Serial Concatenated TCM With an Inner Accumulate Code—Part I: Maximum-Likelihood Analysis

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***Abstract*—In this paper, we propose a serial concatenated trellis-coded modulation system using one or more inner rate-1 accumulate codes and a mapping to a higher order, Gray-labeled signal constellation. As outer codes, we consider repeat codes, single parity-check codes, and convolutional codes. We show that under maximum-likelihood decoding, there exists a signal-to-noise ratio threshold beyond which the bit-error probability goes to zero as the blocklength goes to infinity. We then evaluate the performance for finite blocklengths using a modified union bound. Computer simulations demonstrate that the proposed system, despite its use of a simple rate-1 inner code, achieves performance in additive white Gaussian noise and Rayleigh fading that is comparable to, or better than, that of more complex systems suggested in the literature.**

***Index Terms*—Accumulate codes, coding theorems, fading channels, iterative decoding, performance analysis, serial concatenation, trellis-coded modulation (TCM).**

I. Introduction

SEVERAL turbo trellis-coded modulation (TTCM) systems have been proposed in order to merge the extraordinary performance of turbo codes [1] with the bandwidth efficiency of trellis-coded modulation (TCM) [2]. Both parallel concatenated TCM (PCTCM) [3], [4] and serial concatenated TCM (SCTCM) [5], [6] have been shown to achieve good performance on additive white Gaussian noise (AWGN) channels.

Bit-interleaved coded modulation (BICM) was initially proposed as a TCM technique that could provide diversity gain on fading channels, while retaining good performance on AWGN channels [7]. BICM with iterative decoding (BICM-ID) has been reported to give almost the same performance as turbo-TCM on AWGN channels, but with lower complexity [8].

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Inspired by the analytical tractability of repeat-accumulate (RA) codes [9] and their generalizations [10], [11], we consider an SCTCM system where the inner code is an accumulate code, or a concatenation of multiple, interleaved accumulate codes, followed by a mapping to a higher order, Gray-labeled signal constellation. We are interested in spectrally efficient schemes, and toward this end, we consider systems with high-rate convolutional codes and parity-check codes as outer codes.

The proposed SCTCM system can be considered to be a generalized BICM system in which the convolutional code is replaced by a serially concatenated convolutional code (SCCC), and where the SCCC may have multiple inner codes.

For maximum-likelihood (ML) decoding of the proposed SCTCM system, we prove coding theorems analogous to those presented in [9]–[12]. Specifically, we show that for an outer code with free distance $d_{\text{free}}^{(o)} \geqslant 3$, there exists a system-specific *threshold*, corresponding to a minimum signal-to-noise ratio (SNR) γ_{ML}^{*}, such that for all SNR $\gamma > \gamma_{\text{ML}}^{*}$, the word-error probability (WEP) P_W goes to zero as the blocklength N goes to infinity. When $d_{\text{free}}^{(o)} = 2$, as in the case of an outer single parity-check (SPC) code, the bit-error probability (BEP) P_b goes to zero as $N \to \infty$ for $\gamma > \gamma_{\text{ML}}^{*}$, but P_W does not. However, by incorporating multiple interleaved inner accumulate codes, we show that a threshold exists for P_W as well.

We prove these results by extending the union-bound techniques of [9] and [12] to higher order constellations. Although adequate for the proof, the union-bound approach gives numerical values for the threshold γ_{ML}^{*} that are so large as to be of limited use in predicting actual system performance.

In practice, ML decoding of the concatenated system is prohibitively complex. Instead, we use an iterative turbo-like receiver architecture incorporating soft-input soft-output (SISO) decoders, such as *a posteriori* probability (APP) decoders based upon the Bahl–Cocke–Jelinek–Raviv (BCJR) algorithm [13], for the component codes. Such decoder structures have been found empirically to provide good performance over a range of channel conditions for systems with a small number of component codes. In a companion paper [14], we use density-evolution techniques to find thresholds for outer repeat and SPC codes that more accurately reflect the performance of an iterative, message-passing decoder.

To analyze the performance of the proposed system for finite blocklengths, we derive an improved union bound for the average Euclidean weight enumerator. Specifically, we use properties of the accumulate code to determine the expected squared Euclidean weight of a transmitted codeword corresponding to an outer codeword of given Hamming weight.

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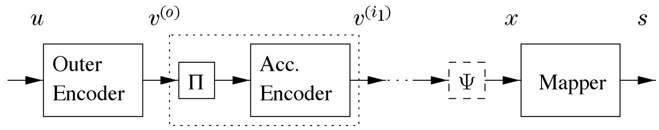

Fig. 1. Basic encoder uses a single inner accumulate code and no channel interleaver Ψ before the mapper. The generalized system uses multiple accumulate codes and nontrivial interleaver Ψ.

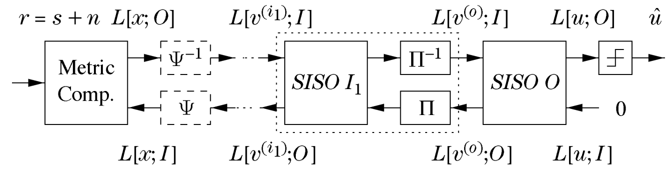

Fig. 2. Basic decoder has two SISO modules and does not feed back any *a priori* information to the metric computation. The general decoder can include more SISO modules and may feed back *a priori* information to the metric computation.

Since the inner code is rate-1, the spectral efficiency of the proposed SCTCM system is determined by the rate of the outer code, the constellation size, and the mapping. Thus, one can use a standard outer convolutional code and puncturing to achieve a range of desired rates. We do not have to find matching high-rate outer and inner convolutional codes, nor do we have to match the inner code to the constellation.

We simulate the performance of the proposed system using an outer convolutional code and a single inner accumulate code, and compare with analogous systems previously reported in the literature. In particular, for AWGN channels, we compare the proposed scheme with the SCTCM system in [6]. In Rayleigh fading, we make our comparison with the BICM-ID system in [8].

The paper is organized as follows. In Section II, we present a detailed description of the proposed system architecture. After some preliminaries, we state and prove the asymptotic coding theorems in Section III. The performance bound for finite blocklengths is derived in Section IV. In Section V, we report simulation results and discuss some design considerations. Our conclusions are presented in Section VI.

II. System Description

A. Encoder

The encoder for the proposed system is shown in Fig. 1. The outer encoder is a block code, formed either by terminating a rate-$r_c = k/n$ convolutional code, or by concatenating short block codes, such as $r_c = (n-1)/n$ SPC codes or $r_c = 1/n$ repeat codes.

