Periodic-finite-type shift spaces

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Abstract

We introduce the class of periodic-finite-type (PFT) shift spaces. They are the subclass of shift spaces defined by a finite set of periodically forbidden words. Examples of PFT shifts arise naturally in the context of distance-enhancing codes for partial-response channels. We show that the class of PFT shifts represent a proper superset of the finite-type shift spaces and a proper subset of almost-finite-type shift spaces. We prove several properties of labeled graphs that present PFT shifts. For a given PFT shift space, we identify a finite set of forbidden words – referred to as "periodic first offenders" – that define the shift space and that satisfy certain minimality properties. Finally, we present an efficient algorithm for constructing labeled graphs that present PFT shift spaces.

1 Introduction

Magnetic recording systems often make use of binary codes that disallow the appearance of certain sequences that are problematic in the data recording or retrieval process. In systems using partial-response equalization and sequence detection, so-called "distance-enhancing" constrained codes have been proposed to increase the minimum distance at the output of the underlying intersymbol-interference channel by forbidding a finite set of binary patterns, e.g., [1, 2, 3]. The set of allowable code sequences are generated by paths in a labeled, directed graph. Such sets of constrained sequences are referred to as

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sofic shift spaces in the symbolic dynamics literature and form a subset of the class of shift spaces [4], often referred to simply as shifts.

Recently, several distance-enhancing constrained codes have been introduced that forbid the appearance of certain patterns in a periodic manner. For example, the time-varying maximum-transition-run (TMTR) constraint forbids the string 111 from beginning at odd indices of a sequence, e.g., [5, 6, 7]. In this paper, we examine properties of sets of sequences that satisfy such time-varying constraints, and establish their relationship to, and position within, the more familiar class of shift-invariant code constraints.

In Section 2, we review basic concepts and terminology from the theory of constrained systems, and we formally define the class of periodic-finite-type (PFT) sequence spaces. In Section 3, we show that PFT sequence spaces are sofic shift spaces, and we prove several properties of labeled graphs that present them. In particular, we demonstrate that finite-type shift spaces are a proper subclass of PFT shift spaces, which, in turn, are a proper subclass of almost-finite-type shifts. In Section 4, for a given PFT shift space, we identify a finite set of forbidden words – referred to as "periodic first offenders" – that define the shift space and that satisfy certain minimality properties. Finally, in Section 5, we present an efficient algorithm for constructing labeled graphs that present a PFT shift space specified by a given list of forbidden words. The complexity of the method grows linearly with the length of the longest word and the number of words in the list. Section 6 concludes the paper.

2 Periodic-Finite-Type Sequence Spaces

2.1 Background and notation

Let $\mathcal{A}^{\mathbb{Z}}$ denote the set of bi-infinite sequences

$$x = \dots x_{-3}x_{-2}x_{-1}x_0x_1x_2\dots$$

whose symbols are drawn from a finite alphabet $\mathcal{A}$,

$$\mathcal{A}^{\mathbb{Z}} \stackrel{\text{def}}{=} \{x | x_i \in \mathcal{A}, \ \ \forall i \in \mathbb{Z}\}.$$

A finite block of consecutive symbols in such a sequence x will be referred to as a *word*. To specify the position of a word within a sequence x, we use the notation

$$x_{[i,j]} \stackrel{\text{def}}{=} x_i x_{i+1} \cdots x_j,$$

where $i \leq j$. (When the context is clear, we will also sometimes use x to denote a word.)

We define $\mathcal{A}^*$ to be the collection of all words over $\mathcal{A}$, including the empty For $x \in \mathcal{A}^*$, we denote by $|x|$ the *length* of x, i.e., the number of x *contains* w, if there exist indices $i \leq j$ such that $w = x_{[i,j]}$. The *shift map* σ takes a sequence x to the sequence $y = \sigma(x)$ with ith coordinate $y_i = x_{i+1}$. The inverse of the shift map takes a sequence y to $x = \sigma^{-1}(y)$ with ith coordinate $x_i = y_{i-1}$.

Let $\mathcal{F}$ be a collection of words over $\mathcal{A}$ and let $\mathsf{X}_{\mathcal{F}}^{\mathcal{A}}$ denote the subset of $\mathcal{A}^{\mathbb{Z}}$ consisting of all bi-infinite sequences that do not contain a word from $\mathcal{F}$. In this context $\mathcal{F}$ is referred to as a *forbidden list*. A *shift space* is a set $X = \mathsf{X}_{\mathcal{F}}^{\mathcal{A}}$. This terminology reflects the fact that X is invariant under the operation of the shift map, i.e., $\sigma(X) = X$. A shift space is a *shift of finite type* if there exists a finite set $\mathcal{F}$ such that $X = \mathsf{X}_{\mathcal{F}}^{\mathcal{A}}$.

It will be useful to have a concept for the collection of words contained in a shift space. Let $\mathcal{B}_n(X)$ denote the set of all length-n words that occur in elements of a shift space X. The *language* of X is the collection

$$\mathcal{B}(X) \stackrel{\text{def}}{=} \bigcup_{n=0}^{\infty} \mathcal{B}_n(X),$$

where $\mathcal{B}_0(X) = \epsilon$, the empty word. The language of a shift space completely determines the space [4, Proposition 1.3.4], i.e., two shift spaces are equal if and only if they have the same language.

Consider $\mathcal{B}_N(X)$ as an alphabet and define the *Nth higher power code* $\gamma_N : X \to (\mathcal{B}_N(X))^{\mathbb{Z}}$ by

$$(\gamma_N(x))_{[i]} = x_{[iN, iN+N-1]}.$$

The image of X under γ_N, $X^N \stackrel{\text{def}}{=} \gamma_N(X)$, is referred to as the Nth *higher power shift* of X.

Let X be a shift space over $\mathcal{A}$, and let $\Psi : \mathcal{B}_{m+a+1}(X) \to \Gamma$ be a map from allowed $(m+a+1)$-blocks in X to symbols in an alphabet Γ. The *sliding block code* with *memory*

m and *anticipation* a *induced by* Ψ is the map $\psi : X \to \Gamma^{\mathbb{Z}}$ defined by

$$y = \psi(x),$$

where, for $x \in X$,

$$y_i = \Psi(x_{[i-m,i+a]}).$$

A sliding block code $\psi : X \to Y$ is a *conjugacy* from X to Y if it is invertible. In such a case, the shifts X and Y are said to be conjugate.

2.2 Periodic-Finite-Type Sequence Spaces

We now introduce the notion of a PFT sequence space. Let $\mathcal{F}$ be a finite collection of words over a finite alphabet $\mathcal{A}$ where each $w_j \in \mathcal{F}$ is associated with a non-negative integer index n_j. We write $\mathcal{F} = \left\{w_1^{(n_1)}, w_2^{(n_2)}, \ldots, w_{|\mathcal{F}|}^{(n_{|\mathcal{F}|})}\right\}$ and associate with the indexed list $\mathcal{F}$ a *period* T, where T is a positive integer satisfying

$$T \geq \max\{n_1, n_2, \ldots, n_{|\mathcal{F}|}\} + 1.$$

We will refer to an indexed list with its period as the pair $\{\mathcal{F}, T\}$.

Definition 1 For a pair $\{\mathcal{F}, T\}$ and alphabet $\mathcal{A}$, define the sequence space $\mathsf{X}^{\mathcal{A}}_{\{\mathcal{F},T\}} \stackrel{\text{def}}{=} \{x | x \in \mathcal{A}^{\mathbb{Z}}$ and $\exists k \in [0, T-1]$ such that for each $w_j^{(n_j)} \in \mathcal{F}$ and for all $m \in \mathbb{Z},$ if $m \bmod T = n_j$ then $\sigma^k(x)_{[m,m+|w_j|-1]} \neq w_j\}$ where, for $m < 0$,

$$m \bmod T \stackrel{\text{def}}{=} \begin{cases} 0 & \text{, if } |m| \bmod T = 0 \\ T - |m| \bmod T & \text{, otherwise.} \end{cases}$$

For $\{\mathcal{F}, T\}$ finite, we say $\mathsf{X}^{\mathcal{A}}_{\{\mathcal{F},T\}}$ is a *periodic-finite-type* sequence space. □

We will often simply write $\mathsf{X}_{\{\mathcal{F},T\}}$ in place of $\mathsf{X}^{\mathcal{A}}_{\{\mathcal{F},T\}}$ when the particular alphabet is clear or not relevant. $\mathsf{X}_{\{\mathcal{F},T\}}$ denotes the set of bi-infinite sequences which can be shifted such that the shifted sequence does not contain a word $w_j^{(n_j)} \in \mathcal{F}$ starting at any index m with $m \bmod T = n_j$. Note that $\mathsf{X}_{\{\mathcal{F},T\}}$ is shift-invariant, i.e., $\sigma(\mathsf{X}_{\{\mathcal{F},T\}}) = \mathsf{X}_{\{\mathcal{F},T\}}$.

A sequence space $\mathsf{X}_{\{\mathcal{F},T\}}$ can be described by many different pairs of indexed lists and periods. For example, if one forms a list $\mathcal{F}'$ from $\mathcal{F}$ by adding one, modulo T, to the index of each word in $\mathcal{F}$ then $\mathsf{X}_{\{\mathcal{F},T\}} = \mathsf{X}_{\{\mathcal{F}',T\}}$. We can also construct a pair $\{\mathcal{F}', T\}$ such that $\mathsf{X}_{\{\mathcal{F},T\}} = \mathsf{X}_{\{\mathcal{F}',T\}}$ and all words in $\mathcal{F}'$ have the same length, or the same index, or both. The first may be accomplished by replacing each w in $\mathcal{F}$ with the words obtained by appending all $|\mathcal{A}|^{l-|w|}$ suffixes to w, where $l > \max_{w\in\mathcal{F}} |w|$, so that each word in $\mathcal{F}'$ has length l. The second may be accomplished by replacing each $w^{(n)}$ in $\mathcal{F}$ for $n \neq 0$ with the words obtained by prepending all $|\mathcal{A}|^n$ prefixes of length n to w and associating index 0 with each resulting word. A list that satisfies both properties can be constructed by applying the second transformation followed by the first.

