fig. 6.   Graph presenting the (d, k) shifts for $d > 0$.

Fig. 7.   Shannon cover of the even shift.

forbidden list, we may choose $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1)$ such that $\mathsf{X}_{\mathcal{G}} = \mathsf{X}_{\mathcal{F}}$ is PFT (2). Since $\mathrm{per}(\mathcal{G}) = 2$, $\gcd(\mathrm{per}(\mathcal{G}), T) = 2$. This demonstrates that the converse of Corollary 6 is not true. ∎

Example 5: Fig. 6 illustrates a graph that presents valid (d, k) sequences for $d > 0$. Aside from the trivial case where $d = k$, we find $\mathrm{per}(\mathcal{G}) = 1$; hence (d, k) shifts with $d > 0$ are not proper PFT. (A similar argument holds for the case $d = 0$.) ∎

The following example shows that not all AFT shifts are PFT shifts.

Example 6: Fig. 7 is the Shannon cover of the *even shift*, so called because its bi-infinite sequences contain only even numbers of consecutive 0's. It is easily verified that the even shift is AFT but not FT. By inspection, we see that $\mathrm{per}(\mathcal{G}) = 1$. Therefore, by Corollary 6, the even shift is not $\mathrm{PFT}(T)$ for any $T \geqslant 1$. ∎

Example 6 shows that the PFT shift spaces are a proper subset of the AFT shift spaces.

Remark: Manada and Kashyap [29], [30] have examined the relationship between the period T inherent in the definition of a PFT shift $X = \mathsf{X}_{\{\mathcal{F}, T\}}$ and properties of the shift. They also study the relationship of the smallest such period, which they call the *descriptive* period, T^X_{desc}, to the periods of periodic sequences in X and to the periods of its graphical presentations. In particular, they showed that if X is an irreducible PFT shift, and $\mathcal{G}$ is an irreducible presentation of X, then $\gcd\left(\mathrm{per}(\mathcal{G}), T^X_{desc}\right) = T^X_{desc}$.

IV. CHARACTERIZATION AND DECIDABILITY

In this section, we further characterize PFT shifts in terms of properties of their presentations. The characterizations imply the decidability of the PFT property, and they suggest a testing algorithm that is quadratic in the number of states of the Shannon cover.

A. Graphical Characterization

The following proposition proves the decidability of the PFT property for an irreducible sofic shift.

Proposition 7: Let X be an irreducible sofic shift, $\mathcal{G}$ its Shannon cover of period q, and T a positive integer. Then the following assertions are equivalent.
 1) X is $\mathrm{PFT}(T)$.
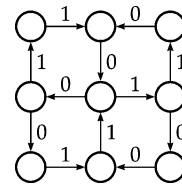 2) The irreducible components of $\mathcal{G}^{\gcd(T, q)}$ are definite graphs.

Fig. 8.   Shannon cover of interleaved-biphase shift.

Proof: Let us assume that X is $\mathrm{PFT}(T)$. Let q be the period of the Shannon cover of X and $d = \gcd(T, q)$. By Lemma 5, X is $\mathrm{PFT}(d)$. We prove that the irreducible components of $\mathcal{G}^d$ are definite. Let C be one of these components. Let us suppose that C is not definite over the alphabet Σ^d. Then C has two distinct cycles with the same label, one around a state I, another around a state J distinct from I. Hence there is in $\mathcal{G}$ a cycle around I (resp. J) labeled by a word u of length nd for some positive integer n. Since I and J belong to a common irreducible component of $\mathcal{G}^d$, there is a path labeled by z from I to J in $\mathcal{G}$ of length md for some positive integer m. Let v be a left-infinite sequence ending with a synchronizing word that focuses to I in $\mathcal{G}$. Since $\mathcal{G}$ is the Shannon cover of X, the states I and J have different follower sets. Let f_J be a right-infinite sequence generated by some path in $\mathcal{G}$ starting at J that is not the label of a path starting at I. For any nonnegative integer N, the bi-infinite sequence $x = v u^N z u^N f_J$ belongs to X. Since X is $\mathrm{PFT}(d)$, this implies that, for a large enough N, $x' = v u^N f_J$ belongs to X, which is a contradiction of the fact that f_J is not generated by a path starting at I.

Conversely, let us assume that each irreducible component C of $\mathcal{G}^d$ is a definite graph. Since $\mathcal{G}$ has period q, one can order the irreducible components of $\mathcal{G}^d$ into $(C_0, C_1, \ldots, C_{d-1})$, such that there is at least one edge from some state in C_i to some state in $C_{i+1 \bmod d}$ in $\mathcal{G}$. Each component C_i presents a shift of finite type $\mathsf{X}_{\mathcal{F}_i}$ over the alphabet $B = \Sigma^d$, where $\mathcal{F}_i$ is a finite subset of B^*. Let $\mathcal{E}_i$ be the set of words in $\mathcal{F}_i$ with symbols in the alphabet Σ. Let $Y = \mathsf{X}_{\mathcal{E}}$ with $\mathcal{E} = (\mathcal{E}_0, \mathcal{E}_1, \ldots, \mathcal{E}_{d-1})$. By construction $X = Y$. It follows that X is $\mathrm{PFT}(d)$ and also, by Lemma 4, $\mathrm{PFT}(T)$. ∎

Corollary 8: Let X be an irreducible sofic shift and p be the period of the Shannon cover $\mathcal{G}$ of X. Then the following assertions are equivalent.
 1) X is PFT.
 2) X is $\mathrm{PFT}(p)$.
 3) The irreducible components of $\mathcal{G}^p$ are definite graphs.

Proof: $(2) \Leftrightarrow (3)$ comes from Proposition 7. We prove $(1) \Rightarrow (2)$. If X is $\mathrm{PFT}(T)$ for some positive integer T, we get from Lemma 5 that X is $\mathrm{PFT}(\gcd(p, T))$. It is then also $\mathrm{PFT}(p)$ by Lemma 4. Finally $(2) \Rightarrow (1)$ follows from the definition of a PFT shift. ∎

Example 7: The Shannon cover of the *interleaved-biphase* shift is illustrated in Fig. 8. The period of the graph is 4, and one can show that the irreducible components of $\mathcal{G}^4$ are finite-type. If $\mathcal{H}$ denotes the irreducible component consisting of the central state in Fig. 8, then $\mathsf{X}_{\mathcal{H}} = \mathsf{X}_{\mathcal{F}'}$, where

$$\mathcal{F}' = \{0000, 0001, 0010, 0100, 0101, 0111,$$
$$1000, 1010, 1011, 1101, 1110, 1111\}.$$

Hence the interleaved-biphase shift is PFT(4), with $\mathcal{F}_0 = \mathcal{F}'$ and $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_3 = \emptyset$. ∎

This example suggests the following conjecture.

Conjecture 1: Let $\mathcal{G}$ be irreducible with period T. If an irreducible component $\mathcal{H}$ of $\mathcal{G}^T$ is FT with $\mathsf{X}_{\mathcal{H}}^{\Sigma^T} = \mathsf{X}_{\mathcal{F}'}^{\Sigma^T}$, then $\mathsf{X}_{\mathcal{G}} = \mathsf{X}_{\{\mathcal{F}, T\}}$ where $\mathcal{F}_0 = \mathcal{F}'$ and $\mathcal{F}_i = \emptyset$, for $i = 1, \ldots, T-1$. ∎

B. Decidability of PFT Property

We now derive from the previous propositions a quadratic-time algorithm to check whether an irreducible sofic shift presented by its Shannon cover is PFT.