The interleaver Π is a random or S-random [15] bit interleaver, and acts on all bits in a block (in contrast to separate interleavers applied to separate subsets of the encoded bits, as in the original BICM architecture [7]). The size of the interleaver is N, and it is assumed that n divides N. Though it is possible to design an S-random interleaver with $S \approx \sqrt{N}$, it has been found empirically that an S-constraint on the order of $S = 10$ yields satisfactory interleaver performance.

The inner code in the concatenation is a rate-1 accumulate code. The accumulate code can be thought of as a block code with an $N \times N$ upper triangular generator matrix of all ones [9], or as a convolutional code with rational generator matrix

$$G(D) = \left(\frac{1}{1 \oplus D}\right). \quad (1)$$

In the companion paper [14], we will describe codes by graphs, and the latter description of the accumulate code allows for a simpler graph representation.

The memoryless mapper, μ, maps an m-tuple of bits to a constellation point $s \in \mathcal{S}$, where $\mathcal{S}$ is a Gray-labeled constellation of size $|\mathcal{S}| = M = 2^m$. Note that m is not necessarily equal to n. The Gray labeling allows for a simple relationship between the Hamming distance d_H of binary sequences and the squared Euclidean distance (SED) d_E^2 of the corresponding outputs of the mapper. This property of the Gray labeling is used in the derivation of the coding theorems in Section III.

The proposed system resembles BICM, in that there is no optimization of the inner code with respect to the signal constellation. The introduction of the rate-1 accumulate code is important, in that it acts as a weight transformer. If we consider a uniform interleaver and the accumulate code as one unit, we get a random weight transformer [11]. We will use the structure imposed by the accumulate code on its output to derive an improved performance bound for finite blocklengths.

The generalization of the encoder to include multiple interleaved accumulate codes is straightforward. The random weight transformer, i.e., the interleaver and accumulate code (the dotted section in Fig. 1), is repeated a specified number of times. We can also introduce a nontrivial interleaver Ψ before the mapper. With an outer convolutional code, no accumulate codes, and a nontrivial interleaver Ψ, we get the BICM system of [7] as a special case.

B. Decoder

The decoder depicted in Fig. 2 consists of a demodulator (bit metric calculator), SISO module(s) [16] matched to the inner code(s), and a SISO module matched to the outer code, separated by appropriate interleavers and deinterleavers. The final, binary decision is given by a slicer.

The demodulator computes the extrinsic probability that the ith bit of the m-tuple x making up the received symbol r has the value b, given all available information except the information about the ith bit itself. Let x^i denote the ith bit in the m-tuple x, and let $\ell^j(s)$ denote the jth bit in the binary label of the symbol s. Following the notation in [16], we let I and O denote quantities at the input and output of a device, respectively. We then define the extrinsic probability as

$$P[x^i = b; O] := \frac{P[x^i = b | r]}{P[x^i = b; I]}. \quad (2)$$

If the interleaver Ψ is present, we can assume that the bits making up the transmitted symbol s are independent, and get

$$P[x^i = b; O] = \frac{\prod_{j=1}^{m} P\left[x^j = \ell^j(s); I\right]}{P[x^i = b; I]} \quad (3)$$

$$= \prod_{\substack{j=1 \\ j \neq i}}^{m} P\left[x^j = \ell^j(s); I\right]. \quad (4)$$

By Bayes' rule, the APP of the symbol s, given the received sample r, $\Pr[s|r]$, can be expressed as $\Pr[s|r] = f[r|s]\Pr[s]/f[r]$. The extrinsic probability is then

$$P[x^i = b; O] = \sum_{s \in \mathcal{S}_b^i} P[r|s] \prod_{\substack{j=1 \\ j \neq i}}^{m} P\left[x^j = \ell^j(s); I\right] \tag{5}$$

where $\mathcal{S}_b^i = \{s \in \mathcal{S} | \ell^i(s) = b\}$ is the set of points in the constellation $\mathcal{S}$, such that the ith bit, $i \in \{1, \ldots, m\}$, in the binary label of the symbol s has the value b, $b \in \{0, 1\}$. The output of the demodulator is a log-likelihood ratio (LLR)

$$L[x^i; O] = \log \frac{P[x^i = b; O]}{P[x^i = \bar{b}; O]} \tag{6}$$

for each bit in the symbol. The LLRs are fed from the demodulator to the innermost SISO, and the SISO modules perform bitwise decoding. The flow of LLRs is indicated in Fig. 2.

If we do not iterate over the metric computation, and assume that all constellation points are transmitted equally often, the decoding metric in (5) simplifies to

$$P[x^i = b; O] = \sum_{s \in \mathcal{S}_b^i} P[r|s]. \tag{7}$$

Iterative demodulation of Gray-labeled signal constellations provides only minor performance improvements, which can be explained by studying the extrinsic information transfer (EXIT) charts for different labelings [17], [18]. When *a priori* knowledge is fed back to the demodulator, set-partition labeling performs better [19], [20]. We use Gray labeling, since the labeling allows for a simple relationship between Hamming and SEDs.

With multiple inner codes (or with iterative demodulation), decoder scheduling becomes an issue. That is, the order in which we decode the different codes will affect the overall performance, as we see in Section V-A.

III. Coding Theorems

In this section, we state coding theorems for ML decoding of the system in Fig. 1 on the Gaussian channel and the Rayleigh fading channel. We will concentrate on systems in which the outer code is a q-repeat code, an SPC code, or a convolutional code with free distance $d_{\text{free}} \geqslant 3$. Our results are obtained by extending the union-bound technique of [9] and [12] to higher order constellations.

We make use of a union bound on the WEP P_W that takes the form

$$P_W \leqslant \sum_{h>0} A_h^{(c)} z^h \tag{8}$$

where $\{A_h^{(c)}\}$ is the Hamming weight enumerator (WE) for the concatenated code, and z is a parameter that depends on the channel SNR and the Gray-labeled constellation.

We are also interested in the probability of bit error, P_b, and a similarly derived upper bound that depends on the input–output Hamming weight enumerator (IOWE), $\{A_{w,h}^{(c)}\}$

$$P_b \leqslant \sum_{h=1}^{N} \sum_{w=1}^{rN} \frac{w}{rN} A_{w,h}^{(c)} z^h. \tag{9}$$

By expressing the bound on P_W in the form

$$P_W \leqslant \sum_{h>0} e^{h(F(\cdot)+\ln z)} \tag{10}$$

where $F(\cdot)$ is a function of the WE, we conclude that if $\ln z < -F(\cdot)$, then P_W asymptotically approaches zero, as $N \to \infty$. For a single inner accumulate code and outer q-repeat code with $q \geqslant 3$, and a single inner accumulate code and outer convolutional code with $d_{\text{free}} \geqslant 3$, we will prove the existence of a finite threshold value z^* such that if $z < z^*$, the WEP P_W approaches zero asymptotically in the blocklength N. For the outer SPC code, we can prove a similar result, but only for the BEP P_b. By incorporating a cascade of two or more interleaved accumulate codes in the inner code, we can extend the result for the outer SPC code to the WEP P_W, as well.