3 Graph Presentations of Periodic-Finite-Type Spaces

In this section, we show that periodic-finite-type (PFT) sequence spaces are sofic shifts and discuss some properties of the graphs that present them. We begin by introducing some relevant definitions and notation from symbolic dynamics and the theory of constrained systems. A more thorough discussion of these topics may be found in [4].

3.1 Background on Labeled Graphs and Sofic Systems

A *graph* $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{L})$ consists of a finite set $\mathcal{V} = \mathcal{V}(\mathcal{G})$ of vertices (or states), a set $\mathcal{E} = \mathcal{E}(\mathcal{G})$ of edges connecting the states, and a *labeling* $\mathcal{L} : \mathcal{E} \to \mathcal{A}$, that assigns a label to each edge. Each edge e will be directed, with an initial state, $\mathsf{i}(e) \in \mathcal{V}$, and terminal state, $\mathsf{t}(e) \in \mathcal{V}$. A *path* in the graph denotes a finite block of edges $\pi = e_1 e_2 \cdots e_N$ such that $\mathsf{t}(e_j) = \mathsf{i}(e_{j+1})$ for $j = 1, \ldots, N-1$. For a path $\pi = e_1 e_2 \cdots e_N$, we say the initial state $\mathsf{i}(\pi) = \mathsf{i}(e_1)$, and the terminal state $\mathsf{t}(\pi) = \mathsf{t}(e_N)$. A path π is a *cycle* if $\mathsf{i}(\pi) = \mathsf{t}(\pi)$. The label of π is the word $\mathcal{L}(\pi) = \mathcal{L}(e_1)\mathcal{L}(e_2)\ldots\mathcal{L}(e_N)$.

A *walk* on $\mathcal{G}$ is a bi-infinite sequence of edges $\xi = \cdots e_{-1} e_0 e_1 \cdots$ such that $\mathsf{t}(e_j) = \mathsf{i}(e_{j+1})$ for all j. The label of a walk ξ is the sequence

$$\mathcal{L}_\infty(\xi) \stackrel{\text{def}}{=} \cdots \mathcal{L}(e_{-1})\mathcal{L}(e_0)\mathcal{L}(e_1)\cdots.$$

A graph $\mathcal{G}$ is *irreducible* if for any pair of states $I, J \in \mathcal{V}$ there exists a path π in $\mathcal{G}$ with $\mathsf{i}(\pi) = I$ and $\mathsf{t}(\pi) = J$. An *irreducible component* of a graph $\mathcal{G}$ is a maximal (with respect to inclusion of vertices) irreducible subgraph of $\mathcal{G}$.

A vertex $I \in \mathcal{V}$ is *stranded* if either no edges start at I or no edges terminate at I. A graph is *essential* if no vertex is stranded.

A graph has *local anticipation* a if a is the smallest nonnegative integer such that, for each $i \in \mathcal{V}(\mathcal{G})$, all paths of length $a+1$ that start at i and have the same label start with the same edge. Similarly, a graph has *local memory* m if m is the smallest nonnegative integer such that, for each $i \in \mathcal{V}(\mathcal{G})$, all paths of length $m+1$ that end at i and have the same label end with the same edge. $\mathcal{G}$ is *deterministic* if it has local anticipation 0, i.e., if edges with the same initial state have distinct labels.

A *sofic shift* $\mathsf{X}_{\mathcal{G}}$ is the set of sequences obtained by reading the labels of walks on $\mathcal{G}$,

$$\mathsf{X}_{\mathcal{G}} \stackrel{\text{def}}{=} \{x \,|\mathcal{L}_\infty(\xi) = x \text{ for some } \xi \text{ a walk on } \mathcal{G}\}.$$

Any graph $\mathcal{G}$ has a unique essential subgraph $\mathcal{H}$ such that $\mathsf{X}_{\mathcal{G}} = \mathsf{X}_{\mathcal{H}}$ [4, Proposition 2.2.10].

We say $\mathcal{G}$ is a *presentation* of $\mathsf{X}_{\mathcal{G}}$, or $\mathcal{G}$ *presents* $\mathsf{X}_{\mathcal{G}}$. Every sofic shift has a deterministic presentation [4, Theorem 3.3.2] and we say a sofic shift is irreducible if it has an irreducible presentation. We say a sofic shift is *almost-finite-type* if it has a presentation with finite local anticipation and finite local memory. Since all sofic shifts have a deterministic presentation, a sofic shift is almost-finite-type if and only if it has a deterministic presentation with finite local memory.

All sofic shifts are shift spaces [4, Theorem 3.1.4]. Hence, for every $\mathsf{X}_{\mathcal{G}}$ there exists a set of words $\mathcal{F}$ such that $\mathsf{X}_{\mathcal{G}} = \mathsf{X}_{\mathcal{F}}$.

Although there are many graphs which present the same sofic shift, there is a unique, up to graph isomorphism, deterministic graph presenting an irreducible sofic shift with the minimal number of states [4, Theorem 3.3.18]. This graph is referred to as the *Shannon cover* of the shift. Given a graph presenting an irreducible sofic shift, one can obtain the Shannon cover via determinizing and state-minimizing algorithms, e.g., [4, pp. 92], [8, pp. 68].

The *follower set* $\mathsf{F}_{\mathcal{G}}(I)$ of state I in $\mathcal{G}$ is the collection of labels of paths starting at I,

$$\mathsf{F}_{\mathcal{G}}(I) \stackrel{\text{def}}{=} \{\mathcal{L}(\pi)|\mathcal{L}(\pi) \in \mathcal{B}(\mathsf{X}_{\mathcal{G}}) \text{ and } \mathsf{i}(\pi) = I\}.$$

Note that

$$\bigcup_{I \in \mathcal{V}(\mathcal{G})} \mathsf{F}_{\mathcal{G}}(I) = \mathcal{B}(\mathsf{X}_{\mathcal{G}}).$$

The Nth *higher power graph* $\mathcal{G}^N$ of $\mathcal{G}$ is the graph with vertex set $\mathcal{V}(\mathcal{G}^N) = \mathcal{V}(\mathcal{G})$, and one edge in $\mathcal{G}^N$ from I to J with label $\mathcal{L}(\pi)$ for each path π of length N from I to J in $\mathcal{G}$. The Nth higher power graph presents the Nth higher power shift, $\mathsf{X}_{\mathcal{G}^N} = (\mathsf{X}_{\mathcal{G}})^N$.

$\mathcal{G}$ is T*-partite* if the vertices of $\mathcal{G}$ may be divided into T disjoint subsets $D_0, D_1, \ldots, D_{T-1}$ such that any edge that begins in D_i terminates in $D_{(i+1) \bmod T}$. For $I, J \in \mathcal{V}(\mathcal{G})$, let A_{IJ} denote the number of edges from I to J in $\mathcal{G}$. The *adjacency matrix* of $\mathcal{G}$ is the $|\mathcal{V}(\mathcal{G})| \times |\mathcal{V}(\mathcal{G})|$ matrix $A_{\mathcal{G}} = [A_{IJ}]$. The *period of state* I, denoted per(I), is the greatest common divisor of those integers n for which $(A^n_{\mathcal{G}})_{II} > 0$. The *period of a matrix* A, denoted per(A), is the greatest common divisor of the periods of the states in A. The *period of a graph* $\mathcal{G}$, denoted per$(\mathcal{G})$, is the period of its adjacency matrix; i.e., $\text{per}(\mathcal{G}) \stackrel{\text{def}}{=} \text{per}(A_{\mathcal{G}})$. For irreducible $\mathcal{G}$, the periods of the states are the same [4, Lemma 4.5.3], hence if $\text{per}(A_{\mathcal{G}}) = T$ then $\mathcal{G}$ is T-partite, and the sets $D_0, D_1, \ldots, D_{T-1}$ are referred to as the T *period-classes* of the graph.

The T*-cascade* of a graph $\mathcal{G}$ is the T-partite graph with vertex set given by T copies $\mathcal{V}_0, \mathcal{V}_1, \ldots, \mathcal{V}_{T-1}$ of the vertex set $\mathcal{V}(\mathcal{G})$ and exactly one edge e from $I \in \mathcal{V}_i$ to $J \in \mathcal{V}_{(i+1) \bmod T}$ for each edge e from I to J in $\mathcal{G}$.

Finally, we note that if $\mathcal{G}$ is irreducible with $\text{per}(A_{\mathcal{G}}) = p$, and T is a positive integer, then $\mathcal{G}^T$ decomposes into q irreducible components $\mathcal{H}_0, \mathcal{H}_1, \ldots, \mathcal{H}_{q-1}$, where $q = \gcd(p, T)$ [9, Theorem 3.5]. Moreover, it is easy to verify that each component $\mathsf{X}_{\mathcal{H}_i}$ has period p/q, and all of the components are conjugate.

3.2 Graph Presentations of Periodic-Finite-Type Sequence Spaces

The following theorem, an analogue to [4, Theorem 3.1.5] for shifts of finite type, establishes that PFT sequence spaces are sofic shifts.

Theorem 1 *Every periodic-finite-type sequence space is a sofic shift space.*

Proof: Let $\mathsf{X}_{\{\mathcal{F},T\}}$ be a PFT sequence space. Assume, without loss of generality, that all elements of $\mathcal{F}$ have length l and index 0. Construct a labeled graph $\mathcal{G}$ as follows.