Proposition 9: Let X be an irreducible sofic shift presented by its n-state Shannon cover. It is decidable in time $O(n^2 \times |\Sigma|)$ whether X is PFT.

Proof: Let $\mathcal{G}$ be the Shannon cover of X. One first computes the period p of $\mathcal{G}$. This operation can be performed with one depth-first search of the graph of $\mathcal{G}$ in time $O(n \log n \times |\Sigma|)$ (see [31] and [32]).

Since $\mathcal{G}$ has period p, one can define a coloring function c from $\mathcal{V}(G)$ to $\{0, 1, \ldots, p-1\}$ such that, whenever there is an edge from a state I to a state J, $c(J) = c(I) + 1 \bmod p$. The color of each state can be computed through a depth-first-search of the graph of $\mathcal{G}$ in time $O(n)$.

One then computes the *fiber product* graph $\mathcal{H} = \mathcal{G} * \mathcal{G}$ whose set of states is the set of pairs (I, J), where I, J are states of $\mathcal{G}$ [17]. There is an edge labeled by a from (I, J) to (I', J') if and only if there are two edges labeled by a from I to I' and from J to J'. The graph $\mathcal{H}$ is deterministic over Σ and has at most n^2 states. Then X is PFT if and only if there is no cycle in $\mathcal{H}$ going through a state (I, J) with $I \neq J$ and I, J having the same color. Indeed, the existence of such a cycle is equivalent to the existence of two identically labeled cycles in $\mathcal{G}^p$, one starting at I, the other one at J with $I \neq J$ and I, J in the same irreducible component of $\mathcal{G}^p$. The existence of such cycles can be determined in time that is linear in the size n^2 of $\mathcal{H}$, for instance by inspection of the irreducible components of $\mathcal{H}$. The final worst-case time-complexity is therefore $O(n^2 \times |\Sigma|)$. ∎

Example 8: Let us consider again the biphase shift of Example 2. The Shannon cover, shown in Fig. 9, has period 2. For any 2-coloring, the states 0 and 2 have the same color while 1 has a different color, as illustrated. The fiber product $\mathcal{H}$ is represented in Fig. 10. (States $(\mathbf{0}, \mathbf{2})$ and $(\mathbf{2}, \mathbf{0})$ are not shown, as there are no edges in $\mathcal{H}$ starting or ending in these states.) Since the cycles go only through pairs of states (I, J) with I and J of different colors or through pairs with I and J of the same color but also with $I = J$, we conclude that the biphase shift is PFT.

V. PERIODIC FIRST OFFENDERS

In this section, we define a notion of minimal periodic forbidden list of a PFT shift for a given period.

Let $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_{T-1})$ be a periodic forbidden list of a shift X for some positive period T. We say that $\mathcal{F}$ is *periodic antifactorial* if and only if for any $0 \leqslant i \leqslant T-1$ and any $j \geqslant 0$,

$$w \in \mathcal{F}_i \text{ and } u \prec_j w \text{ with } u \neq w \implies u \notin \mathcal{F}_{i+j \bmod T}.$$
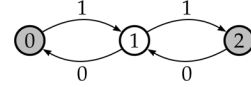

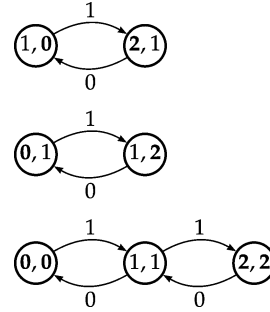Fig. 9.   A 2-coloring of the Shannon cover of the biphase shift.

Fig. 10.   Graph $\mathcal{H}$ for checking if the biphase constraint is PFT. Names of shaded states are shown in bold font. Stranded states are not shown.

The notion of periodic antifactorial list was introduced in [13]. It generalizes the notion of antifactorial language (see [25]). In particular, the sets $\mathcal{F}_i$ of a periodic antifactorial list are prefix-free and suffix-free codes.

Example 9: The list

$$\mathcal{F}_0 = \{00, 11\}$$
$$\mathcal{F}_1 = \{00, 11, 010\}$$

with $T = 2$ is periodic antifactorial, while the list

$$\mathcal{F}_0 = \{00, 11, 010\}$$
$$\mathcal{F}_1 = \{00, 10\}$$

with $T = 2$ is not periodic antifactorial. Indeed, in the latter list, $010 \in \mathcal{F}_0$, $10 \in \mathcal{F}_1$, and $10 \prec_1 010$.

For any regular periodic forbidden list $\mathcal{F}$ of a shift X, there is a regular and periodic antifactorial forbidden list $\mathcal{F}'$ of X such that $\mathcal{F}'_i \subseteq \mathcal{F}_i$ for any $0 \leqslant i \leqslant T-1$. Indeed, one can choose

$$\mathcal{F}'_i = \mathcal{F}_i - \mathcal{F}_i \Sigma^+ - (\Sigma^T)^+ \mathcal{F}_i \Sigma^*$$
$$- \bigcup_{j=1}^{T-1} (\Sigma^T)^* \Sigma^j \mathcal{F}_{i+j \bmod T} \Sigma^*.$$

Periodic antifactorial lists do not seem to satisfy any useful kind of minimality property among periodic forbidden lists of a PFT shift. We consider, instead, periodic forbidden lists based upon sets of periodic forbidden words called periodic first offenders that were introduced in [14], [15]. Their definition is intended to mimic that of the *first offenders* of a shift X [18] and to refine the notion of periodic antifactorial list. A key difference, however, is that their definition is not intrinsic; rather, it refers specifically to a presentation of the sofic shift.

We first recall the key properties of the set of first offenders. A word w is a *first offender* for a shift X if $w \notin \mathcal{B}(X)$ but every proper subword of w is in $\mathcal{B}(X)$. The collection of first offenders, $\mathcal{O}$, describes the space, $X = \mathsf{X}_{\mathcal{O}}$, and satisfies the following minimality properties [18], [16, Exercises 1.3.8, 2.1.10]:

1) if $\mathcal{F} \subseteq \mathcal{O}$ and $\mathsf{X}_{\mathcal{F}} = X$, then $\mathcal{F} = \mathcal{O}$,
2) if $\mathcal{F}$ is finite and $\mathsf{X}_{\mathcal{F}} = X$, then $\sum_{w \in \mathcal{O}} |w| \leqslant \sum_{w \in \mathcal{F}} |w|$.

Clearly, the words in $\mathcal{O}$ form an antifactorial list.

We now introduce an analogous construction for the periodic scenario. Let $\mathcal{G}$ be an irreducible presentation of period p of an irreducible sofic shift X. The states $\mathcal{V}$ of $\mathcal{G}$ are colored in p colors by a coloring function $c : \mathcal{V} \to \{0, 1, \ldots, p-1\}$. One has $c(J) = c(I) + 1 \bmod p$ whenever there is an edge from I to J. We denote by $\mathcal{V}_i$ the set of states of color i, for $0 \leqslant i \leqslant p-1$. We also say that these states are *in phase* i. We denote by $\mathcal{F}(\mathcal{G}, c)$ the list $\mathcal{F} = (\mathcal{F}_i)_{0 \leqslant i \leqslant p-1}$ where the sets $\mathcal{F}_i$ are the sets of finite words $w = w_{[0,|w|-1]}$ such that

1) $w \notin \mathsf{F}_{\mathcal{G}}(\mathcal{V}_i)$;
2) for any $0 \leqslant j < |w| - 1$, $w_{[0,j]} \in \mathsf{F}_{\mathcal{G}}(\mathcal{V}_i)$;
3) for any $0 < j \leqslant |w| - 1$, $w_{[j,|w|-1]} \in \mathsf{F}_{\mathcal{G}}(\mathcal{V}_{i+j \bmod p})$.