To state the coding theorems and compute numerical values for the thresholds, we need expressions for the channel parameter z for the channels of interest, and for the WE of the concatenated codes. We address these requirements in the following subsections.

A. Channel Parameters

1) Gaussian Channels: For higher order, Gray-labeled constellations, let d_h^2 denote the minimum SED between symbols whose labels differ in h positions. From the Law of Cosines, it follows that for Gray-labeled phase-shift keying (PSK) and quadrature amplitude modulation (QAM) constellations

$$d_h^2 \geqslant h \cdot d_1^2 = h \cdot d_{\min}^2 \tag{11}$$

where $d_{\min}^2$ is the minimum SED between constellation symbols.

For the ML symbol decoder, the probability of correct decision P_c is found by integrating the noise probability density function (pdf) over the Voronoi region surrounding the transmitted symbol. In the two-dimensional (2-D) case, we can inscribe a circle with radius equal to half the minimum Euclidean distance centered around the transmitted symbol in the Voronoi region. We find a lower bound on P_c by integrating the noise pdf over this circle. The probability of error P_E is then $P_E = 1 - P_c$ [21, p. 261]. Applying (11), we obtain an upper bound on the probability of h bit errors $P_{E,h}$

$$P_{E,h} \leqslant e^{-\frac{d_h^2}{8\sigma^2}} \tag{12}$$

$$\leqslant \left(e^{-\frac{d_{\min}^2}{8\sigma^2}}\right)^h \tag{13}$$

where σ^2 is the channel noise variance.

Assuming unit signal energy, we can express σ^2 in terms of E_b/N_0, r_c, and the number of bits per symbol m. Applying the union bound, we get

$$P_W \leqslant \sum_{h>0} A_h^{(c)} z^h \tag{14}$$

where

$$z = e^{-\frac{r_c m E_b}{4N_0} d_{\min}^2}. \tag{15}$$

Note that this derivation is applicable to any 2-D Gray-labeled constellation.

2) Rayleigh Fading Channels: For an independently fading Rayleigh channel with unit energy, consider two codewords at Hamming distance h. Let η_i be the index set for the symbols differing in i bits, $1 \leqslant i \leqslant m$, and $\sum_{i=1}^{m} |\eta_i| i = h$.

An upper bound on the pairwise error probability (PEP), conditioned on the fading power vector $\boldsymbol{\rho}$, is given by

$$z_{\boldsymbol{\rho},h} = \prod_{i=1}^{m} \left(\prod_{j \in \eta_i} e^{-\rho_j \frac{i r_c m d_{\min}^2}{4} \frac{E_b}{N_0}} \right). \quad (16)$$

Assuming independent fading, we obtain an upper bound on the unconditional PEP by integrating over the fading power pdf, in the form

$$z = \prod_{i=1}^{m} \left(\frac{1}{1 + \frac{i r_c m d_{\min}^2}{4} \frac{E_b}{N_0}} \right)^{\frac{|\eta_i|}{h}}. \quad (17)$$

For small h, we assume there exists an interleaver such that $|\eta_1| = h$ and $|\eta_i| = 0$, $i \neq 1$. This leads to a union bound on P_W for the independently fading Rayleigh channel, in which the channel parameter z is given by

$$z = \frac{1}{1 + \frac{r_c m d_{\min}^2}{4} \frac{E_b}{N_0}}. \quad (18)$$

B. Code WEs

The IOWE for a binary linear code is an array of numbers $\{A_{w,h}\}$, where $A_{w,h}$ is the number of codewords with input Hamming weight w and output Hamming weight h. By summing over the input weight w, we retrieve the WE, $\{A_h\}$, where A_h denotes the total number of codewords of Hamming weight h.

For a rate-$r_c = (n-1)/n$ SPC code, the codeword weight h is always even. An easily obtained upper bound for A_h is given by

$$A_h^{(\mathrm{spc})} \leqslant \binom{\lfloor \frac{n}{2} \rfloor \cdot \frac{N}{n}}{\frac{h}{2}} \binom{n}{\lfloor \frac{n}{2} \rfloor}^{\frac{h}{2}}. \quad (19)$$

We note that, for the $r_c = 1/2$ and $r_c = 2/3$ SPC codes, the upper bound in (19) is tight, providing an exact expression for the WE.

For a rate-$1/n$ repeat code, the IOWE is given by

$$A_{w,h}^{(\mathrm{rep})} = \begin{cases} \binom{\frac{N}{n}}{w}, & h = nw \\ 0, & \text{otherwise} \end{cases} \quad (20)$$

and the IOWE for an accumulate code is

$$A_{w,h}^{(\mathrm{acc})} = \binom{n-h}{\lfloor \frac{w}{2} \rfloor} \binom{h-1}{\lceil \frac{w}{2} \rceil - 1}. \quad (21)$$

(See [9].)

To simplify the performance analysis, we will assume a uniform interleaver [22] between the outer and inner encoders in our system. The average IOWE for the concatenated code is then given by

$$A_{w,h}^{(c)} = \sum_{h_o=0}^{N} \frac{A_{w,h_o}^{(o)} A_{h_o,h}^{(i)}}{\binom{N}{h_o}} \quad (22)$$

TABLE I
E_b/N_0 THRESHOLDS IN DECIBELS FOR P_W FOR RA CODES OF DIFFERENT RATES OVER AWGN AND RAYLEIGH FADING CHANNELS (ML DECODING)

	AWGN			
	8-PSK		16-QAM	
Rate	C^*	γ^*	C^*	γ^*
1/3	0.12	5.77	0.69	6.18
1/4	-0.36	5.50	0.07	5.91
1/5	-0.62	5.37	-0.29	5.78
1/6	-0.80	5.29	-0.52	5.70
	Rayleigh fading			
	8-PSK		16-QAM	
Rate	C^*	γ^*	C^*	γ^*
1/3	1.41	7.03	2.07	7.44
1/4	0.62	6.37	1.16	6.78
1/5	0.16	6.04	0.62	6.45
1/6	-0.13	5.84	0.26	6.25

where $A_{w,h_o}^{(o)}$ and $A_{h_o,h}^{(i)}$ are the IOWE coefficients for the outer and inner codes, respectively.

We now have the tools needed to prove the following coding theorems.