Let $\mathcal{H}$ be the graph with vertex set $\mathcal{V}(\mathcal{H}) = \mathcal{A}^l$, the set of all l-blocks of letters from $\mathcal{A}$. For any two states $I = a_1a_2 \ldots a_l$ and $J = b_1b_2 \ldots b_l$ in $\mathcal{V}(\mathcal{H})$, if $a_2a_3 \ldots a_l = b_1b_2 \ldots b_{l-1}$ then there is an edge in $\mathcal{E}(\mathcal{H})$ from I to J with label b_l.

Let $\mathcal{G}'$ be the T-cascade of $\mathcal{H}$ with vertex sets $\mathcal{V}_0, \mathcal{V}_1, \ldots \mathcal{V}_{T-1}$. Form the graph $\mathcal{G}$ from $\mathcal{G}'$ by deleting each vertex, as well as the edges starting and terminating at the vertex, $I = a_1 a_2 \ldots a_l \in \mathcal{V}_{l \bmod T}$ such that $I = w$ for some $w \in \mathcal{F}$. We will show that $\mathsf{X}_{\{\mathcal{F},T\}} = \mathsf{X}_{\mathcal{G}}$.

Choose $x = \cdots x_{-1} x_0 x_1 \cdots \in \mathsf{X}_{\mathcal{G}}$. Let $\mathcal{V}_k \subseteq \mathcal{V}(\mathcal{G})$ be the set of vertices such that $\mathsf{i}(\mathcal{L}^{-1}(x_0)) \in \mathcal{V}_k$. Then $\forall m \in \mathbb{Z}$ and each $w \in \mathcal{F}$, if $m \mod T = 0$ then $\sigma^k(x)_{[m,m+l-1]} \neq w$. Therefore $x \in \mathsf{X}_{\{\mathcal{F},T\}}$ and $\mathsf{X}_{\mathcal{G}} \subseteq \mathsf{X}_{\{\mathcal{F},T\}}$.

To show the reverse inclusion, choose $x \in \mathsf{X}_{\{\mathcal{F},T\}}$, and let k be an integer such that $\forall m \in \mathbb{Z}$ and each $w \in \mathcal{F}$, if $m \mod T = 0$ then $\sigma^k(x)_{[m,m+l-1]} \neq w$. Since $\mathcal{G}'$ presents $\mathcal{A}^{\mathbb{Z}}$, $\sigma^k(x)$ is the label of a walk on $\mathcal{G}'$. Let ξ be the walk on $\mathcal{G}'$ such that $\mathcal{L}_\infty(\xi) = \sigma^k(x)$ and $\mathsf{i}(\xi_o) \in \mathcal{V}_0$. Suppose an edge in ξ is deleted when constructing $\mathcal{G}$. Then $\sigma^k(x)_{[m,m+l-1]} = w$ for some $w \in \mathcal{F}$ and $m \in \mathbb{Z}$ with $m \bmod T = 0$, a contradiction. Therefore $x \in \mathsf{X}_{\mathcal{G}}$ and $\mathsf{X}_{\{\mathcal{F},T\}} \subseteq \mathsf{X}_{\mathcal{G}}$. □

The constructive proof of Theorem 1 provides a straightforward method to obtain a graph presenting a PFT shift. The drawback of using this method in practice is the size of the initial representation, which grows exponentially with the length of the longest element in $\mathcal{F}$. In Section 5 we discuss an algorithm whose complexity grows linearly in $\mathcal{F}$.

In view of Theorem 1, we make the following definitions.

Definition 2 A shift space X is a shift of *periodic-finite-type* if there exists a pair $\{\mathcal{F}, T\}$ with $|\mathcal{F}|$ and T finite such that $X = \mathsf{X}_{\{\mathcal{F},T\}}$.

Definition 3 X is a *proper* periodic-finite-type shift space if X is periodic-finite-type but there is no pair $\{\mathcal{F}, T\}$ with $|\mathcal{F}|$ finite and $T = 1$ such that $X = \mathsf{X}_{\{\mathcal{F},T\}}$.

The proper PFT shifts are those that have a non-trivial description as a finite indexed forbidden list. For any proper PFT shift there exists a word that is allowed in some, but not all, positions.

Example 1 The well-known "biphase shift" is a PFT shift with $\mathcal{F} = \{00^{(0)}, 11^{(0)}\}$, and $T = 2$. Let $\mathcal{G}$ be the corresponding graph constructed by following the steps described in the proof of Theorem 1. Fig. 1 illustrates $\mathcal{G}$, where the cascade is represented by re-drawing $\mathcal{V}_1$ and deletions are shown as dashed lines. The Shannon cover, illustrated in Fig. 2, may be obtained by applying a state-minimization algorithm to Fig. 1. One can verify that $\mathsf{X}_{\mathcal{G}}$ is not finite-type [4, Theorem 3.4.17]; i.e., there is no finite list $\mathcal{F}$ such that $\mathsf{X}_{\mathcal{G}} = \mathsf{X}_{\mathcal{F}}$. Therefore, the biphase shift is a proper PFT shift space $\square$

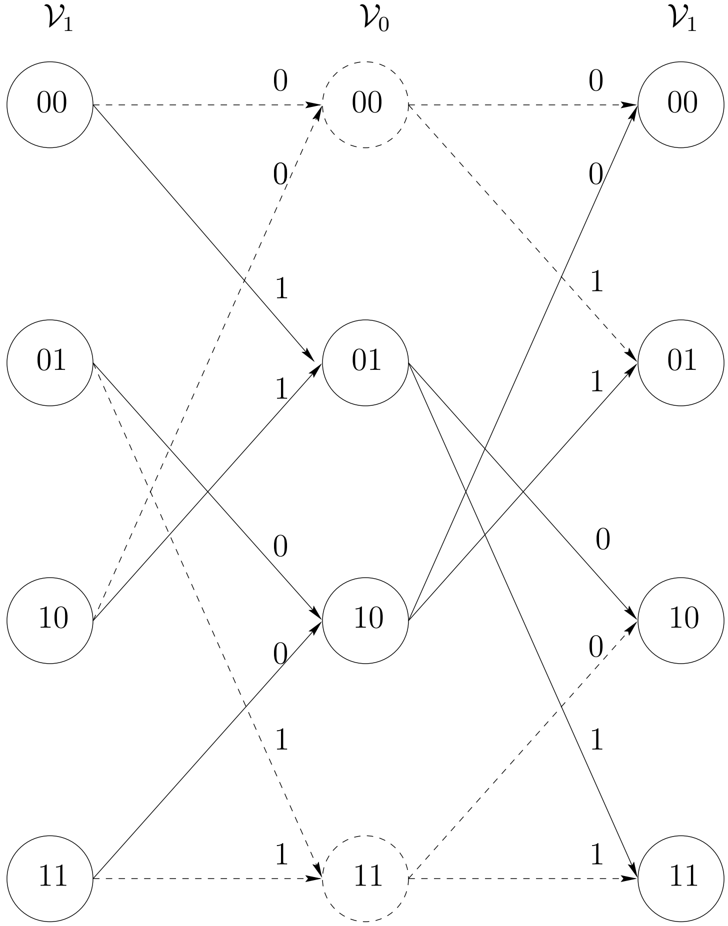

Figure 1: $\mathcal{G}$ presenting $\mathsf{X}_{\{\{00^{(0)},11^{(0)}\},2\}}$.

How can one, in general, determine if a graph presents a PFT shift? The following propositions address this question. Proposition 1 gives a necessary condition for an irreducible sofic shift to be a proper PFT shift. Proposition 2 gives a sufficient condition for an irreducible sofic shift to be PFT, and a method to determine a corresponding indexed list and period. Proposition 3 is a converse to Proposition 2.

Proposition 1 *Let $\mathcal{G}$ be an irreducible presentation of $\mathsf{X}_{\mathcal{G}}$. If $\mathsf{X}_{\mathcal{G}}$ is a proper periodic-finite-type shift with $\mathsf{X}_{\mathcal{G}} = \mathsf{X}_{\{\mathcal{F},T\}}$ then gcd(per($A_{\mathcal{G}}$), T) $\neq 1$.*

Proof: Let $\mathcal{G}$ be an irreducible presentation of a proper PFT shift. Choose $\{\mathcal{F}, T\}$ with $T > 1$ and $\mathcal{F}$ finite such that $\mathsf{X}_{\mathcal{G}} = \mathsf{X}_{\{\mathcal{F},T\}}$. Suppose that $\gcd(\text{per}(A_{\mathcal{G}}), T) = 1$. Choose a word $w^{(n)} \in \mathcal{F}$ and state $I \in \mathcal{V}(\mathcal{G})$ such that $w \in \mathsf{F}_{\mathcal{G}}(I)$. From irreducibility of $\mathcal{G}$, we can choose a word v such that $\mathcal{L}^{-1}(wv)$ is a cycle. Choose a cycle π with $\mathsf{i}(\pi) = \mathsf{t}(\pi) = I$

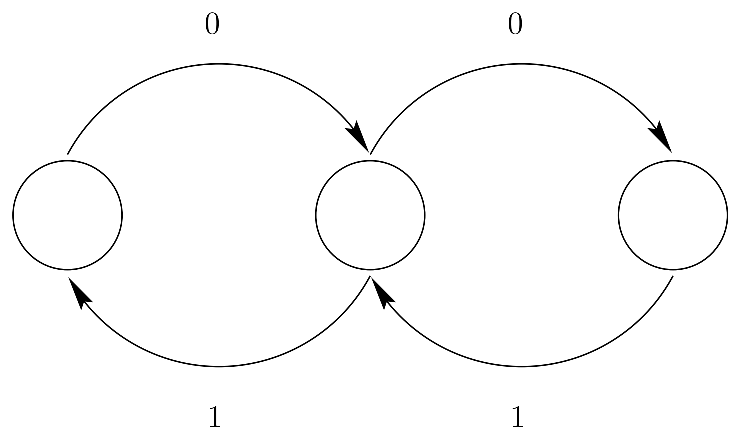

Figure 2: Shannon cover of $\mathsf{X}_{\{\{00^{(0)},11^{(0)}\},2\}}$.