Note that the second condition can be replaced by $w_{[0,|w|-2]} \in \mathsf{F}_{\mathcal{G}}(\mathcal{V}_i)$, and the third one can be replaced by $w_{[1,|w|-1]} \in \mathsf{F}_{\mathcal{G}}(\mathcal{V}_{i+1 \bmod p})$. Hence, for $0 \leqslant i \leqslant p - 1$, the sets $\mathcal{F}_i$ can also be defined by

$$\mathcal{F}_i = (\Sigma^* - \mathsf{F}_{\mathcal{G}}(\mathcal{V}_i)) \cap (\mathsf{F}_{\mathcal{G}}(\mathcal{V}_i)\Sigma) \cap (\Sigma \mathsf{F}_{\mathcal{G}}(\mathcal{V}_{i+1 \bmod p})).$$

Note also that, when c is changed into another coloring of the graph in p colors, the list $\mathcal{F}(\mathcal{G}, c) = (\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_{p-1})$ is changed into one of its conjugates $(\mathcal{F}_j, \mathcal{F}_{j+1}, \ldots, \mathcal{F}_{p-1}, \mathcal{F}_0, \ldots \mathcal{F}_{j-1})$.

Proposition 10: Let $\mathcal{G}$ be an irreducible presentation with a coloring of its states c in p colors. The list $\mathcal{F}(\mathcal{G}, c)$ is a regular and antifactorial periodic forbidden list of the sofic shift presented by $\mathcal{G}$.

Proof: Let $\mathcal{F} = \mathcal{F}(\mathcal{G}, c)$ and let X be the sofic shift presented by $\mathcal{G}$. It follows from the definitions that $X \subset \mathsf{X}_{\mathcal{F}}$. Conversely, let $x \in \mathsf{X}_{\mathcal{F}}$. We will show that every subword of x is in $\mathcal{B}(X)$. Up to a power of the shift of the sequence x, for any integers i, j, we have $x_{[i,j]} \notin \mathcal{F}_{i \bmod p}$. We prove by induction on j that $x_{[i,j]} \in \mathsf{F}_{\mathcal{G}}(\mathcal{V}_i)$ and $x_{[i+1,j]} \in \mathsf{F}_{\mathcal{G}}(\mathcal{V}_{i+1})$ for any $j \geqslant i$. Since $x_{[i]} \notin \mathcal{F}_i$ and $x_{[i+1]} \notin \mathcal{F}_{i+1}$, we have $x_{[i]} \in \mathsf{F}_{\mathcal{G}}(\mathcal{V}_i)$ and $x_{[i+1]} \in \mathsf{F}_{\mathcal{G}}(\mathcal{V}_{i+1})$. By definition of $\mathcal{F}_i$, from $x_{[i,i+1]} \notin \mathcal{F}_i$, we get $x_{[i,i+1]} \in \mathsf{F}_{\mathcal{G}}(\mathcal{V}_i)$. Let us now assume that $x_{[i,j]} \in \mathsf{F}_{\mathcal{G}}(\mathcal{V}_i)$ and $x_{[i+1,j]} \in \mathsf{F}_{\mathcal{G}}(\mathcal{V}_{i+1})$. By definition of $\mathcal{F}_i$, we get $x_{[i,j+1]} \in \mathsf{F}_{\mathcal{G}}(\mathcal{V}_i)$. This implies also $x_{[i+1,j+1]} \in \mathsf{F}_{\mathcal{G}}(\mathcal{V}_{i+1})$. Thus, any subword of x belongs to $\mathcal{B}(X)$. This shows that $x \in X$. It is clear that $\mathcal{F}(\mathcal{G}, c)$ is antifactorial. ∎

We denote by $\text{size}(\mathcal{F})$ the size of a periodic forbidden list $\mathcal{F}$ for a period p. It is defined by

$$\text{size}(\mathcal{F}) = \sum_{0 \leqslant i \leqslant p-1} \sum_{w \in \mathcal{F}_i} |w|.$$

Proposition 11: Let X be an irreducible sofic shift and $\mathcal{G}$ be an irreducible presentation of X with a p-coloring c. Let $\mathcal{F}$ be any regular periodic forbidden list of X for the period p. If $\mathcal{F}$ is finite, $\mathcal{F}(\mathcal{G}, c)$ is finite and $\text{size}(\mathcal{F}(\mathcal{G}, c)) \leqslant \text{size}(\mathcal{F})$.

Let $\mathcal{G}'$ be another irreducible presentation of X with a p-coloring c' of its states. Up to a conjugacy, $\mathcal{F}(\mathcal{G}, c)$ and $\mathcal{F}(\mathcal{G}', c')$ are equal.

Proof: We first prove that, up to a conjugacy of $\mathcal{F}$, we have $\mathsf{F}_{\mathcal{G}}(\mathcal{V}_i) \cap \mathcal{F}_i = \emptyset$ for all $0 \leqslant i \leqslant p-1$. Let us assume that this is false. For any j such that $0 \leqslant j \leqslant p - 1$, there exists an integer $f(j)$, with $0 \leqslant f(j) \leqslant p - 1$, such that there is a word $w_j \in \mathsf{F}_{\mathcal{G}}(\mathcal{V}_{f(j)}) \cap \mathcal{F}_{f(j)+j \bmod p}$. That is, the word w_j is the label of a path π_j starting at some state in $\mathcal{V}_{f(j)}$ and $w_j \in \mathcal{F}_{f(j)+j \bmod p}$.

Since $\mathcal{G}$ is irreducible, there is a bi-infinite path of $\mathcal{G}$ labeled by x which contains all the paths π_j as subpaths. Moreover, since $\mathcal{G}$ has a p-coloring and $w_j \in \mathsf{F}_{\mathcal{G}}(\mathcal{V}_{f(j)})$, one can choose the path such that $w_j \prec_{g(j)} x$ and $g(j) = f(j) \bmod p$ for all integers j. Since $X = \mathsf{X}_{\mathcal{F}}$, there is an integer N such that, for any integer ℓ, $w \prec_\ell x \Rightarrow w \notin \mathcal{F}_{N+\ell \bmod p}$. By taking $\ell = g(N)$, we get that $w_N \nprec_{g(N)} x$, which is a contradiction.