C. Coding Theorems for Higher Spectral Efficiencies

Theorem 1: Consider the ensemble of codes represented by the system with an outer repeat code of rate $r_c \leqslant 1/3$, followed by a uniform interleaver, a single accumulate code, and a mapping to a higher order, Gray-labeled constellation. For ML decoding, there exists a threshold z_A^* for the AWGN channel, such that for $z < z_A^*$, the ensemble average WEP $\bar{P}_W$ approaches zero as the blocklength N approaches infinity, and a threshold z_R^* for the independently fading Rayleigh channel, such that for $z < z_R^*$, $\bar{P}_W \to 0$ as $N \to \infty$.

Proof: The proof is a straightforward application of the technique in [9], combined with the channel parameters in (15) and (18). ■

We refer to this class of codes as RA codes. The theorem provides a method to compute an upper bound on the E_b/N_0 threshold γ^*, above which $P_W \to 0$ as $N \to \infty$. Bounds for different RA code rates are given in Table I. In the table, C^* denotes the minimum SNR needed to achieve the corresponding spectral efficiency for the constellation under consideration.

Theorem 2: For ML decoding of the ensemble of codes represented by a system consisting of an outer convolutional code with $d_{\mathrm{free}} \geqslant 3$, followed by a uniform interleaver, a single accumulate code, and a mapping to a higher order Gray-labeled constellation, there exists a threshold z_A^* for the AWGN channel, such that for $z < z_A^*$, the ensemble average WEP $\bar{P}_W \to 0$ as $N \to \infty$, and a threshold z_R^* for the independently fading Rayleigh channel, such that for $z < z_R^*$, $\bar{P}_W \to 0$ as $N \to \infty$.

Proof: The theorem follows from [23, Th. 5.1] combined with the channel parameters in (15) and (18). ■

Theorem 3: For ML decoding of the ensemble of codes represented by a system consisting of an outer SPC code, a uniform interleaver, one inner accumulate code, and a mapping to a higher order Gray-labeled constellation, there exists a threshold z_A^* for the AWGN channel, such that for $z < z_A^*$, the ensemble

TABLE II
E_b/N_0 THRESHOLDS IN DECIBELS FOR P_b FOR PA CODES OF DIFFERENT RATES OVER AWGN AND RAYLEIGH FADING CHANNELS (ML DECODING)

AWGN				
	8-PSK		16-QAM	
Rate	C^*	γ^*	C^*	γ^*
1/2	1.28	18.28	2.11	18.69
2/3	2.75	20.04	3.68	20.45
3/4	3.66	28.56	4.54	28.97
Rayleigh fading				
	8-PSK		16-QAM	
Rate	C^*	γ^*	C^*	γ^*
1/2	3.18	70.76	3.93	71.17
2/3	5.37	133.69	6.13	134.10
3/4	6.81	389.90	7.57	390.31

average BEP $\bar{P}_b \to 0$ as $N \to \infty$, and a threshold z_R^* for the independently fading Rayleigh channel, such that for $z < z_R^*$, $\bar{P}_b \to 0$ as $N \to \infty$.

Proof: Since the minimum distance for the outer SPC code is $d_{\min}^{(o)} = 2$, P_W does not go to zero as N goes to infinity [24], [25]. However, by expressing the upper bound on P_b as

$$P_b \leqslant \sum_{h=1}^{N} \sum_{w=1}^{rN} \frac{w}{rN} A_{w,h}^{(c)} z^h \tag{23}$$

$$\leqslant \sum_{h=1}^{N} \sum_{h_o=0}^{2h} \frac{h_o}{rN} \frac{A_{h_o}^{(o)} A_{h_o,h}^{(i)}}{\binom{N}{h_o}} z^h \tag{24}$$

we show in the Appendix that $P_b \to 0$ as $N \to \infty$. ■

We refer to this class of codes as parity-accumulate (PA) codes. Table II gives upper bounds on the E_b/N_0 thresholds γ^*, above which $P_b \to 0$ as $N \to \infty$ for PA codes of different rates. The numerical values are very large, since several loose upper bounds are used in the computation of the thresholds.

Theorem 4: For ML decoding of the ensemble of codes represented by a system consisting of an outer parity-check code, and two (or more) uniformly interleaved inner accumulate codes followed by a mapping to a higher order Gray-labeled constellation, there exists a threshold z_A^* for the AWGN channel, such that for $z < z_A^*$, the ensemble average WEP $\bar{P}_W \to 0$ as $N \to \infty$, and a threshold z_R^* for the independent Rayleigh channel, such that for $z < z_R^*$, $\bar{P}_W \to 0$ as $N \to \infty$.

Proof: For a concatenation with m accumulate codes, the WEP is given by

$$P_W \leqslant \sum_{h=1}^{N} \sum_{h_1=0}^{N} \cdots \sum_{h_m=0}^{N} A_{h_1}^{(o)} \frac{A_{h_1,h_2}^{(i_1)}}{\binom{N}{h_1}} \cdots \frac{A_{h_m,h}^{(i_m)}}{\binom{N}{h_m}} z^h. \tag{25}$$

Using similar techniques as in the proof of *Theorem 3*, one can show that this bound approaches zero as the blocklength grows to infinity. (The interested reader can find the details of the proof for PA^2 codes in [26].) ■

We refer to this class of codes as parity m-accumulate codes, or PA^m codes. As in the case of PA codes, the numerical values for the computed thresholds are so large as to be of little practical use. The values are pessimistic, because the derivations are based upon weak union bounds. In [14], we give tight thresholds for iterative message-passing decoding of RA and PA^i codes, $1 \leqslant i \leqslant 3$, computed by density-evolution techniques.

IV. Finite Blocklength Analysis

In this section, we bound the performance of the proposed system for finite blocklengths. We use properties of the accumulate code to find the expected minimum SED between codewords of a given Hamming distance. Under the assumption of uniform interleaving, we determine the average IOWE over the ensemble of codes generated by our serial concatenation architecture.

We also modify the IOWE to reflect the minimum distance of a particular concatenated code in order to better estimate the system performance at high SNRs, as suggested in [27] and [28].

The union bounds on WEP and BEP for ML decoding, as given in (8) and (9), are based upon rather loose upper bounds on the PEPs for codewords of Hamming distance h. In the following section, we derive a new bound on the SED corresponding to such pairs of codewords, leading to a tighter bound on the PEP and a better bound on the system performance.

A. Expected SED

Consider two binary codewords c, c' at Hamming distance h. When the mapping μ is used to translate these codewords to a sequence of symbols in a higher order Gray-labeled constellation, the SED between the symbol sequences $\mu(c)$ and $\mu(c')$, denoted $d_E^2(\mu(c), \mu(c'))$, is $h \cdot d_{\min}^2$ only if the sequences differ in h constellation symbols. If the two symbol sequences differ in less than h constellation symbols, then $d_E^2(\mu(c), \mu(c')) > h \cdot d_{\min}^2$.