such that T and $l = |\pi|$ have no common divisors greater than 1. Put $u = \mathcal{L}(\pi)$. One can choose $q_0, q_1, \ldots, q_{T-1}$ such that

$$x = \cdots w^{(n)} v u^{q_0} w^{(n)} v u^{q_1} \cdots w^{(n)} v u^{q_{T-1}} \cdots$$

is the label of a path on $\mathcal{G}$ and w appears in x at all indices $0, \ldots, T-1$, mod T. Therefore, $x \notin \mathsf{X}_{\{\mathcal{F},T\}}$, contradicting the assumption that $\mathsf{X}_{\mathcal{G}} = \mathsf{X}_{\{\mathcal{F},T\}}$. We conclude that $\gcd(\text{per}(A_{\mathcal{G}}), T) \neq 1$. □

Example 2 Magnetic recording channels often use a code to control the maximum and minimum spacing between transitions in the two-level recorded waveform. The corresponding code constraints are referred to as (d, k) run-length-limited (RLL) constraints, where d and k denote the minimum and maximum number of 0's between 1's in the allowed set of binary sequences. In this context, referred to as NRZI notation, a 0 denotes no transition in the recorded signal and a 1 denotes a transition. Hence d constrains the minimum spacing between transitions and k constrains the maximum spacing between transitions. Fig. 3 illustrates a graph which presents the shift space containing all valid (d, k) sequences. Aside from the trivial case where $d = k$, we find $\text{per}(A_{\mathcal{G}}) = 1$; hence (d, k) shifts are not proper PFT. □

Example 3 The graph $\mathcal{G}$ in Fig. 4 is the Shannon cover of a finite-type constraint that we will refer to as the *abcd shift*. Clearly $\text{per}(A_{\mathcal{G}}) = 2$, but the shift is not proper PFT.□

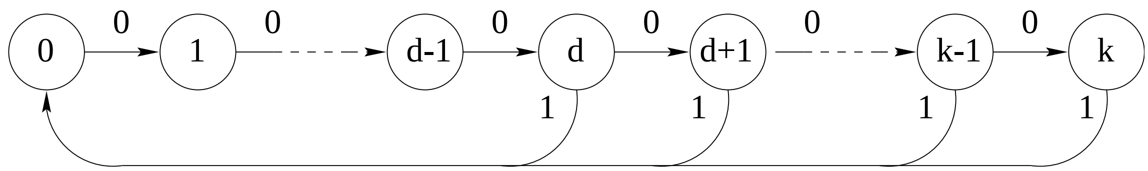

Figure 3: Graph presenting the (d, k) shifts.

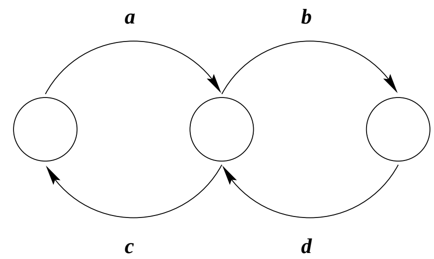

Figure 4: Graph presenting the *abcd* shift.

Example 4 The *time-varying maximum-transition-run (TMTR) shift* [5, 6, 7] is a PFT shift with $\mathcal{F} = \{111^{(0)}\}$, and $T = 2$. The Shannon cover $\mathcal{G}$ is shown in Fig. 5. Note that $\text{per}(A_{\mathcal{G}}) = 2$. It is easy to verify that the TMTR shift is not finite-type; for example, note that $\mathcal{G}$ contains the Shannon cover of the biphase shift, Fig. 2, as a subgraph. Hence, the TMTR shift is proper PFT. $\square$

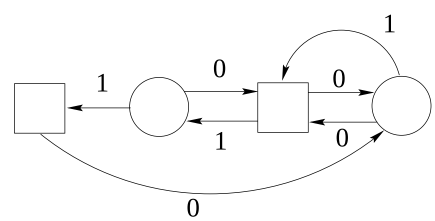

Figure 5: Shannon cover of the TMTR shift.

Example 5 The *even shift* is presented by the irreducible graph in Fig. 6. It is well-known and easily verified that this shift is not finite-type. By inspection, we see that $\text{per}(A_{\mathcal{G}}) = 1$. Therefore the even shift is not proper PFT. $\square$

Let ϕ denote the mapping that assigns indices to words, $\phi(w, k) \stackrel{\text{def}}{=} w^{(k)}$, and $\phi^{-1}(w^{(k)}) \stackrel{\text{def}}{=} w$. We use the notation $\phi(\mathcal{F}, k)$ to denote the indexed list of words obtained by assigning the index k to each of the words in $\mathcal{F}$.

Proposition 2 *Let $\mathcal{G}$ be an irreducible presentation of $\mathsf{X}_{\mathcal{G}}$. If there exists an integer $T \geq 1$ such that an irreducible component of $\mathcal{G}^T$ is finite-type over $\mathcal{A}^T$ then $\mathsf{X}_{\mathcal{G}}$ is periodic-finite-type. Furthermore, if $\mathcal{H}$ is an irreducible component of $\mathcal{G}^T$ and $\mathcal{F}' \subseteq (\mathcal{A}^T)^*$ is a*

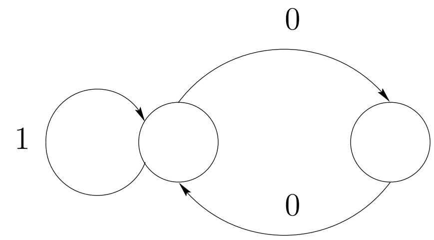

Figure 6: A presentation of the even shift.

finite list such that $\mathsf{X}_{\mathcal{F}'}^{\mathcal{A}^T} = \mathsf{X}_{\mathcal{H}}$*, then* $\mathsf{X}_{\mathcal{G}} = \mathsf{X}_{\{\mathcal{F},T\}}$ *where* $\mathcal{F} = \phi(\mathcal{F}', 0)$.

Proof: Let $\mathcal{G}$ be an irreducible graph, and $\mathcal{H}$ an irreducible component of $\mathcal{G}^T$ such that $\mathsf{X}_{\mathcal{H}} = \mathsf{X}_{\mathcal{F}'}^{\mathcal{A}^T}$. Assume, without loss of generality, that all words in $\mathcal{F}'$ have length l. Let $\mathcal{F} = \phi(\mathcal{F}', 0)$. We will show that $\mathsf{X}_{\{\mathcal{F},T\}} = \mathsf{X}_{\mathcal{G}}$.

Choose $x \in \mathsf{X}_{\mathcal{G}}$ and an integer k such that $y = \gamma_T(\sigma^k(x))$ is presented by a walk on $\mathcal{H}$. The sequence y does not contain any $w \in \mathcal{F}'$. Hence $\forall m \in \mathbb{Z}$ and each $w \in \mathcal{F}$, if m mod $T = 0$ then $\sigma^k(x)_{[m,m+l-1]} \neq w$. Therefore $x \in \mathsf{X}_{\{\mathcal{F},T\}}$ and $\mathsf{X}_{\mathcal{G}} \subseteq \mathsf{X}_{\{\mathcal{F},T\}}$.

For the reverse inclusion, choose $x \in {\mathsf{X}_{\mathcal{G}}}^c$. For each integer k, put

$$y(k) = \gamma_T(\sigma^k(x)).$$

Suppose that, for some k, $y(k)$ is presented by a walk on $\mathcal{H}$. Then $y(k) \in \mathsf{X}_{\mathcal{H}} \subseteq \mathsf{X}_{\mathcal{G}^T}$, and $\sigma^k(x) = \gamma_T^{-1}(y(k)) \in \mathsf{X}_{\mathcal{G}}$, a contradiction. Hence, for all k, $y(k) \in \mathsf{X}_{\mathcal{H}}^c$, and each $y(k)$ contains a word in $\mathcal{F}'$. Therefore, for all k, there exists $m \in \mathbb{Z}$ and $w \in \mathcal{F}$ such that m mod $T = 0$ and $\sigma^k(x)_{[m,m+l-1]} = w$. Hence $x \in \mathsf{X}_{\mathcal{F}}^c$ and ${\mathsf{X}_{\mathcal{G}}}^c \subseteq {\mathsf{X}_{\{\mathcal{F},T\}}}^c$, implying $\mathsf{X}_{\{\mathcal{F},T\}} \subseteq \mathsf{X}_{\mathcal{G}}$. □

Example 6 Let $\mathcal{G}$ be the Shannon cover of the *interleaved-biphase* shift illustrated in Fig. 7. $\mathcal{G}$ has four period classes, emphasized in the figure by use of different symbols for the corresponding vertices. One can show that $T = 4$ is the smallest integer for which an irreducible component of $\mathcal{G}^T$ is finite-type over $\mathcal{A}^T$. Letting $\mathcal{H}$ denote the irreducible component of $\mathcal{G}^4$ consisting of the central state in Fig. 7, we have $\mathsf{X}_{\mathcal{H}} = \mathsf{X}_{\mathcal{F}'}$, where

$$\begin{aligned} \mathcal{F}' = \ & \{0000, 0001, 0010, 0100, 0101, 0111, \\ & 1000, 1010, 1011, 1101, 1110, 1111\}. \end{aligned}$$