Next, we change $\mathcal{F}$ into another list $\mathcal{E}$ such that each proper prefix of a word in $\mathcal{E}_i$ belongs to $\mathsf{F}_{\mathcal{G}}(\mathcal{V}_i)$. For this, one replaces each word in $\mathcal{F}_i$ by its shortest prefix which is not in $\mathsf{F}_{\mathcal{G}}(\mathcal{V}_i)$. Thus we define $\mathcal{E}$ by the formula

$$\mathcal{E}_i = (\mathsf{F}_{\mathcal{G}}(\mathcal{V}_i)\Sigma) \cap (\Sigma^* - \mathsf{F}_{\mathcal{G}}(\mathcal{V}_i)) \cap (\mathcal{F}_i(\Sigma^*)^{-1})$$

where $\mathcal{F}_i(\Sigma^*)^{-1} = \{u \mid \exists v \in \Sigma^*, uv \in \mathcal{F}_i\}$. Note that the new list $\mathcal{E}$ is still a regular periodic forbidden list of X for the period p. Indeed, it is clear that $\mathsf{X}_{\mathcal{E}} \subseteq X$. Conversely, let $x \in X$. Up to some shift, the word x is the label of a path $(I_j)_{j \in \mathbb{Z}}$ in $\mathcal{G}$ such that $I_j \in \mathcal{V}_{j \bmod p}$. Hence, for any $k \geqslant j$, we have $x_{[j,k]} \in \mathsf{F}_{\mathcal{G}}(\mathcal{V}_j)$ and thus $x_{[j,k]} \notin \mathcal{E}_j$. Thus, $X \subseteq \mathsf{X}_{\mathcal{E}}$ and $X = \mathsf{X}_{\mathcal{E}}$.

Now, we remove each word $w \in \mathcal{E}_i$ which is not in $\mathcal{F}(\mathcal{G}, c)_i$ and add at most one word shorter than w into some $\mathcal{E}_j$ as follows, in order to still have a periodic forbidden list of X. If $w \notin \mathcal{F}(\mathcal{G}, c)_i$, there are indices j, j' such that $w_{[j,j']} \in \mathcal{F}(\mathcal{G}, c)_{i+j \bmod p}$. We add $w_{[j,j']} \in \mathcal{E}_{i+j \bmod p}$ and remove w from $\mathcal{E}_i$. It is important to note that j, j' are unique in this case. Indeed, let us assume that there are two factors v_1 and v_2 of w, both shorter than w, with $v_1 = w_{[j,j']}$ in $\mathcal{F}(\mathcal{G}, c)_{i+j \bmod p}$ and $v_2 = w_{[k,k']}$ in $\mathcal{F}(\mathcal{G}, c)_{i+k \bmod p}$. Since $w_{[0,|w|-2]} \in \mathsf{F}_{\mathcal{G}}(\mathcal{V}_i)$, $j' = k' = |w| - 1$ and v_1 is a suffix of v_2, or vice-versa. This contradicts the fact that $\mathcal{F}(\mathcal{G}, c)$ is periodic antifactorial. Hence at most one word is added whenever one is removed.

The new list $\mathcal{D}$ that we get satisfies $\mathcal{D}_i \subseteq \mathcal{F}(\mathcal{G}, c)_i$. Hence $X \subseteq \mathsf{X}_{\mathcal{D}}$. Conversely, let $x \notin X = \mathsf{X}_{\mathcal{E}}$. For any integer N, there is an integer $\ell(N)$ such that there is a word w_N with $w_N \prec_{\ell(N)} x$ and $w_N \in \mathcal{E}_{N+\ell(N) \bmod p}$. If $w_N \in \mathcal{F}(\mathcal{G}, c)_{N+\ell(N) \bmod p}$, then $w_N \in \mathcal{D}_{N+\ell(N) \bmod p}$. If $w_N \notin \mathcal{F}(\mathcal{G}, c)_{N+\ell(N) \bmod p}$, then there is $u_N \prec_{r(N)} w_N$ (and hence $u_N \prec_{\ell(N)+r(N)} x$) such that $u_N \in \mathcal{D}_{N+\ell(N)+r(N) \bmod p}$. As a consequence $x \notin \mathsf{X}_{\mathcal{D}}$. Hence, $\mathsf{X}_{\mathcal{D}} \subseteq X$ and we conclude that $X = \mathsf{X}_{\mathcal{D}}$.

We now show that $\mathcal{D}_i = \mathcal{F}(\mathcal{G}, c)_i$. Assume the contrary and let w be a word in $\mathcal{F}(\mathcal{G}, c)_i - \mathcal{D}_i$. By definition of $\mathcal{F}(\mathcal{G}, c)_i$, if $w = ua = bv$ with $a, b \in \Sigma$, we have $u \in \mathsf{F}_{\mathcal{G}}(\mathcal{V}_i)$, $ua \notin \mathsf{F}_{\mathcal{G}}(\mathcal{V}_i)$, and $v \in \mathsf{F}_{\mathcal{G}}(\mathcal{V}_{i+1})$. Hence u is the label of a path in $\mathcal{G}$ starting at a state $I \in \mathcal{V}_i$ and v is the label of a path ending in a state $J \in \mathcal{V}_{i+|w| \bmod p}$. For any left-infinite word z labeling a path ending at I, and any right-infinite word y labeling a path starting at J, the word zwy is in $\mathsf{X}_{\mathcal{D}}$. It is possible to choose z and y such that $zwy \notin \mathsf{X}_{\mathcal{F}(\mathcal{G}, c)}$, which contradicts the fact $X = \mathsf{X}_{\mathcal{D}}$. Hence $\mathcal{D} = \mathcal{F}(\mathcal{G}, c)$. By construction, if $\mathcal{F}$ is finite, then $\mathcal{D}$ is also, and $\text{size}(\mathcal{D}) \leqslant \text{size}(\mathcal{E}) \leqslant \text{size}(\mathcal{F})$. Thus, $\text{size}(\mathcal{F}(\mathcal{G}, c)) \leqslant \text{size}(\mathcal{F})$.

We now prove the second statement of the proposition. We first transform $\mathcal{F}(\mathcal{G}', c')$ into $\mathcal{E}'$ as above. The size of $\mathcal{E}'$ is less than the size of $\mathcal{F}(\mathcal{G}', c')$ if $\mathcal{E}' \neq \mathcal{F}(\mathcal{G}', c')$. We then transform $\mathcal{E}'$ into $\mathcal{D} = \mathcal{F}(\mathcal{G}, c)$. Again, the size of $\mathcal{D}$ is less than the size of $\mathcal{E}'$ if $\mathcal{D} \neq \mathcal{E}'$. It follows that $\text{size}(\mathcal{F}(\mathcal{G}, c)) \leqslant \text{size}(\mathcal{F}(\mathcal{G}', c'))$
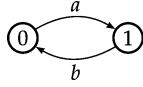
Fig. 11. A shift of finite type X over the alphabet $\Sigma = \{a, b, c, d, e\}$. We have $X = \mathsf{X}_{\mathcal{F}}$ with $\mathcal{F} = \{c, d, e, aa, bb\}$ for the period $p = 1$. We also have $X = \mathsf{X}_{\mathcal{E}}$ with $\mathcal{E}_0 = \{b, c, d, e\}$, $\mathcal{E}_1 = \{a, c, d, e\}$ for the period $p = 2$. The size of $\mathcal{F}$ is less than the size of $\mathcal{E}$ and the period of the Shannon cover of X is 2.

and the two sets are equal whenever the sizes are equal. By reversing the roles played by $\mathrm{size}(\mathcal{F}(\mathcal{G}, c))$ and $\mathrm{size}(\mathcal{F}(\mathcal{G}', c'))$, we conclude that equality holds and that the two lists are equal, up to some conjugacy. ∎

Example 10: The Shannon cover of the interleaved-biphase shift, Fig. 8, has period 4. The periodic first offenders are

$$\mathcal{F}_0 = \{000, 010, 101, 111\},$$
$$\mathcal{F}_1 = \{000, 010, 101, 111\},$$
$$\mathcal{F}_2 = \emptyset,$$
$$\mathcal{F}_3 = \emptyset.$$

∎

The following corollary, a direct consequence of Proposition 11, provides a method to determine whether a sofic shift is PFT, based on the periodic first offenders.