Consider the relationship between the inputs and outputs of the accumulate code. The accumulate code computes the running modulo-2 sum of the input word. Thus, the symbols "1" at the input to the accumulate code alternately start and terminate runs of symbols "1" at the output. Therefore, the number of runs, t, is $t = \lceil h_o/2 \rceil$, where h_o is the Hamming weight of the input to the accumulate code. Note that if h_o is odd, the output codeword will end with a run of ones.

To count the number of runs, we augment the IOWE for the serial concatenation with a "run counter," denoted T

$$A^{(c)}(W,H,T) = \sum_{w,h,t} A_{w,h,t}^{(c)} W^w H^h T^t \tag{26}$$

$$= \sum_{w,h} \sum_{h_o}^{N} \frac{A_{w,h_o}^{(o)} A_{h_o,h}^{(i)}}{\binom{N}{h_o}} W^w H^h T^{\lceil \frac{h_o}{2} \rceil}. \tag{27}$$

The coefficient $A_{w,h,t}^{(c)}$ is the number of codewords with input weight w and output weight h distributed among t runs.

Each run must have at least one 1, so the remaining $h-t$ ones are to be distributed among the t runs. This is equivalent to solving the equation

$$x_1 + x_2 + \dots x_t = h - t \quad x_i \geq 0, \quad 1 \leqslant i \leqslant t. \tag{28}$$

The number of solutions to this equation is [29, pp. 24, 26]

$$\binom{h-t+t-1}{t-1} = \binom{h-1}{t-1}. \tag{29}$$

TABLE III
PATTERNS AND DISTANCES FOR RUNS OF LENGTHS 1–5 FOR AN 8-PSK GRAY-LABELED CONSTELLATION

l	Patterns	SED	Min. SED	$h \cdot d^2_{\min}$
1	001	0.5858		
	010	3.4141	0.5858	0.5858
	100	0.5858		
2	001,100	1.1716		
	011	4	1.1716	1.1716
	110	2		
3	001,110	2.5858		
	011,100	4.5858	2.5858	1.7574
	111	3.4142		
4	001,111	4		
	011,110	6	4.000	2.3431
	111,100	4		
5	001,111,100	4.5858		
	011,111	7.4142	4.5858	2.9289
	111,110	5.4142		

The number of solutions such that a particular $x_i = j$ is given by

$$\binom{h-t-j+t-2}{t-2} = \binom{h-2-j}{t-2}. \quad (30)$$

The probability of a run of length $l = j+1$, given h and t, is therefore

$$\Pr[l|h,t] = \frac{\binom{h-1-l}{t-2}}{\binom{h-1}{t-1}} \quad 1 \leqslant l \leqslant h-t+1. \quad (31)$$

By applying the identity $\binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}$, we verify that $\sum_{l=1}^{h-t+1} \binom{h-1-l}{t-2} = \binom{h-1}{t-1}$ and that (31) is indeed a probability distribution.

We now know the probability of a run of length l, given a codeword of weight h in a total of t runs. It remains to determine the SED corresponding to a run of length l. The SED depends on the constellation $\mathcal{S}$, and since not all possible binary patterns of a given length can occur in a single run, the SED also depends on the constellation labeling. In Table III, we list possible patterns and their SED for runs of length $l = 1, \ldots, 5$ for a Gray-labeled 8-PSK constellation. The SED is given with point "000" as reference, but the minimum SED for a run of length l is computed over all constellation points. The lower bound $d_H \cdot d^2_{\min}$ is given for comparison.

The patterns containing a run of length five can be obtained from the patterns with runs of length two by appending or inserting an additional 3-tuple of the form "111." More generally, when dealing with a constellation of size $|\mathcal{S}| = 2^m$, we will use the fact that the patterns with a run of length $2m-1$ or greater can be generated from patterns containing a run of lower weight, by appending or inserting additional m-tuples of ones.

Let $d^2_E(l)$ denote the minimum SED corresponding to a run of length l. The expected minimum SED, given h and t, can be written as

$$d^2_E(h,t) = \sum_{j=1}^{h-t+1} \Pr[j|h,t] d^2_E(j). \quad (32)$$

An improved upper bound on the PEP can now be expressed as a function $f_{\text{PEP}}(\mathcal{S}, h, t)$ of the constellation $\mathcal{S}$ and its labeling, the codeword difference weight h, and the number of runs t in the codeword difference. For any Gray-labeled constellation, we can define such a function by

$$f_{\text{PEP}}(\mathcal{S}, h, t) = e^{-\frac{r_c m E_b}{4N_0} d^2_E(h,t)}. \quad (33)$$

This bound should be compared to the bound z^h, with z given by the expression in (15). For PSK constellations, we can use the function, taken from [21, p. 273]

$$f_{\text{PEP}}(\mathcal{S}, h, t) = 2Q\left(\sqrt{\frac{r_c m E_b}{2N_0} d^2_E(h,t)}\right) \quad (34)$$

to obtain a bound even tighter than (33).

B. Modified WE

The average WEs given in (22) and (27) correspond to uniform interleavers, and therefore, the entries may take on noninteger values. However, for a code generated by a specific interleaver π, the WE values, which we denote by A^π_h, must be integers. In particular, if there exists a codeword of weight h, the corresponding term in the WE, A^π_h, is at least one. Therefore, if the term A_h in the average WE is less than one, there must be a code in the ensemble corresponding to some interleaver π, such that $A^\pi_h = 0$.

Following the approach in [27], we now find the probability that a particular code has a minimum Hamming distance $d_{\min}$, i.e., the probability that $A^\pi_h = 0$ for $0 < h < d_{\min}$.

Let $\mathcal{I}_h$ be the set of interleavers such that $A^\pi_h \neq 0$. Then, for the average WE, we have [27]

$$A_h \geqslant \frac{|\mathcal{I}_h|}{N!}. \quad (35)$$

For a particular Hamming distance h', we define

$$P_{h'} := \sum_{h=1}^{h'} A_h. \quad (36)$$

If h' is chosen such that $P_{h'} \leqslant 1$, then $P_{h'}$ can be interpreted as a probability. Now let $\mathcal{I}_{h>h'}$ be the set of interleavers yielding a minimum distance greater than h'. The size of $\mathcal{I}_{h>h'}$ is then lower bounded by [27]

$$|\mathcal{I}_{h>h'}| \geqslant N!(1 - P_{h'}) \quad (37)$$

or, expressed as a fraction of all possible interleavers

$$\frac{|\mathcal{I}_{h>h'}|}{N!} \geqslant 1 - P_{h'}. \quad (38)$$

Hence, the probability of randomly choosing an interleaver such that the resulting SCCC has a minimum Hamming distance $d_{\min}$ exceeding h' is lower bounded by

$$\Pr[d_{\min} > h'] > 1 - P_{h'} \quad (39)$$

where $P_{h'}$ is given by (36). We can then use (36) and (39) to determine the largest possible $d_{\min}$ such that a desired fraction, say 1/2, of all codes have a minimum distance of $d_{\min}$.