Hence, by Proposition 2, the interleaved-biphase shift is PFT, i.e., $\mathsf{X}_{\mathcal{G}} = \mathsf{X}_{\{\mathcal{F},T\}}$, where

$$\begin{aligned}\mathcal{F} = \; & \{0000^{(0)}, 0001^{(0)}, 0010^{(0)}, 0100^{(0)}, 0101^{(0)}, 0111^{(0)}, \\ & 1000^{(0)}, 1010^{(0)}, 1011^{(0)}, 1101^{(0)}, 1110^{(0)}, 1111^{(0)}\},\end{aligned}$$

and $T = 4$, which may be verified via the constructive proof of Theorem 1. □

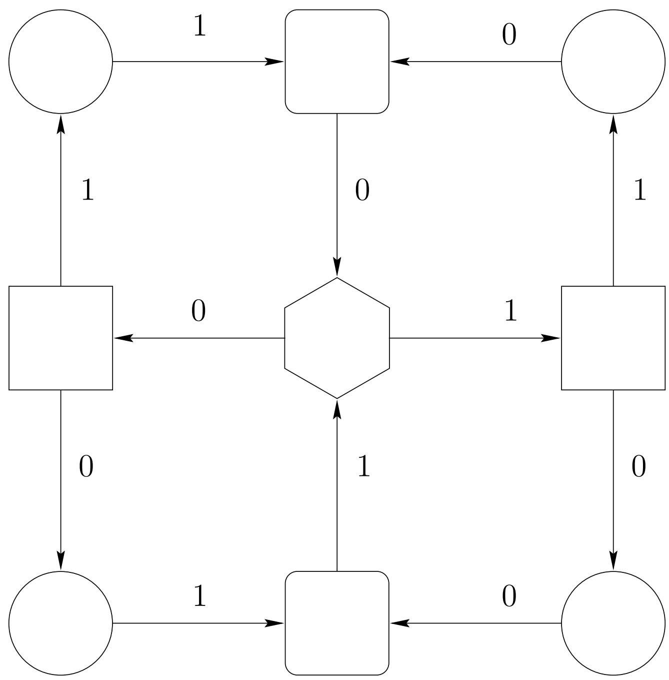

Figure 7: Shannon cover of interleaved-biphase shift.

Proposition 3 *Let $\mathsf{X}_{\{\mathcal{F},T\}}$ be periodic-finite-type. Then there exists a presentation $\mathcal{G}$ of $\mathsf{X}_{\{\mathcal{F},T\}}$ such that a component of $\mathsf{X}_{\mathcal{G}^T}$ is finite-type over $\mathcal{A}^T$.*

Proof: Assume, without loss of generality, all elements of $\mathcal{F}$ have length kT, for some integer k, and index 0. Let $\mathcal{G}$ be the T-partite graph constructed following the proof of Theorem 1 with vertex sets $\mathcal{V}_0, \ldots, \mathcal{V}_{T-1}$. Let $\mathcal{H}$ be the component of $\mathcal{G}^T$ with vertex set $\mathcal{V}_0$. We claim that

$$\mathsf{X}_{\mathcal{H}} = \mathsf{X}_{\mathcal{F}'}^{\mathcal{A}^T}$$

where $\mathcal{F}' = \phi^{-1}(\mathcal{F})$ is the list obtained by removing the indices of words in $\mathcal{F}$.

Indeed, the vertices of $\mathcal{H}$ are the allowed kT-blocks in $\mathsf{X}_{\mathcal{F}'}^{\mathcal{A}^T}$, and there is an edge e from $a_1 a_2 \ldots a_k$ to $b_1 b_2 \ldots b_k$ if and only if $a_2 a_3 \ldots a_k = b_1 b_2 \ldots b_{k-1}$ and $a_1 a_2 \ldots a_k b_k$ is in $\mathcal{B}(\mathsf{X}_{\mathcal{F}'}^{\mathcal{A}^T})$, where $a_i, b_i \in \mathcal{A}^T$. Hence $\mathcal{H}$ is a presentation of $\mathsf{X}_{\mathcal{F}'}^{\mathcal{A}^T}$, e.g., [4, Theorem 3.1.5]. □

The following lemma provides a sufficient condition for a sofic shift to be almost-finite-type.

Lemma 1 *Let $\mathcal{G}$ be an irreducible, deterministic presentation of a shift X. If there exists an integer T such that an irreducible component of $\mathcal{G}^T$ presents a finite-type shift over $\mathcal{A}^T$, then X is almost-finite-type.*

Proof: Suppose the shift space presented by an irreducible component of $\mathcal{G}^T$ is finite-type over $\mathcal{A}^T$. The spaces presented by the components of $\mathcal{G}^T$ are conjugate, hence they are all finite-type [4, Theorem 2.1.10]. Choose $i \in \mathcal{V}(\mathcal{G})$ and let $\mathcal{H}$ be the component of $\mathcal{G}^T$ with $i \in \mathcal{V}(\mathcal{H})$. Since $\mathcal{H}$ is deterministic and finite-type, there exists an integer m such that for any sequence $x = x_{-m}, \cdots, x_0 \in (\mathcal{A}^T)^{m+1}$ the set of paths $e = e_{-m}, \cdots, e_0$ on $\mathcal{H}$ that generate x all agree in edge e_0. Now let $\gamma = \gamma_{-(m+1)T+1}, \cdots, \gamma_0$ and $\gamma' = \gamma'_{-(m+1)T+1}, \cdots, \gamma'_0$ be paths of length $(m+1)T$ on $\mathcal{G}$ that terminate at i and generate the same sequence. Then $\gamma_0 = \gamma'_0$, hence $\mathcal{G}$ has finite local memory and X is almost-finite-type. □

We will make use of the following lemma, which is a restatement of [9, Lemma 2.8].

Lemma 2 *Let X be an irreducible sofic shift, and let $\mathcal{G}$ be such that $X \subseteq \mathsf{X}_{\mathcal{G}}$. Then there exists an irreducible component $\mathcal{G}'$ of $\mathcal{G}$ such that $X \subseteq \mathsf{X}_{\mathcal{G}'}$.* □

The following Theorem is an immediate consequence of Proposition 3 and Lemmas 1 and 2.

Theorem 2 *The irreducible periodic-finite-type shift spaces are almost-finite-type.* □

Remark The biphase shift and interleaved-biphase shift of Examples 1 and 6, respectively, are both proper PFT. The theorem confirms the well-known fact that both shifts are almost-finite-type. On the other hand, the even shift of Example 5 is almost-finite-type, but it is not PFT.

It is known that the sliding block coding theorem [4, Theorem 5.5.6] holds for almost-finite-type systems [10]. Therefore there exist sliding-block-decodable finite-state codes into irreducible PFT shifts at rational rates less than or equal to the Shannon capacity of the shift.

Fig. 8 illustrates how PFT shift spaces fit into the family of subclasses of shift spaces.

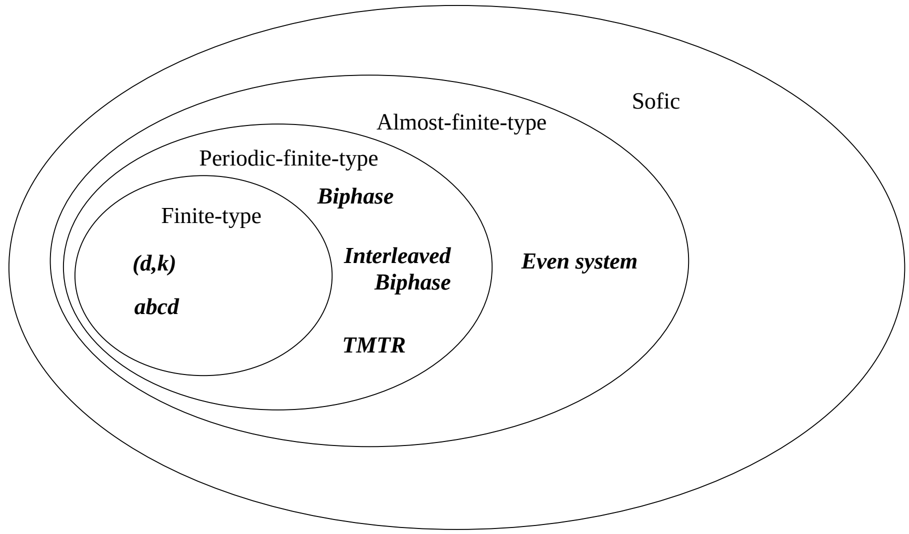

Figure 8: Relations between shift spaces.

4 Periodic First Offenders

A shift space may be represented by various forbidden lists. For example, the space $\mathsf{X}^{\{0,1\}}_{\{11111,0111\}}$ is equivalent to $\mathsf{X}^{\{0,1\}}_{\{111\}}$, since 11111 and 0111 are forbidden to occur in a sequence if and only if 111 is forbidden to occur. It is useful to have a unique, minimal forbidden list that describes a shift space. Such a minimal description is well known for general shift spaces, e.g., [11],[4, Exercises 1.3.8,2.1.20].

Result 1 For any shift space X, there exists a unique forbidden list $\mathcal{O}$ that is minimal relative to any other list $\mathcal{F}$ in the sense that

$$\begin{aligned}
&1)\text{ if } \mathcal{F} \subseteq \mathcal{O} \text{ and } X = \mathsf{X}_{\mathcal{F}} = \mathsf{X}_{\mathcal{O}},\text{ then } \mathcal{F} = \mathcal{O},\\
&2)\text{ if } \mathcal{F} \text{ is finite and } X = \mathsf{X}_{\mathcal{F}} = \mathsf{X}_{\mathcal{O}}, \text{then} \sum_{w\in\mathcal{O}} |w| \le \sum_{w\in\mathcal{F}} |w|.
\end{aligned}$$

The set satisfying these properties is referred to as the list of *first offenders*. The set is composed of all words w such that w does not appear in any word of X, but every sub-word of w does. □

We now extend the concept of first offenders to PFT shift spaces.