Corollary 12: Let X be an irreducible sofic shift with presentation $\mathcal{G}$ that admits a coloring c. Then the following assertions are equivalent.

- X is PFT.
- $\mathcal{F}(\mathcal{G}, c)$ is finite.

∎

Moreover, a list of periodic first offenders provides a minimal description of a PFT shift.

Proposition 13: Let X be an irreducible PFT shift with presentation $\mathcal{G}$ that has period p. For d that divides p, let $\mathcal{F}(\mathcal{G}, c(d))$ denote the periodic first offenders for the d-coloring of $\mathcal{G}$. Then

$$\min_{d|p} \mathrm{size}(\mathcal{F}(\mathcal{G}, c(d))) \leqslant \min_{\mathcal{F}: X = \mathsf{X}_{\mathcal{F}}} \mathrm{size}(\mathcal{F}).$$

Proof: Let $\mathcal{F}$ be a finite periodic forbidden list of an irreducible PFT shift X for a period T. By Proposition 5, one can obtain from $\mathcal{F}$ a finite periodic forbidden list $\mathcal{F}'$ of X for period $d = \gcd(p, T)$ such that the size of $\mathcal{F}'$ is less than or equal to the size of $\mathcal{F}$. By Proposition 11, $\mathrm{size}(\mathcal{F}') \geqslant \mathrm{size}(\mathcal{F}(\mathcal{G}, c))$. Hence $\mathrm{size}(\mathcal{F}) \geqslant \mathrm{size}(\mathcal{F}(\mathcal{G}, c))$, which completes the proof. ∎

It was conjectured in [14], [15] that the periodic first offenders of the Shannon cover, with a period corresponding to the period of the cover, are the minimal periodic forbidden list for *any* period. The following example shows that this is not true.

Example 11: Let X be the shift on the alphabet $\Sigma = \{a, b, c, d, e\}$ presented by the Shannon cover of Fig. 11. The shift X is FT and its minimal periodic forbidden list for the period $p = 1$, i.e., its list of first offenders, is $\mathcal{F} = \{c, d, e, aa, bb\}$. For the period $p = 2$, which is the period of the Shannon cover, the periodic first offenders are $\mathcal{F}(\mathcal{G}, c(p)) = \{\mathcal{F}_0 = \{c, d, e, b\}, \mathcal{F}_1 = \{c, d, e, a\}\}$. Hence $\mathrm{size}(\mathcal{F}(\mathcal{G}, c(p))) > \mathrm{size}(\mathcal{F})$. ∎

VI. CAPACITY OF PFT SHIFTS

The *base-2 capacity*, or simply *capacity* of a sofic shift space X over an alphabet Σ is defined as

$$C(X) = \lim_{n \to \infty} \frac{1}{n} \log_2 |\mathcal{B}_n(X)|.$$

It measures the growth rate of the number of words of length n in X. In this section, we discuss methods for computing the capacity of a PFT shift.

It is well known that the capacity of a sofic shift is the logarithm of the largest real eigenvalue of the adjacency matrix of a lossless presentation of the shift [17]. Hence it is straightforward to determine the capacity from a lossless presentation. In Section VI-A, we review techniques for generating lossless presentations of PFT shifts.

In Section VI-B, we present a combinatorial technique for computing the capacity directly from a periodic forbidden list. It extends to PFT shifts the computation of the capacity of FT shifts presented by Pimentel and Uchôa-Filho in [21], relying on the well-known *Inclusion-Exclusion Principle* from enumerative combinatorics [19], [20]. It is also known as the Goulden–Jackson Cluster Method [34]–[36, III.7.4] (see also [37]). This combinatorial method provides a much more efficient means to compute the capacity than the conventional graph-based method when the lengths of the periodically forbidden words are large compared to the number of words.

A. Constructing a Lossless Presentation

Suppose one is given a finite, antifactorial list $\mathcal{F}$ of forbidden words over an alphabet Σ. The proof of Theorem 2 provides a method to construct a presentation of the shift $\mathsf{X}_{\mathcal{F}}^{\Sigma}$ with $|\Sigma|^{\ell_{\max}-1}$ states, where $\ell_{\max}$ is the length of the longest word in $\mathcal{F}$. Of course, this construction has complexity exponential in $\mathrm{size}(\mathcal{F})$.

An alternative algorithm was described in the unpublished masters thesis of Sindhushayana [38]. The construction makes use of the close connections between symbolic dynamics and automata theory, a theme that underlies several of the other techniques we will mention. Although generally more practical than the straightforward approach, it is not computationally efficient in the sense of guaranteed time complexity polynomial in $\mathrm{size}(\mathcal{F})$. A similar construction appeared in the unpublished doctoral dissertation of McEwen [39]. In [25], Crochemore *et al.* gave an efficient, automata-theoretic construction of a deterministic presentation that requires time only linear in $\mathrm{size}(\mathcal{F})$.

These algorithms for FT shifts can be extended, often naturally, to PFT shifts. McEwen [39] includes such an extension, and [15] described a generalization of the procedure in [38]. Although neither of these run in polynomial time, for many applications they are convenient to implement and give insights into the properties of the PFT shift.

Constrained systems with unconstrained positions, introduced by Wijngaarden and Immink [10] and further studied by de Souza *et al.* [11], represent a natural example of PFT shift spaces. Given a sofic shift X, a positive integer T, and a subset $\mathcal{U}$ of integers modulo T, the authors of [11] construct a presentation of the unique maximal subsystem such that any position modulo T in $\mathcal{U}$ is unconstrained. Beginning with a finite-state
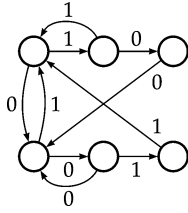
Fig. 12. Shannon cover corresponding to $\Sigma = \{0, 1\}$, $T = 2$, $\mathcal{F}_0 = \{101\}$, $\mathcal{F}_1 = \{010\}$.

presentation of the underlying shift X, their algorithm in general has exponential time and space complexity. However, for FT shifts, under a certain *gap condition* that restricts $|\mathcal{U}|$ relative to the memory of the shift, their algorithm is efficient, requiring only quadratic complexity in space and time. They also provide an efficient construction for Maximum-Transition-Run (MTR) constraints with parameter $j \geqslant 1$ [8], the systems in which the maximum allowable length of a run of consecutive 1's is j.

Béal *et al.* [13] also recognized the connection between PFT shifts and constraints with unconstrained systems. Their construction of a presentation for such a system consists of two steps. First, they derive a periodic list of forbidden words that define a maximal subsystem for T and $\mathcal{U}$, given a prefix-free list $\mathcal{F}$ of forbidden words defining the underlying FT shift. The description of $\mathcal{F}$ must be in the form of a tree-like deterministic automaton called a *trie* [13]. (A linear time and space algorithm for this step has recently been given in [33].)

In the second step, they invoke a general procedure for constructing a finite-state presentation of a PFT shift defined by a periodic forbidden list. The input to the algorithm is a collection of T tries representing the periodically forbidden words associated with the phases $0, 1, \ldots, T - 1$. They show that this step has time and space complexity that is linear in the size of the periodic forbidden list.