To reflect the case where the minimum distance is $d_{\min}$, the entries of A_h for $0 < h < d_{\min}$ are set to zero. The original values of A_h can either be added to $A_{d_{\min}}$ or discarded. Adding the entries overestimates the multiplicity of codewords of weight $d_{\min}$, yielding a slightly high error-probability bound.

Discarding the low-weight values keeps the original multiplicity, and yields a lower error-probability bound.

To reflect the (expected) minimum distance of the SCCC, we compute a modified WE as

$$\tilde{A}_h = \begin{cases} 1, & h = 0 \\ 0, & 1 < h < d_{\min} \\ \sum_{j=1}^{d_{\min}} A_j, & h = d_{\min} \\ A_h, & h > d_{\min}. \end{cases} \tag{40}$$

The construction of the modified IOWE and modified augmented IOWE is similar.

Using the modified augmented IOWE and the PEP function (33) or (34), we get the improved bound

$$P_b \leqslant \sum_{h=1}^{n} \sum_{w=1}^{k} \sum_{t=1}^{\lceil \frac{h}{2} \rceil} \frac{w}{k} \tilde{A}^{(c)}_{w,h,t} f_{\mathrm{PEP}}(\mathcal{S}, h, t) \tag{41}$$

where $\tilde{A}^{(c)}_{w,h,t}$ is the coefficient of the modified augmented IOWE of the concatenated code.

V. Simulation Results

In this section, we first show the simulated performance of a PA^2 system. We then show the performance of a system with an outer convolutional code and a single inner accumulate code in AWGN and correlated Rayleigh fading, and compare with the performance of several other systems presented in the literature. Finally, we show an example of the improved bound derived in Section IV.

A. *Performance of PA^2 System*

The PA^2 system consists of an outer SPC code followed by two inner interleaved accumulate codes. At the receiver, we now have three decoders, one for each constituent code. Since we have more than two decoders, the scheduling becomes an issue, i.e., the order in which we decode the different codes affects the overall system performance. We have simulated two of the decoding schedules proposed in [15], referred to as the "master–slave" and "serial" schedules. Let D_O, D_{I_1}, and D_{I_2} denote the decoders corresponding to the outer code, the first, and second inner code, respectively. In the master–slave setup, we let D_{I_1} act as master, and D_O and D_{I_2} as slaves. The decoding can then be visualized as

$$\underbrace{D_{I_2} \to D_{I_1} \to D_O}_{\text{Initialization}} \to \underbrace{D_{I_1} \to \begin{matrix} D_O \\ D_{I_2} \end{matrix}}_{\text{Main decoding loop}} \to \cdots.$$

In the serial decoding schedule, the decoding is performed in the following order:

$$\underbrace{D_{I_2} \to D_{I_1} \to D_O}_{\text{Initialization}} \to \underbrace{D_{I_1} \to D_{I_2} \to D_{I_1} \to D_O}_{\text{Main decoding loop}} \to \cdots.$$

In Fig. 3, we show the simulated performance of a PA^2 system with an outer $r_c = 2/3$ SPC, blocklength $N = 12288$, and 8-PSK modulation over an AWGN channel. The maximum number of decoding iterations is 500. According to *Theorem 4*, both the BER and WER should go to zero as the SNR exceeds a certain threshold, and this is verified by the simulations. Using the master–slave decoding schedule, the WER and BER decreases sharply at about 3.85 dB.[1] With the serial decoding schedule, we observe a significant performance degradation relative to the master–slave schedule, namely 0.8 dB at bit-error rate (BER) 10^{-5}. If the maximum number of decoding iterations is reduced to 200, a decoder using the serial decoding schedule suffers an additional performance loss of about 0.25 dB. For the master–slave decoding schedule, we can reduce the maximum number of decoding iterations to about 100 without any noticeable performance degradation.

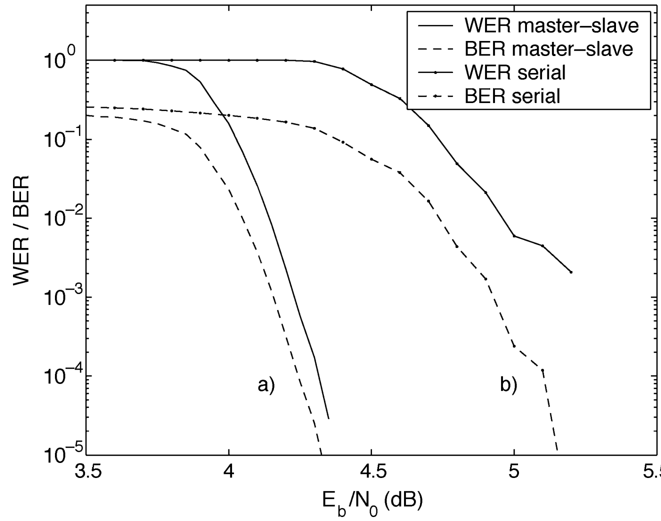

Fig. 3. Performance comparison between two decoding schedules for a PA^2 system in AWGN using an 8-PSK constellation and blocklength 4096 channel symbols. a) Master–slave decoding schedule. b) Serial decoding schedule.

B. *Performance Comparisons*

In the following, we use an outer convolutional code, a single accumulate code, an S-random interleaver with $S = 10$, and no channel interleaver Ψ before the mapper.

We first consider a comparison of our SCTCM architecture with that reported in [6] on an AWGN channel. Both systems use 16-QAM modulation, with a target spectral efficiency of 3 b/symbol. In contrast to our system, which incorporates an inner accumulate code and a Gray mapping, the system in [6] uses an inner code specifically matched to the constellation, along with set-partition labeling.

Both systems use a terminated outer $r_c = 3/4$, memory-two convolutional code with generator matrix

$$G(D) = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1+D & D & D & 0 \\ 1+D & 1 & 0 & 0 \end{pmatrix}. \tag{42}$$

The blocklength is 12288 information symbols, corresponding to 4096 channel symbols.

Fig. 4 shows the BER performance for the two systems, based upon computer simulation of an iterative decoder. Despite the simpler inner code and corresponding decoder structure, the proposed system achieves performance comparable to the alternative system. Also, it is worth noting that after 19 iterations, the BER of our scheme drops sharply at about 5.25 dB, even for this comparatively short blocklength.