Let $\mathcal{G}$ be the Shannon cover of an irreducible sofic shift, $T = \text{per}(A_{\mathcal{G}})$, and $D_0, D_1, \ldots, D_{T-1}$ the period classes of $\mathcal{G}$. An indexed word $w^{(n)} = (w_0, w_1, \ldots, w_{l-1})^{(n)}$ is a *periodic first offender* for period class n if $w \notin \bigcup_{I\in D_n} \mathsf{F}_{\mathcal{G}}(I)$ but, for all $i, j \in [0, l-1]$, with $i \le j$ and $w_{[i,j]} \ne w$, $w_{[i,j]} \in \bigcup_{I\in D_{(n+i)\ mod\ T}} \mathsf{F}_{\mathcal{G}}(I)$. Note that if $T = 1$ the periodic first offenders are the first offenders.

Theorem 3 *Let $\mathsf{X}_\mathcal{G}$ be an irreducible sofic shift, T the period of the Shannon cover, and $\mathcal{O}$ the collection of all periodic first offenders. Then $\mathsf{X}_\mathcal{G} = \mathsf{X}_{\{\mathcal{O},T\}}$.*

Proof: Let $\mathcal{G}$ be the Shannon cover of an irreducible sofic shift. Put $T = \mathrm{per}(A_\mathcal{G})$ and let $D_0, D_1, \ldots, D_{T-1}$ be the period classes of $\mathcal{G}$. Let $\mathcal{O}$ be the collection of periodic first offenders for $\mathsf{X}_\mathcal{G}$.

Choose $x \in \mathsf{X}_\mathcal{G}$. Let D_{T-k} be the period class such that $\mathsf{i}(\mathcal{L}^{-1}(x_0)) \in D_{T-k}$. Choose $w_j^{(n_j)} \in \mathcal{O}$ and an integer m such that $m \mod T = n_j$. Note that $\mathsf{i}(\mathcal{L}^{-1}(x_{k+m})) \in D_{(T-k+k+m) \mod T} = D_{n_j}$. Hence $\sigma^k(x)_{[m,m+|w_j|-1]} = x_{[k+m,k+m+|w_j|-1]} \neq w_j$. Therefore $x \in \mathsf{X}_{\{\mathcal{O},T\}}$ and $\mathsf{X}_\mathcal{G} \subseteq \mathsf{X}_{\{\mathcal{O},T\}}$.

To show the reverse inclusion, choose

$$v \in \mathcal{B}(\mathsf{X}_\mathcal{G})^c = \bigcap_{I \in \mathcal{V}(\mathcal{G})} \mathsf{F}_\mathcal{G}(I)^c.$$

Put $v = v_0 v_1 v_2 \cdots v_{l-1}$. For each D_k, v is either a periodic first offender for D_k or a sub-word $v_{[p,q]}$ is a periodic first offender for $D_{(k+p) \mod T}$, i.e., for each $k \in [0, T-1]$, $\exists w_j^{(n_j)} \in \mathcal{O}$ and $p \in \mathbb{Z}$ such that $(k+p) \mod T = n_j$ and $v_{[p,p+|w_j|-1]} = w_j$. Let x be a bi-infinite sequence that contains v. Suppose $x_{[n,n+l-1]} = v$. Then for any $(n-k) \mod T$, there exists $p \in \mathbb{Z}$ and $w_j^{(n_j)} \in \mathcal{O}$ such that $(n-k+p) \mod T = n_j$ and $\sigma^k(x)_{[n+p-k,n+q-k]} = v_{[p,q]} = w_j$. In other words, for each $k \in [0, T-1]$, $\exists w_j^{(n_j)} \in \mathcal{O}$ and $m \in \mathbb{Z}$ such that $m \mod T = n_j$ and $\sigma^k(x)_{[m,m+|w_j|-1]} = w_j$. Hence $v \in \mathcal{B}(\mathsf{X}_{\{\mathcal{O},T\}})^c$, $\mathcal{B}(\mathsf{X}_\mathcal{G})^c \subseteq \mathcal{B}(\mathsf{X}_{\{\mathcal{O},T\}})^c$, and $\mathsf{X}_\mathcal{G} \supseteq \mathsf{X}_{\{\mathcal{O},T\}}$. □

Example 7 Let $\mathcal{G}$ be the Shannon cover of the interleaved-biphase shift discussed in Example 6 and illustrated in Fig. 7. In this case, $T = \mathrm{per}(A_\mathcal{G}) = 4$, and one can show that the list of periodic first offenders is given by

$$\begin{aligned} \mathcal{O} = \ & \{000^{(0)}, 000^{(1)}, 010^{(0)}, 010^{(1)}, \\ & 101^{(0)}, 101^{(1)}, 111^{(0)}, 111^{(1)}\}. \end{aligned}$$

□

The following theorem establishes that the periodic first offenders are the unique minimal forbidden word description of the space for the given period.

Theorem 4 *Let $\mathcal{O}, T$ be the periodic first offenders of the irreducible shift $\mathsf{X}_{\{\mathcal{O},T\}}$. Suppose $\mathsf{X}_{\{\mathcal{F},T\}} = \mathsf{X}_{\{\mathcal{O},T\}}$ for some pair $\mathcal{F}, T$. If $\mathcal{F} \subseteq \mathcal{O}$, then $\mathcal{F} = \mathcal{O}$. If $\mathcal{F}$ is finite then $|\mathcal{O}| \leq |\mathcal{F}|$ and $\sum_{w\in\mathcal{O}} |w| \leq \sum_{w\in\mathcal{F}} |w|$.*

Proof: Construct the Shannon cover and associate the period classes $D_0, D_1, \ldots, D_{T-1}$ with the corresponding indices of words in $\mathcal{O}$ and $\mathcal{F}$, i.e., such that there is no path starting from period class D_k with label $w^{(k)}$. Choose $w^{(k)} \in \mathcal{O}$, $w = w_0 w_1 \ldots w_{l-1}$. Construct a right infinite path with $\mathsf{i}(\mathcal{L}^{-1}(w_1)) \in D_{k+1 \bmod T}$ and corresponding label

$$w_1 w_2 \cdots w_{l-1} u_l u_{l+1} \cdots$$

and a left infinite path with label

$$\cdots u_{-2} u_{-1} w_0 w_1 \cdots w_{l-2}.$$

By definition, there is no bi-infinite path on the graph with $\mathsf{i}(\mathcal{L}^{-1}(w_1)) \in D_{k+1 \bmod T}$ and label

$$\cdots u_{-2} u_{-1} w_0 w_1 \cdots w_{l-2} w_{l-1} u_l u_{l+1} \cdots$$

Therefore, a sub-word of this sequence is not allowed from the associated period class. Hence there exists $v \in \mathcal{F}$ with appropriate index that contains w and no other element of $\mathcal{O}$ with matching index. This establishes a bijective mapping of $\mathcal{O}$ to $\mathcal{F}$. $\square$

We have seen that the periodic first offenders may offer a finite description of shifts that are not finite-type. It is possible that the list of periodic-first-offenders provides a minimal description of an irreducible sofic shift among all forbidden lists, in the sense of the following conjecture.

Conjecture 1 *Let X be an irreducible periodic-finite-type shift and $\mathcal{O}$ the list of periodic first offenders. For any pair $\{\mathcal{F}, T\}$ such that $X = \mathsf{X}_{\{\mathcal{F},T\}}$, $|\mathcal{O}| \leq |\mathcal{F}|$ and $\sum_{w\in\mathcal{O}} |w| \leq \sum_{w\in\mathcal{F}} |w|$.*

Example 8 The list of first offenders of the *abcd* shift of Example 3 is:

$$\mathcal{O} = \{aa, ad, ba, bb, bc, cb, cc, cd, da, dd\}.$$

For $T = 2$, the list of periodic first offenders, which we denote by $\{\mathcal{O}, 2\}$ is:

$$\{\mathcal{O}, 2\} = \{\{a^{(0)}, d^{(0)}, ba^{(0)}, cd^{(0)}, b^{(1)}, c^{(1)}\}, 2\}.$$

Clearly, $|\{\mathcal{O}, 2\}| \leq |\mathcal{O}|$. □

5 Graph Construction

The constructive proof of Theorem 1 describes one method to generate a graph presenting a PFT shift from a list of forbidden words. However, the complexity of the procedure, measured by the number of states in the graph, grows exponentially with the length of the longest word in the forbidden list. This section describes a method to construct a graph whose complexity grows linearly with the length of the longest word and the number of words in the list.

An algorithm with linear complexity for constructing a graph presenting a shift of finite algorithm draws on the connections between symbolic dynamics and automata theory. Similar construction methods have been described in [12, 13]. In this section we present a generalization of the procedure from [14] to construct graphs presenting PFT shifts. An alternative construction method for PFT shifts may be found in [13]. Note that the class of PFT shifts include some almost-finite-type shifts, which were not included in the constructions in [14, 12]. We begin by defining some notation and stating a few results from automata theory relevant to the construction procedure. A more detailed exposition on automata theory may be found in [8].

5.1 Background on Automata Theory

As before, let $\mathcal{A}$ denote a finite set of symbols. A *language* over $\mathcal{A}$ is a subset $L \subseteq \mathcal{A}^*$. A *finite automaton* is given by the quadruple $M = (\mathcal{G}, \mathcal{A}, q_0, F)$, where $\mathcal{A}$ is referred to as the *input alphabet*, $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{L})$ is a finite-state labeled graph, $q_0 \in \mathcal{V}$ is the *initial* state, and $F \subseteq \mathcal{V}$ is the set of *final* states. We refer to elements of F as *accepting* states of the automaton. Any other state is a *non-accepting* state.