Example 12: Suppose we would like to determine the capacity of the PFT(2) shift space over the alphabet $\{0, 1\}$ with $\mathcal{F}_0 = \{101\}$ and $\mathcal{F}_1 = \{010\}$. Following the procedure in [13], one may construct a lossless presentation of the shift, with Shannon cover illustrated in Fig. 12. The corresponding adjacency matrix

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

has largest real eigenvalue

$$\lambda = \frac{1}{3}\left(1 + (19 - 3\sqrt{33})^{1/3} + (19 + 3\sqrt{33})^{1/3}\right)$$

and the capacity is

$$C = \log(\lambda) = 0.6093\ldots$$

B. Combinatorial Determination of Capacity

The method we describe here is a computation of the capacity directly from the periodic forbidden list. As mentioned in the Introduction, it extends to periodic-finite-type shifts the computation of the capacity of shifts of finite type presented by Pimentel and Uchôa-Filho in [21], based upon the combinatorial Inclusion-Exclusion Principle [19], [20], also known as the Goulden-Jackson Cluster Method [34, Sec. III.7.4]–[37].

Let us assume that $X = \mathsf{X}_\mathcal{F}$, where $\mathcal{F}$ is some finite antifactorial periodic forbidden list for a period T. (Note that if the given list is not antifactorial, it can be changed into one that is in linear time [13].) Denoting the cardinality of $\mathcal{B}_n(X)$ by x_n for convenience, we define the generating series counting the number of factors of X:

$$f(z) = \sum_{n \geqslant 0} x_n z^n. \tag{1}$$

It is known (see for instance [40]) that $f(z)$ is a rational series and that $C(X)$ is $\log 1/\rho$, where ρ is the radius of convergence of $f(z)$.

Recalling the definition of the set X_0 in Section III-A, we denote by $\mathcal{B}^{(i)}(X)$ (for $0 \leqslant i < T$) the set of factors u of X such that $u \prec_i x$, for some $x \in X_0$.

We set $x_n^{(i)} = |\mathcal{B}^{(i)}(X) \cap \Sigma^n|$, and define the generating series of the integers $\sum_{i=0}^{T-1} x_n^{(i)}$:

$$g(z) = \sum_{n \geqslant 0} \sum_{i=0}^{T-1} x_n^{(i)} z^n. \tag{2}$$

For an irreducible PFT shift X, it is known that

$$C(X) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{i=0}^{T-1} x_n^{(i)}. \tag{3}$$

and $C(X)$ is $\log 1/\rho$, where ρ is the radius of convergence of $g(z)$.

Let $0 \leqslant i < T$ and let $k \geqslant 0$. If $u \in \Sigma^*$, we denote by $n(u, i)$ the number of occurrences of a factor v of u such that $v \prec_{i+j} u$ and $v \in \mathcal{F}_{j \bmod T}$. We denote by $d(u, i, k)$ the number of ways to choose k indices j such that there is a factor v of u with $v \prec_{i+j} u$ and $v \in \mathcal{F}_{j \bmod T}$. Note that $d(u, i, k) = \binom{n(u,i)}{k}$. Finally we define

$$\Delta(n, i, k) = \sum_{u : |u| = n} d(u, i, k).$$

By the Inclusion-Exclusion Principle, each word u of length n contributes 0 to $\sum_{k \geqslant 0} (-1)^k \Delta(n, i, k)$ if it contains at least one word $v \prec_{i+j} u$, where $v \in \mathcal{F}_{j \bmod T}$. It contributes 1 otherwise, i.e., when it belongs to $\mathcal{B}^{(i)}(X)$. Indeed, let u be a word of length n with $n(u, i) = r$ occurrences of words v such that $v \prec_{i+j} u$ and $v \in \mathcal{F}_{j \bmod T}$. Then $\sum_{k \geqslant 0} (-1)^k d(u, i) = \sum_{k \geqslant 0} (-1)^k \binom{r}{k} = (1 - 1)^r$. It is equal to 0 if $r > 0$ and to 1 if $r = 0$.

We deduce that

$$x_n^{(i)} = \sum_{k \geqslant 0} (-1)^k \Delta(n, i, k). \tag{4}$$

We define the following bivariate generating series:

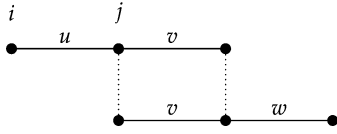$$h(z, y, i) = \sum_{u \in \Sigma^*} \sum_{k \geqslant 0} d(u, i, k) z^{|u|} y^k, \tag{5}$$

Fig. 13. The word u belongs to $(f,i) \triangleright (g,j)$ since $f = uv$, $g = vw$ and $v \prec_{j-i \bmod T} f$.

$$= \sum_{n \geqslant 0} \sum_{k \geqslant 0} \Delta(n,i,k) z^n y^k, \qquad (6)$$

$$h(z,y) = \sum_{i=0}^{T-1} h(z,y,i). \qquad (7)$$

It follows from (2), (4), and (7) that

$$g(z) = h(z,-1).$$

Example 13: We consider the PFT shift $X = X_{\mathcal{F}}$ over the alphabet $\Sigma = \{0,1\}$ for a period $T = 4$ with

$$\mathcal{F}_0 = \{111\},$$
$$\mathcal{F}_1 = \{111\},$$
$$\mathcal{F}_2 = \{1111\},$$
$$\mathcal{F}_3 = \emptyset.$$

This list of periodically forbidden words defines the TMTR(2,2,3,3) constraint. This constraint can be described as follows. The number of consecutive 1's ending at the time indices 0 mod 4 and 1 mod 4 is at most 2, while the number of consecutive 1's ending at the time indices $2 \bmod 4$ and $3 \bmod 4$ is at most 3. It is not difficult to see that this description is equivalent to saying that the block 111 is forbidden when it begins at the time indices 0 or 1 mod 4, and the block 1111 is forbidden when it begins at the time indices 2 mod 4. Hence the TMTR(2,2,3,3) constraint is described by the shift $X_{\mathcal{F}}$.

Let $u = 000011111100$. It has the word 111 of $\mathcal{F}_0$ as a factor at position 4, the word 111 of $\mathcal{F}_1$ as a factor at position 5, and the word 1111 of $\mathcal{F}_2$ as a factor at position 6. Hence it contributes 1 to $\Delta(12,0,0)$, 3 to $\Delta(12,0,1)$, $\binom{3}{2}$ to $\Delta(12,0,2)$, 1 to $\Delta(12,0,3)$, and 0 to $\Delta(12,0,k)$ for $k > 3$. Its total contribution to $\sum_{k \geqslant 0} (-1)^k \Delta(n,0,k)$ is $1 - \binom{3}{1} + \binom{3}{2} - 1 = 0$.

Now let $u = 000000000000$. It contributes 1 to the sum $\sum_{k \geqslant 0} (-1)^k \Delta(n,0,k)$ since it contributes 1 to $\Delta(12,0,0)$ and 0 to $\Delta(12,0,k)$ for $k > 0$.

We now describe how to compute the bivariate series $h(z,y)$. Let $\mathcal{F} = (\mathcal{F}_0, \ldots, \mathcal{F}_{T-1})$ be a finite periodic forbidden list. If $\mathcal{F}_i$ is a nonempty set, we define the set $\tilde{F}_i = \{(f,i) \mid f \in \mathcal{F}_i\}$. If $\mathcal{F}_i$ is the empty set we denote by $\tilde{F}_i$ the singleton containing the integer i. We denote by $\tilde{F}$ the union of the $\tilde{F}_i$. Note that the size of $\tilde{F}$ is at most size$(\mathcal{F}) + T - 1$.