[1]In [14], we compute the threshold for a message-passing decoder to be 3.82 dB for the AWGN channel.

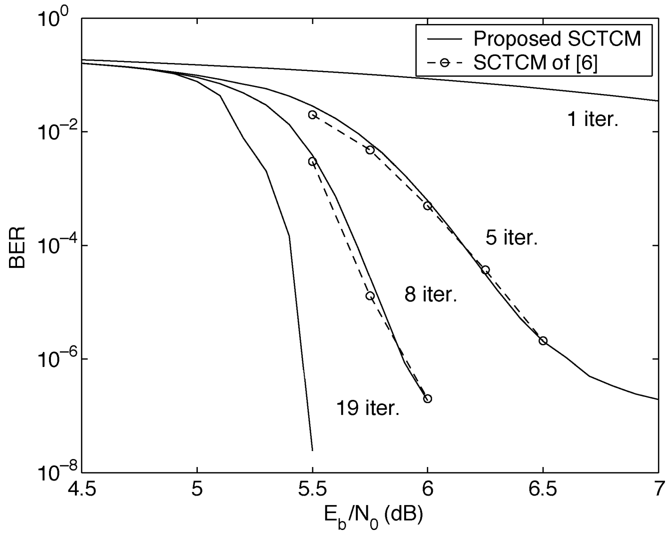

Fig. 4. Performance comparison in AWGN between the proposed SCTCM system and the SCTCM system of [6]. 16-QAM constellation and blocklength 4096 channel symbols.

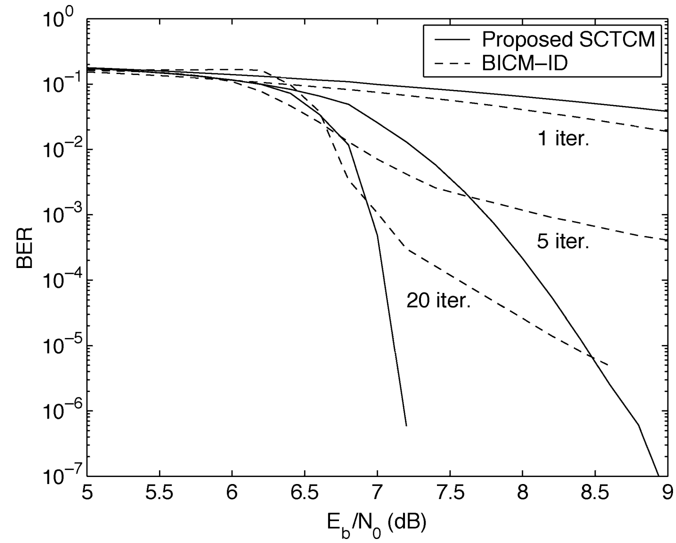

Fig. 5. Performance comparison in fading between the proposed SCTCM system and BICM-ID. 8-PSK constellation and blocklength 16384 channel symbols.

We next compare the BER performance of the proposed SCTCM system to that of BICM-ID [8] on a correlated flat Rayleigh fading channel. Both systems use 8-PSK modulation with a target spectral efficiency of 2 b/symbol. The BICM-ID system uses an outer rate-2/3, eight-state convolutional code with generator matrix

$$G(D) = \begin{pmatrix} 1 & 0 & \frac{1+D+D^2+D^3}{1+D+D^3} \\ 0 & 1 & \frac{1+D^2+D^3}{1+D+D^3} \end{pmatrix} \tag{43}$$

and a signal constellation with set-partition labeling.

The proposed SCTCM system uses an outer rate-2/3, four-state convolutional code with generator matrix

$$G(D) = \begin{pmatrix} 1 & 0 & \frac{1+D}{1+D+D^2} \\ 0 & 1 & \frac{1+D^2}{1+D+D^2} \end{pmatrix} \tag{44}$$

an inner (two-state) accumulate code, and a constellation with Gray labeling.

The receiver for the BICM-ID systems incorporates iterative demodulation, and a sequence detector for the outer code. The proposed system uses noniterative demodulation, and an iterative decoder for the concatenated codes.

Fig. 5 shows the results of a performance simulation using an interleaver of length $N = 49152$ b, corresponding to 16384 channel symbols. After 20 iterations, the proposed system shows a 1-dB performance gain, compared with the BICM-ID system at a BER of 10^{-5}. The BICM system exhibits a pronounced error floor, reflecting the low free Hamming distance of the constituent eight-state convolutional code, $d_{\text{free}} = 4$. In contrast, any error floor for the proposed SCTCM system is not visible in the simulation. This behavior is consistent with the observation that the expected minimum distance of the system, calculated according to the procedure outlined in Section IV-B, is 25.

It is difficult to make a rigorous comparison of the implementation complexity of the two systems. However, we note that the BICM-ID system uses an eight-state SISO sequence detector with 32 branches per stage, and the SCTCM system incorporates a four-state SISO detector with 16 branches per stage, and a two-state SISO detector with four branches per stage. The required storage in the SISO decoder(s) is given by the blocklength N times the number of states. Hence, for the BICM system, the storage requirement is $8N$, and for the SCTCM system, $(4+2)N$. Although the SCTCM system requires two SISO decoders, the computational complexity and storage requirements are comparable to, if not smaller than, those of the BICM system. Moreover, the BICM system, unlike the proposed SCTCM system, requires recomputation of the decoding metrics in each decoding iteration.

Finally, we remark that in our simulation of SCTCM systems, we obtained good performance using optimum distance profile (ODP) codes, which have fewer low-weight codewords than other convolutional codes of the same complexity [30, p. 112]. We also found that in the performance simulations, an S-random interleaver $10 \leqslant S \leqslant 15$ was sufficient to avoid low-weight codewords in the concatenated code.

C. Modified Union Bound

To illustrate the new bound derived in Section IV, we consider a system with an outer $r_c = 2/3$ terminated convolutional code with generator matrix

$$G(D) = \begin{pmatrix} 1 & 0 & \frac{1+D}{1+D+D^2} \\ 0 & 1 & \frac{1+D^2}{1+D+D^2} \end{pmatrix} \tag{45}$$

an interleaver of length $N = 384$ b, an inner accumulate code, and 8-PSK modulation. We compute the WE for the average code using a uniform interleaver, and list the WE coefficients for $h \leqslant 7$ in the upper row of Table IV. Using (39), we compute $\Pr[d_{\min} > 4] > 1 - P_4 = 1 - \sum_{h=1}^{4} A_h = 0.576$. Hence, the probability that $d_{\min} \geqslant 5$ is at least 0.576. In the second row of Table IV, we give the modified WE according to (40) for a concatenated code where we have set $h' = 4$, yielding $d_{\min} = 5$.