We say an automaton is *deterministic* if $\mathcal{G}$ is deterministic. A word w is *accepted* by the automaton $M = (\mathcal{G}, \mathcal{A}, q_0, F)$ if there exists a path e on $\mathcal{G}$ with $\mathsf{i}(e) = q_0$, $\mathsf{t}(e) \in F$, and $\mathcal{L}(e) = w$. The *language accepted by the automaton*, denoted $L(M)$, is the set of words accepted by the automaton. A *regular* language is a language accepted by a finite automaton. Note that in a deterministic automaton, for all $w \in L(M)$ there exists a unique path from the initial state to an accepting state that generates w.

There are clearly strong connections between regular languages and sofic shifts, e.g., [15, 16]. Although sofic shifts are sets of bi-infinite sequences that have no designated starting or accepting states, there is a natural correspondence between languages of sofic shifts and regular languages. The language of a sofic shift is a regular language [11, A.12]. However, all regular languages are not languages of sofic shifts. In particular, if $M = (\mathcal{G}, \mathcal{A}, q_0, F)$, then $\mathcal{B}(\mathsf{X}_\mathcal{G})$ is not necessarily equal to $L(M)$.

The following well-known result, e.g., [8, Theorem 3.2], will be important to our construction.

Result 2 *The class of regular languages is closed under complementation, i.e., if L is accepted by a finite automaton, then L^c is accepted by a finite automaton.* □

In the following section we describe an algorithm to construct a graph presenting a PFT shift. The method, a generalization of the construction in [14], proceeds as follows. We first construct a non-deterministic finite automaton that accepts the complement of the language in which we are interested. It turns out to be straightforward to construct this automaton. An automaton accepting the language is formed by following a constructive proof of Result 2. By deleting the non-accepting states of the resulting automaton, we obtain a graph representing the shift space.

5.2 Graph Construction

Fix a pair consisting of an indexed list and period,

$$\mathcal{F} = \{w_1^{(n_1)}, w_2^{(n_2)}, \ldots, w_{|\mathcal{F}|}^{(n_{|\mathcal{F}|})}\}, T.$$

For $i = 0, 1, 2, \ldots, T-1$, define the language

$$\begin{aligned} L_i \stackrel{\text{def}}{=} \{v|\ &\text{putting } v = v_i v_{i+1} \cdots v_{i+|v|-1}, \\ &\forall m, p \in [i, i+|v|-1] \text{ with } m \le p \text{ and all } w_j^{(n_j)} \in \mathcal{F}, \\ &\text{if } m \bmod T = n_j \text{ then } v_{[m,p]} \ne w_j\}, \end{aligned}$$

and, subsequently, its complement,

$$L_i^c = \{v|\text{ putting } v = v_i v_{i+1} \cdots v_{i+|v|-1},$$
$$\exists m, p \in [i, i+|v|-1] \text{ with } m \le p \text{ and } w_j^{(n_j)} \in \mathcal{F},$$
$$\text{such that } m \bmod T = n_j \text{ and } v_{[m,p]} = w_j\}.$$

Note that $\mathcal{B}(\mathsf{X}_{\{\mathcal{F},T\}}) \subseteq \bigcup_{i=0}^{T-1} L_i$.

Construct a non-deterministic graph $\mathcal{G}_{nd}$ as follows. Fix T states labeled $q_0, q_1, \ldots, q_{T-1}$. Draw an edge for each $a \in \mathcal{A}$ and each $i \in [0, T-1]$ from q_i to $q_{(i+1) \bmod T}$ with label a. Fix a state labeled x and draw an edge(cycle) for each $a \in \mathcal{A}$ from x to x with label a. Now draw paths, as shown in Fig. 9, from q_i to x for each word $w_j = w_{j,0} w_{j,1} \cdots w_{j,|w_j|-1}$ in $\mathcal{F}$ with index i.

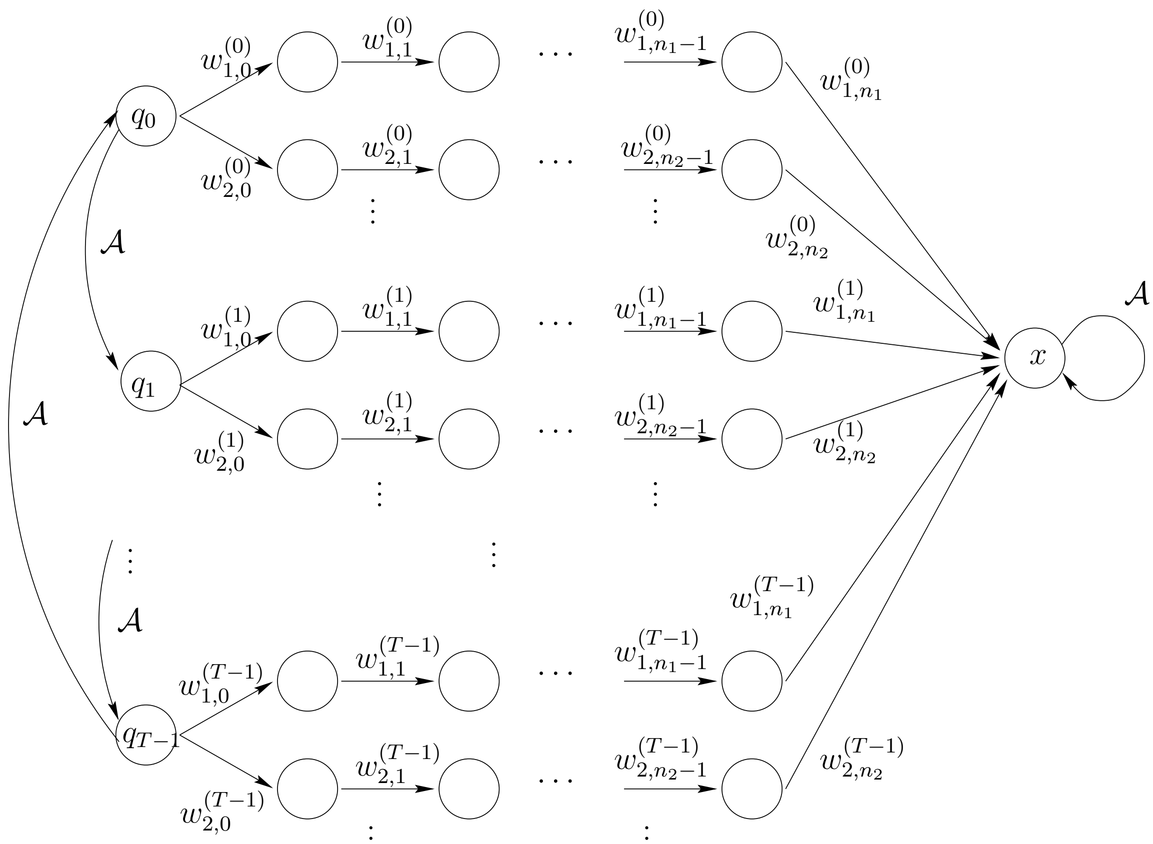

Figure 9: Non-deterministic graph $\mathcal{G}_{nd}$ underlying $M_{nd,i}$.

Note that we may reduce the number of states in $\mathcal{G}_{nd}$ by sharing common suffixes of forbidden words. From this observation, we have a simple relation for the number of states in $\mathcal{G}_{nd}$ when suffixes are shared,

$$|\mathcal{V}(\mathcal{G}_{nd})| = T + 1 + \left(\sum \text{lengths of distinct suffixes of words in } \mathcal{F}\right).$$

Put $M_{nd,i} = (\mathcal{G}_{nd}, \mathcal{A}, q_i, x)$. It is straightforward to show that $L(M_{nd,i}) = L_i^c$. Indeed, a word in L_i^c is of the form uwv, where u and v are arbitrary elements of $\mathcal{A}^*$, $w = w_j^{(n_j)} \in \mathcal{F}$, and $(i + |u|) \bmod T = n_j$. These are precisely the words accepted by $M_{nd,i}$.

Example 9 Put $\mathcal{A} = \{0,1\}, \mathcal{F} = \{101^{(0)}, 010^{(1)}\}, T = 2$. The distinct suffixes are 01 and 10, with length 2. $\mathcal{G}_{nd}$ is illustrated in Fig. 10 and $|\mathcal{V}(\mathcal{G}_{nd})| = 2 + 1 + 2 + 2 = 7$, as predicted.

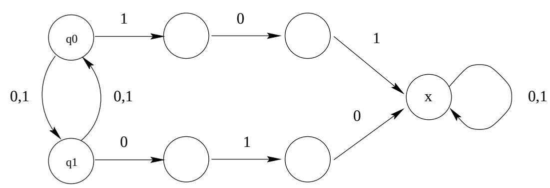

Figure 10: G_{nd} corresponding to $\mathcal{A} = \{0,1\}, \mathcal{F} = \{010^{(0)}, 101^{(1)}\}, T = 2$.

Example 10 Put $\mathcal{A} = \{0,1\}, \mathcal{F} = \{00^{(0)}, 00^{(1)}, 11^{(0)}\}, T = 2$. The distinct suffixes are 0 and 1, with length 1. $\mathcal{G}_{nd}$ is illustrated in Fig. 11 and $|\mathcal{V}(\mathcal{G}_{nd})| = 3 + 1 + 1 = 5$.

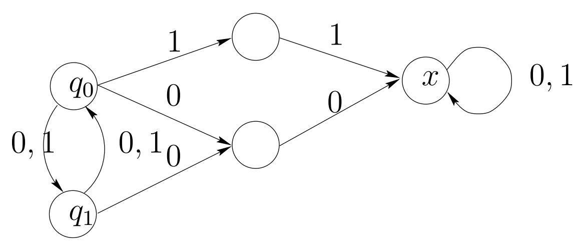

Figure 11: G_{nd} corresponding to $\mathcal{A} = \{0,1\}, \mathcal{F} = \{00^{(0)}, 00^{(1)}, 11^{(0)}\}, T = 2$.