Let (f,i) and (g,j) be in $\tilde{F}$. We denote by $(f,i) \triangleright (g,j)$ the set of nonempty words u such that there are nonempty words v, w with $f = uv$, $g = vw$ and $v \prec_{j-i \bmod T} f$ (see Fig. 13). We define a square matrix $G(z)$ with entries indexed by $\tilde{F} \times \tilde{F}$ as follows. For any (f,i), (g,j), k, r in $\tilde{F}$,

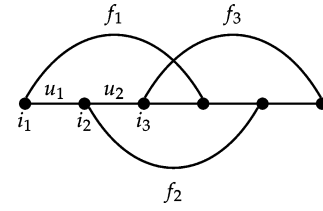$$G(z)_{(f,i)(g,j)} = \sum_{u \in (f,i) \triangleright (g,j)} z^{|u|},$$

Fig. 14. An example of a 3-overlapping decomposition of the word $u = u_1 u_2 f_3$.

$$G(z)_{k(f,i)} = G(z)_{(f,i)k} = G(z)_{kr} = 0.$$

Example 13 (continued): The matrix $G(z)$ for the periodic forbidden list $\mathcal{F}$ of period 4 of Example 13 is the following $|\tilde{F}| \times |\tilde{F}|$ matrix with $\tilde{F} = \{(111,0), (111,1), (1111,2), 3\}$.

$$G(z) = \begin{bmatrix} 0 & z & z^2 & 0 \\ 0 & 0 & z & 0 \\ z^2 & z^3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let $(f_1, i_1), (f_2, i_2), \ldots, (f_k, i_k) \in \tilde{F}$. We denote by $(f_1, i_1) \triangleright (f_2, i_2) \triangleright \cdots \triangleright (f_k, i_k)$ the set of $(k-1)$-tuples of nonempty words $(u_1, \ldots, u_{k-1})$ such that, for $1 \leqslant r \leqslant k-1$, $u_r \in (f_r, i_r) \triangleright (f_{r+1}, i_{r+1})$ (see Fig. 14). The $(k-1)$-tuple $(u_1, \ldots, u_{k-1})$ is called a *k-overlapping decomposition* of the word $u = u_1 \ldots u_{k-1} f_k$. Note that $(f_1, i_1) \triangleright (f_2, i_2) \triangleright \cdots \triangleright (f_k, i_k)$ is obtained by concatenating all words u_1 in $(f_1, i_1) \triangleright (f_2, i_2)$ to all $(k-2)$-tuples in $(f_2, i_2) \triangleright \cdots \triangleright (f_k, i_k)$.

For any word u, we denote by $v(u,k,i,j)$ the number of $(k+1)$-overlapping decompositions of u in all $(f_1, i_1) \triangleright (f_2, i_2) \triangleright \cdots \triangleright (f_{k+1}, i_{k+1})$ such that $i_1 = i$ and $i_{k+1} + |f_{k+1}| = j$.

For $0 \leqslant i, j < T$, we define the bivariate series

$$V_{ij}(z,y) = \sum_{n \geqslant 0} \sum_{k \geqslant 0} v(n,k,i,j) z^n y^{k+1},$$

where $v(n,k,i,j) = \sum_{u:|u|=n} v(u,k,i,j)$.

We define the $T \times T$-matrix $V(z,y)$

$$V(z,y) = (V_{ij}(z,y))_{0 \leqslant i,j < T}.$$

We then define the $|\tilde{F}| \times T$ matrix $\Phi(z)$ as follows: For any (f,i), k in $\tilde{F}$, $0 \leqslant j < T$,

$$\Phi(z)_{(f,i)j} = \begin{cases} z^{|f|} & \text{if } j = i + |f| \bmod T, \\ 0 & \text{otherwise,} \end{cases}$$
$$\Phi(z)_{kj} = 0.$$

Example 13 (continued): The matrix $\Phi(z)$ for the periodic forbidden list $\mathcal{F}$ for period 4 in Example 13 is an $|\tilde{F}| \times T$ matrix with $\tilde{F} = \{(111,0), (111,1), (1111,2), 3\}$.

$$\Phi(z) = \begin{bmatrix} 0 & 0 & 0 & z^3 \\ z^3 & 0 & 0 & 0 \\ 0 & 0 & z^4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We define a $T \times |\tilde{F}|$-matrix $\Psi(z)$ as follows. For any (f,i), k in $\tilde{F}$ and $0 \leqslant j < T$,

$$\Psi(z)_{j(f,i)} = \begin{cases} 1, & \text{if } j = i, \\ 0, & \text{otherwise,} \end{cases}$$

$$\Psi(z)_{jk} = \begin{cases} 1, & \text{if } k = j, \\ 0, & \text{otherwise.} \end{cases}$$

Example 13 (continued): The matrix $\Psi(z)$ for the periodic forbidden list $\mathcal{F}$ for period 4 in Example 13 is an $|\tilde{F}| \times T$ matrix with $\tilde{F} = \{(111, 0), (111, 1), (1111, 2), 3\}$.

$$\Psi(z) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Note that in this example $\Phi(z)$ and $\Psi(z)$ are square matrices since $|\tilde{F}| = T$.

Therefore, for $0 \leqslant i, j < T$, we get

$$V_{ij}(z, y) = \sum_{n \geqslant 0} \sum_{k \geqslant 0} v(n, k, i, j) z^n y^k y,$$
$$= \sum_{k \geqslant 0} \left(\sum_{n \geqslant 0} v(n, k, i, j) z^n \right) y^k y,$$
$$= \sum_{k \geqslant 0} \mathbf{1_i}^T \Psi(z) G(z)^k \Phi(z) \mathbf{1_j} y^k y$$

where $\mathbf{1_i}$ is the column characteristic vector of i.

Hence

$$V(z, y) = \Psi(z) \sum_{k \geqslant 0} (G(z)^k y^k) \Phi(z) y,$$
$$= \Psi(z)(\mathbf{I} - G(z)y)^{-1} \Phi(z) y.$$

where $\mathbf{I}$ is the $|\tilde{F}| \times |\tilde{F}|$ identity matrix.

Finally, we define a $T \times T$ square matrix $P(z)$. For any $0 \leqslant i, j < T$,

$$P(z)_{ij} = \begin{cases} |\Sigma| z, & \text{if } j = i + 1 \bmod T, \\ 0, & \text{otherwise.} \end{cases}$$

Example 13 (continued): The matrix $P(z)$ for the periodic forbidden list $\mathcal{F}$ of period 4 of Example 13 is an $|\tilde{F}| \times |\tilde{F}|$ matrix with $\tilde{F} = \{(111, 0), (111, 1), (1111, 2), 3\}$.

$$P(z) = \begin{bmatrix} 0 & 2z & 0 & 0 \\ 0 & 0 & 2z & 0 \\ 0 & 0 & 0 & 2z \\ 2z & 0 & 0 & 0 \end{bmatrix}.$$

Let $\mathcal{P} = (\{0, 1, \dots, T - 1\}, E)$ be an automaton, i.e, a labeled graph, whose edges are labeled in $\mathbb{N}[\Sigma^*, y]$ and defined as follows. There is in $\mathcal{P}$ an edge labeled by ay^0 for each letter a of the alphabet Σ from the state i to the state $i + 1 \bmod T$. There is an edge labeled by $\sum_k v(u, k, i, j) u y^{k+1}$ from the state i to the state j for each word u. The cover $\mathcal{P}$ is pictured in Fig. 15 for $\Sigma = \{0, 1\}$.