For short interleavers, it is possible to determine $d_{\min}$ by search. We investigated one particular S-random interleaver with $S = 10$, and found that the resulting SCCC has $d_{\min} = 6$.

TABLE IV
WE AND MODIFIED WE FOR A CONCATENATED CODE, $N = 384$

$h =$	0	1	2	3	4	5	6	7
A_h	1	0	0.046	0.121	0.257	0.523	1.080	2.312
$\tilde{A}_h$	1	0	0	0	0	0.948	1.080	2.312
$\tilde{A}'_h$	1	0	0	0	0	0	2.028	2.312

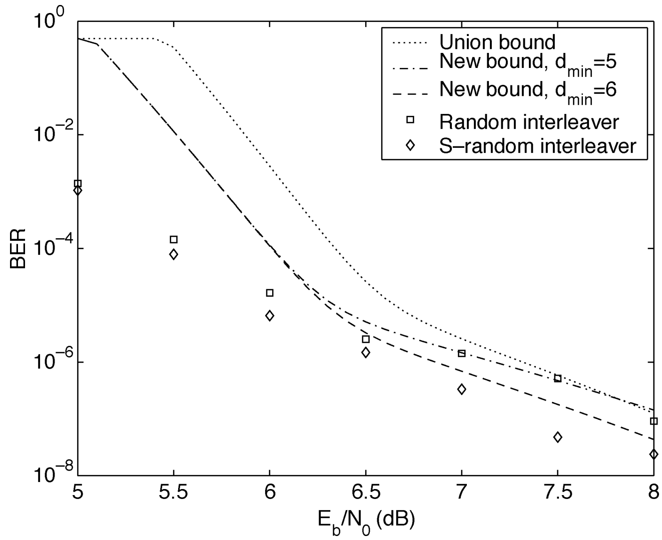

Fig. 6. Comparison of union bound and new bound to simulation results, $N = 384$.

For the code using this interleaver, we set $h' = 5$ and compute the modified weight spectrum, denoted $\tilde{A}'_h$. We show the modified WE for this code in the last row of Table IV.

In Fig. 6, we show the union bound computed based on A_h, and the new bound based on $\tilde{A}_h$ for $h' = 4$, and based on $\tilde{A}'_h$ for $h' = 5$, corresponding to $d_{\min} = 5$ and 6, respectively. We simulated two SCTCM systems, one using a randomly selected interleaver $(S = 0)$, and another using the aforementioned S-random interleaver. With the randomly selected interleaver, the system performance corresponds well with the bound for $h' = 4$. For the S-random interleaver, the simulated performance corresponds well with the bound for $h' = 5$. However, since the new bound is based on the union bound, it diverges for low SNR.

VI. CONCLUSIONS

An SCTCM system with an outer block code, inner accumulate code(s), and a higher-order Gray-labeled constellation has been introduced. We have derived coding theorems yielding SNR thresholds for reliable performance with ML decoding as the blocklength goes to infinity. For finite blocklengths, we have derived a new BER bound, which improves substantially over the conventional union bound. Though the proposed system uses a simple inner accumulate code, in AWGN, it performs comparably to serial concatenation schemes reported in the literature that use more complex rate-1 inner codes [6]. Over a flat Rayleigh fading channel, the proposed SCTCM system performs better than BICM-ID at low BERs, since the proposed system has a much steeper error floor.

APPENDIX
UNION-BOUND THRESHOLD FOR PA CODES

The union bound on P_b is

$$P_b \leqslant \sum_{h=1}^{N} \sum_{w=1}^{rN} \frac{w}{rN} A_{w,h}^{(c)} z^h \tag{46}$$

$$= \sum_{h=1}^{N} \sum_{w=1}^{rN} \sum_{h_o=0}^{N} \frac{w}{rN} \frac{A_{w,h_o}^{(o)} A_{h_o,h}^{(i)}}{\binom{N}{h_o}} z^h. \tag{47}$$

For the accumulate code $h_o \leqslant 2h$, and for the parity-check code $w \leqslant h_o$, h_o is even. Interchanging the order of summations, summing over w and inserting (19) and (21) gives

$$P_b \leqslant \sum_{h=1}^{N} \sum_{h_o=0}^{2h} \frac{h_o}{rN} \frac{\binom{\frac{N}{2}}{\frac{h_o}{2}} \binom{n}{\lfloor \frac{n}{2} \rfloor}^{\frac{h_o}{2}} \binom{N-h}{\lfloor \frac{h_o}{2} \rfloor} \binom{h-1}{\lceil \frac{h_o}{2} \rceil - 1}}{\binom{N}{h_o}} z^h. \tag{48}$$

Let $g := h_o/2$ and use $\binom{h-1}{g-1} \leqslant \binom{h}{g}$ to get

$$P_b \leqslant \sum_{h=1}^{N} \sum_{g=0}^{h} \frac{g}{rN} \frac{\binom{\frac{N}{2}}{g} \binom{n}{\lfloor \frac{n}{2} \rfloor}^{g} \binom{N-h}{g} \binom{h}{g}}{\binom{N}{2g}} z^h. \tag{49}$$

Let $K := \binom{n}{\lfloor n/2 \rfloor}$ and $\delta := h/N$. Consider the inner sum

$$\sum_{g=0}^{h} \frac{2g}{rN} \frac{\binom{\frac{N}{2}}{g} K^g \binom{N-h}{g} \binom{h}{g}}{\binom{N}{2g}} \tag{50}$$

$$\leqslant \sum_{g=0}^{h} \frac{2g}{rN} \frac{\left(\frac{Ne}{2g}\right)^g K^g \left(\frac{(N-h)e}{g}\right)^g \left(\frac{he}{g}\right)^g}{\left(\frac{N}{2g}\right)^{2g}} \tag{51}$$

$$= \sum_{g=0}^{h} \frac{2g}{rN} \left(\frac{2e^3 K h (1-\delta)}{g}\right)^g. \tag{52}$$

Maximizing $((2e^3 K h(1-\delta))/g)^g$ with respect to g gives

$$g = 2Ke^2 h(1-\delta) \tag{53}$$

and the inner sum becomes

$$\sum_{g=0}^{h} \frac{2g}{rN} e^{2e^2 K (1-\delta) h} = \frac{2h(h+1)}{2rN} e^{2Ke^2(1-\delta)h}. \tag{54}$$

Finally

$$P_b \leqslant \sum_{h=1}^{N} \frac{h(h+1)}{rN} e^{2Ke^2 h(1-\delta)} z^h. \tag{55}$$

As long as $e^{2Ke^2(1-\delta)}/z < 1$, then $P_b \to 0$ as $N \to \infty$.

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