Following the constructive proof of Theorem 2 in [8], we will build a deterministic automaton that accepts $L(M_{nd,i})^c = L_i$. First, construct a deterministic graph $\mathcal{G}_d$ from $\mathcal{G}_{nd}$ via the well-known *subset construction algorithm*, e.g., [4, Theorem 3.3.2], as follows. $\mathcal{V}(\mathcal{G}_d)$ is the set of all nonempty subsets of $\mathcal{V}(\mathcal{G}_{nd})$. For every edge e in $\mathcal{E}(\mathcal{G}_{nd})$ from $\mathsf{i}(e)$ to $\mathsf{t}(e)$ put edges in $\mathcal{E}(\mathcal{G}_d)$ with labels $\mathcal{L}(e)$ from each $I \in \mathcal{V}(\mathcal{G}_d)$ to each $J \in \mathcal{V}(\mathcal{G}_d)$ such that $\mathsf{i}(e) \in I$ and $\mathsf{t}(e) \in J$.

Put $M_i' = (\mathcal{G}_d, \mathcal{A}, q_i, F)$, where F is the subset of $\mathcal{V}(\mathcal{G}_d)$ consisting of those states that contain the accepting state of $M_{nd,i}$, i.e., x. M_i' is deterministic and one can show that

$L(M_i') = L(M_{nd,i})$, e.g., [8, Theorem 2.1].

Let $M_i = (\mathcal{G}_d, \mathcal{A}, q_i, \mathcal{V}(\mathcal{G}_d) - F)$, i.e., the automaton constructed from M_i' by switching the roles of the accepting and non-accepting states. Since $\mathcal{G}_d$ is deterministic, M_i accepts a word w if and only if w is in $L^c(M_i')$, therefore $L(M_i) = L^c(M_i') = L_i$. Note that the underlying labeled graph $\mathcal{G}_d$ and the set of accepting states $\mathcal{V}(\mathcal{G}_d) - F$ are the same for each $i \in [0, \dots, T-1]$, i.e., for each automaton M_i.

No accepting state of M_i may be reached from a non-accepting state. Hence we can delete the non-accepting states from $\mathcal{G}_d$ without changing the language accepted by M_i. Let $\mathcal{G}$ denote the graph that results from deleting the non-accepting states from $\mathcal{G}_d$. The following theorem establishes that $\mathsf{X}_{\mathcal{G}} = \mathsf{X}_{\{\mathcal{F},T\}}$.

Theorem 5 *Let $\{\mathcal{F}, T\}$ be an indexed list and period. Let $\mathcal{G}$ be the graph constructed following the method described in this section. Then $\mathsf{X}_{\{\mathcal{F},T\}} = \mathsf{X}_{\mathcal{G}}$.*

Proof: Choose $x \in \mathsf{X}_{\mathcal{G}}$. Since $|\mathcal{V}(\mathcal{G})|$ is finite and every state in $\mathcal{G}$ is reachable from some q_i, choose a starting state q_i such that any sub-word of x lies on a path originating from q_i. Let π be a path starting at q_i and terminating at $\mathsf{i}(\mathcal{L}^{-1}(x_0))$. Put $k = -(|\pi| + i)$. Then for all m and all $w_j^{(n_j)} \in \mathcal{F}$, if $m \mod T = n_j$ then $\sigma^k(x)_{[m,m+|w_j|-1]} \neq w_j$. Therefore $\sigma^k(x) \in \mathsf{X}_{\{\mathcal{F},T\}}$ and $\mathsf{X}_{\mathcal{G}} \subseteq \mathsf{X}_{\{\mathcal{F},T\}}$.

For the reverse inclusion, choose $w \in \mathcal{B}(\mathsf{X}_{\{\mathcal{F},T\}})$. Then there exists i such that $w \in L_i$. In addition, w is left extendible by words in $\mathcal{B}(\mathsf{X}_{\{\mathcal{F},T\}})$. Hence we can choose $uw \in \mathcal{B}(\mathsf{X}_{\{\mathcal{F},T\}})$ such that $uw \in \bigcup_{i=0}^{T-1} L_i$ and $w \in \mathcal{B}(\mathsf{X}_{\mathcal{G}})$, i.e., we can choose some u such that w lies on the essential subgraph of $\mathcal{G}$. Therefore $\mathcal{B}(\mathsf{X}_{\{\mathcal{F},T\}}) \subseteq \mathcal{B}(\mathsf{X}_{\mathcal{G}})$ and $\mathsf{X}_{\{\mathcal{F},T\}} \subseteq \mathsf{X}_{\mathcal{G}}$. □

The construction may be simplified by keeping in mind that all accepting states will be deleted from $\mathcal{G}_d$, hence there is no need to distinguish between different accepting states nor to draw edges between different accepting states when constructing the deterministic automaton. In addition, only the subgraph of $\mathcal{G}$ which may be reached from the starting states needs to be considered.

Finally, take the essential subgraph of $\mathcal{G}$ and apply a state-minimization algorithm, e.g., [4, pp. 92]. If the shift is irreducible, this will result in the Shannon cover. In Table 5.2, we summarize and repeat the construction procedure including these simplifications.

Example 11 Let $\mathcal{F} = \{101^{(0)}, 010^{(1)}\}, T = 2$. $\mathcal{G}_d$, constructed from $\mathcal{G}_{nd}$ in Fig. 10, is

Table 1: Summary of Graph Construction

1. Construct the non-deterministic graph $\mathcal{G}_{nd}$ as described.
2. Construct a deterministic graph $\mathcal{G}_d$ using the subset construction algorithm including only those states which may be reached from one of the starting states, and directing any edge which terminates in an accepting state into a single accepting state.
3. Construct $\mathcal{G}$ by deleting the accepting state and all edges which begin or terminate there.
4. Take the essential subgraph of $\mathcal{G}$, and apply a state-minimization algorithm.

illustrated in Fig. 12, and Fig. 13 illustrates the resulting Shannon cover.

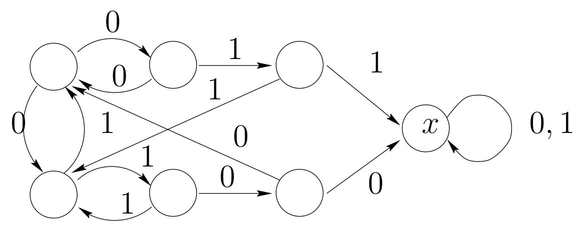

Figure 12: $\mathcal{G}_d$ corresponding to $\mathcal{F} = \{010^{(0)}, 101^{(1)}\}, T = 2$.

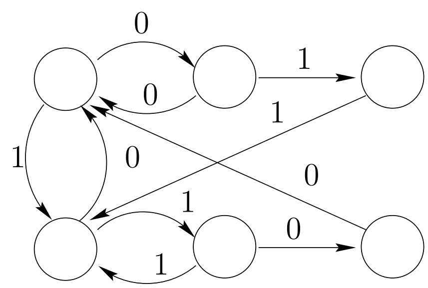

Figure 13: Shannon cover corresponding to $\mathcal{F} = \{010^{(0)}, 101^{(1)}\}, T = 2$.

Example 12 Let $\mathcal{F} = \{00^{(0)}, 00^{(1)}, 11^{(0)}\}, T = 2$. $\mathcal{G}_d$, constructed from $\mathcal{G}_{nd}$ in Fig. 11, is illustrated in Fig. 14, and Fig. 15 illustrates a deterministic graph presenting the shift. Note this shift space is not irreducible.

6 Conclusions

We have introduced the class of periodic-finite-type (PFT) shift spaces. This class of sofic shifts lie between the class of finite-type shifts and almost-finite-type shifts. We proved several properties of graph presentations of these spaces. For a given PFT space, we identified a particular list of periodically forbidden words, the periodic first-offenders,

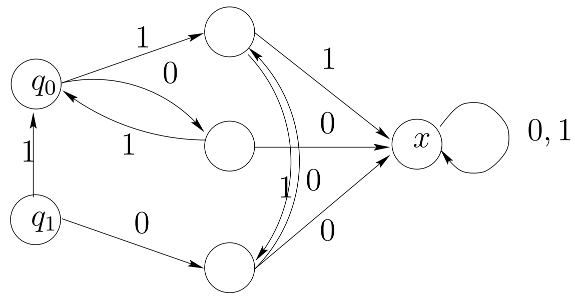

Figure 14: $\mathcal{G}_d$ corresponding to $\mathcal{F} = \{00^{(0)}, 00^{(1)}, 11^{(0)}\}, T = 2$.

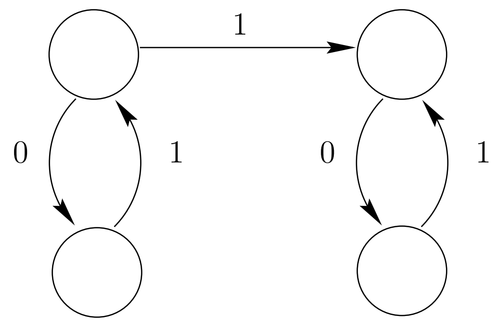

Figure 15: Deterministic graph corresponding to $\mathcal{F} = \{00^{(0)}, 00^{(1)}, 11^{(0)}\}, T = 2$.

that enjoy certain minimality properties with respect to other forbidden lists defining the space. Finally, we developed an algorithm that, for a given periodic forbidden list, produces a graph presenting the corresponding shift space. The complexity of the algorithm grows only linearly in the number, lengths, and period of the given forbidden list.

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