We claim that, for any $0 \leqslant i < T$, the bivariate series $h(z, y, i) = \sum_{u \in \Sigma^*} \sum_{k \geqslant 0} d(u, i, k) z^{|u|} y^k$, seen as a series in z, enumerates the labels of paths in $\mathcal{P}$ starting at the state i. Indeed, let u be a word and let us choose $r \geqslant 0$ indices j such that there is a factor f_j of u with $f_j \prec_{i+j} u$ and $f_j \in \mathcal{F}_j \bmod T$. The sequence (f_j) can be decomposed into ℓ sequences s_m of $k_1, \dots, k_\ell$ elements such that $k_1 + \dots + k_\ell = r$ and, if $s_m = (f_{i_1}, \dots f_{i_{k_m}})$, $u[i_1, i_1 + 1], u[i_1 + 1, i_1 + 2], \dots, u[i_{k_m} - 2, i_{k_m} - 1]$ is an overlapping decomposition in $(f_{i_1} \triangleright \dots \triangleright f_{i_{k_m}})$.
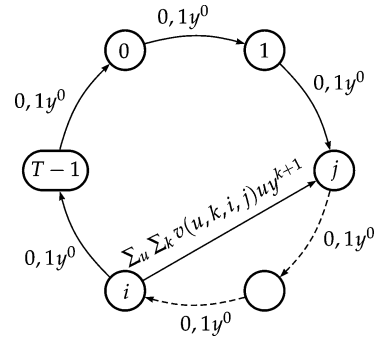

Fig. 15. The automaton $\mathcal{P}$ for the period T.

Note that overlapping occurrences of periodic forbidden words always correspond to a decomposition in some $(f_{i_1} \triangleright \dots \triangleright f_{i_{k_m}})$ since $\mathcal{F}$ is antifactorial.

Thus the paths of $\mathcal{P}$ starting at the state i and labeled by uy^r count all such choices of the indices j. Since $d(u, i, r)$ is the number of choices of r indices j such that there is a factor f of u with $v \prec i + ju$ and $v \in \mathcal{F}_j \bmod T$, the series $h(z, y, i)$ enumerates the labels of paths in $\mathcal{P}$ starting at the state i. Hence the bivariate series $h(z, y)$ enumerates the labels of all paths in $\mathcal{P}$.

As a consequence,

$$h(z, y) = \mathbf{1}^T \sum_{r \geqslant 0} (P(z) + V(z, y))^r \mathbf{1}$$
$$= \mathbf{1}^T (1 - P(z) - \Psi(z)(\mathbf{I} - G(z)y)^{-1} \Phi(z) y)^{-1} \mathbf{1}.$$

We get

$$g(z) = \mathbf{1}^T (1 - P(z) + \Psi(z)(\mathbf{I} + G(z))^{-1} \Phi(z))^{-1} \mathbf{1}. \quad (8)$$

As a consequence, $C(X)$ is $\log 1/\rho$, where ρ is the positive root of minimum modulus of

$$\det(\mathbf{I} - P(z) + \Psi(z)(\mathbf{I} + G(z))^{-1} \Phi(z)). \quad (9)$$

Example 13 (continued): For the periodic forbidden list $\mathcal{F}$ of period 4 of Example 13, the series $g(z)$ is[1]

$$g(z) = \frac{1}{4z^8 - 13z^4 + 1} \left(8z + 16z^2 + 30z^3 + 4z^4 \right.$$
$$\left. + 2z^5 - 8z^6 - 12z^7 + z^8 - 2z^9 + z^{10} + 4 \right).$$

The capacity of X is $\log 1/\rho$, where ρ is the positive root of minimum modulus of

$$13z^4 - 4z^8 - 1 = (3z^2 + 2z^4 - 1)(3z^2 - 2z^4 + 1).$$

We get $\rho = \frac{\sqrt{\sqrt{17} - 3}}{2}$ and $\lambda = 1/\rho = 1.887207676$.

Example 14: We consider the PFT shift $X = \mathsf{X}_{\mathcal{F}}$ over the alphabet $\Sigma = \{0, 1\}$ for a period $T = 2$ with

$$\mathcal{F}_0 = \{111\},$$
$$\mathcal{F}_1 = \emptyset.$$

[1] Obtained with a MuPAD computation.

The $|\tilde{F}| \times |\tilde{F}|$ matrices $G(z)$, $\Phi(z)$ and $P(z)$, with $\tilde{F} = \{(111, 0), 1\}$ are

$$G(z) = \begin{bmatrix} z^2 & 0 \\ 0 & 0 \end{bmatrix}, \quad \Phi(z) = \begin{bmatrix} 0 & z^3 \\ 0 & 0 \end{bmatrix}, \quad P(z) = \begin{bmatrix} 0 & 2z \\ 2z & 0 \end{bmatrix}.$$

The series $g(z)$ is

$$g(z) = \frac{-4z - 2z^2 - 3z^3 - 2}{3z^2 + 2z^4 - 1}.$$

The capacity of X is $\log 1/\rho$, where ρ is the positive root of minimum modulus of $3z^2 + 2z^4 - 1$ We get $\rho = \frac{\sqrt{\sqrt{17}-3}}{2}$. This PFT shift has the same capacity as the PFT shift of Example 13. See [9] for a classification of the capacities of the $\mathrm{TMTR}(\mathbf{m})$ constraints where $\mathbf{m}$ is a positive integral vector with length up to four.

Example 15: We consider the PFT shift $X = \mathsf{X}_{\mathcal{F}}$ over the alphabet $\Sigma = \{0, 1\}$ for a period $T = 2$ with

$$\mathcal{F}_0 = \{101\},$$
$$\mathcal{F}_1 = \{010\}.$$

The $|\tilde{F}| \times |\tilde{F}|$ matrices $G(z)$, $\Phi(z)$ and $P(z)$, with $\tilde{F} = \{(101, 0), (010, 1)\}$ are

$$G(z) = \begin{bmatrix} z^2 & z \\ z & z^2 \end{bmatrix}, \quad \Phi(z) = \begin{bmatrix} 0 & z^3 \\ z^3 & 0 \end{bmatrix}, \quad P(z) = \begin{bmatrix} 0 & 2z \\ 2z & 0 \end{bmatrix}.$$

The series $g(z)$ is

$$g(z) = \frac{-2z - 2z^2 - 2}{z + z^2 + z^3 - 1}.$$

The capacity of X is $\log 1/\rho$, where ρ is the positive root of minimum modulus of $z + z^2 + z^3 - 1$. This time-varying constraint has a capacity approximatively 0.8791464216. This capacity is equal to the capacity of the MTR(2) constraint (see [41] for the relationship between these two constraints).

VII. Conclusions

We have introduced the class of periodic-finite-type (PFT) shift spaces. This class of sofic shifts lie between the class of finite-type shifts and almost-finite-type shifts. We proved several properties of graph presentations of these spaces. For a given PFT space, we identified a particular list of periodically forbidden words, the periodic first offenders, that enjoy certain minimality properties with respect to other forbidden lists defining the space. Finally, we consider the calculation of the capacity of a PFT shift. We present a straightforward algorithm to construct a graph presenting a PFT space that can be used to determine the capacity of the constraints. We also present a quite different method which relies upon techniques from enumerative combinatorics and that appears to be very effective when the size of the periodic forbidden blocks is large compared to the number of blocks in the list.

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