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On Unequal Error Protection of Finite-Length LDPC Codes Over BECs: A Scaling Approach

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Abstract— **In this paper, we explore a novel approach to evaluate the inherent UEP (unequal error protection) properties of irregular LDPC (low-density parity-check) codes over BECs (binary erasure channels). Exploiting the finite-length scaling methodology, suggested by Amraoui *et. al.*, we introduce a scaling approach to approximate the bit erasure rates of variable nodes with different degrees in the waterfall region of the peeling decoder. Comparing the bit erasure rates obtained from Monte Carlo simulation with the proposed scaling approximations, we demonstrate that the scaling approach provides a close approximation for a wide range of code lengths (between** 1000 **and** 8000**). In view of the complexity associated with the numerical evaluation of the scaling approximation, we also derive simpler upper and lower bounds.**

I. Introduction

LDPC codes, with their superb error correction performance [1], [2], are among the most promising forward error correction schemes for EEP (equal error protection). However, many practical applications, such as robust transmission of video/image, require UEP. The near capacity performance of EEP-LDPC codes over BECs suggests LDPC codes could possibly also offer good performance in UEP scenarios [3]. Consequently, the idea of designing LDPC codes for UEP has been studied in many papers, including [3] - [6]. In [4]-[6], UEP is provided with partially regular LDPC codes and unequal density evolution, while Plotkin-type constructions are employed to design UEP-LDPC codes in [3]. Here, exploiting the finite-length scaling methodology, we introduce a novel approach to analytically evaluate the inherent UEP properties in (finite-length) irregular LDPC codes.

It is known that in an irregular LDPC code transmitted over a BEC, variable nodes with larger degrees provide better protection than variable nodes with smaller degrees. In this work, extending the results from [7], [8], we will quantify the effect of the variable-node degree on the bit erasure rate of the variable nodes in the waterfall region.

The rest of this paper is organized as follows. Section II briefly establishes the required background and notation. Section III outlines the derivation of the bit erasure rate (the scaling approximation, upper bounds, and lower bounds) based on the finite-length scaling of LDPC codes. Numerical simulations and performance analysis are presented in Section IV, and finally Section V concludes the paper.

II. Preliminaries

In this section, we first establish some background and notation on LDPC codes, and then we present a concise outline of the peeling decoding algorithm, the traditional algorithm for decoding LDPC codes over BECs.

A. Low-Density Parity-Check Code Ensembles

LDPC codes are linear block codes which have a sparse parity-check matrix. One of the main reasons for the great interest in these codes is their superb error correction performance under low-complexity sub-optimal message-passing algorithms, such as the peeling algorithm [9].

Throughout this paper, we will define LDPC code ensembles by three parameters:

1. The block length of the code, n.
2. The edge perspective variable node degree distribution, $\lambda(x) = \sum_{i=1}^{\mathtt{l}_{\max}} \lambda_i x^{i-1}$ with $\mathtt{l}_{\max}$ representing the maximum variable, or left, degree.
3. The edge perspective check node degree distribution, $\rho(x) = \sum_{i=1}^{\mathtt{r}_{\max}} \rho_i x^{i-1}$, where $\mathtt{r}_{\max}$ denotes the maximum check, or right, degree.

Similar to edge perspective degree distributions, node perspective degree distributions can also be defined for both variable and check nodes. Define $\Lambda(x) \triangleq \sum_{i=1}^{\mathtt{l}_{\max}} \Lambda_i x^i = \frac{\int_0^x \lambda(u)du}{\int_0^1 \lambda(u)du}$ as the variable node perspective degree distribution, where Λ_i for $1 \leq i \leq \mathtt{l}_{\max}$ represents the ratio of degree-i variable nodes relative to the total number of variable nodes in the original graph. Further, let $P(x) \triangleq \sum_{i=1}^{\mathtt{r}_{\max}} P_i x^i = \frac{\int_0^x \rho(u)du}{\int_0^1 \rho(u)du}$ denote the check node perspective degree distribution, where P_i for $1 \leq i \leq \mathtt{r}_{\max}$ represents the ratio of degree-i check nodes relative to the total number of check nodes in the original graph.

B. Peeling Algorithm

The peeling algorithm was first introduced in [10] for decoding graph-based codes over BECs. In this algorithm, upon receiving the channel outputs, the known variable nodes send their values to the check nodes connected to them and are removed from the graph. The decoder proceeds by looking for a degree-one check node, i.e., a check node such that all but one of the variable nodes connected to it are known. If it finds one, it computes the value of the unknown variable node, propagates the value of the variable node to all check nodes connected to the variable node, and then removes it from the graph. If the decoder does not find a degree-one check node, then the decoding stops. At this point, the residual graph does not have a degree-one check node. So, two cases are possible: either the graph is empty, i.e., the decoding is successful, or the graph is not empty and all the remaining check nodes have degrees greater than one, i.e., the decoding has failed.

Consider decoding an LDPC code with the peeling algorithm over a BEC of erasure probability ϵ. The threshold erasure parameter, ϵ^*, is the maximum value of ϵ such that the probability of decoder failure tends to zero for all $\epsilon < \epsilon^*$ as n tends to infinity. Denote the ratio of the number of erased check-to-variable messages to the number of edges in the original graph by y. Let r_1 be the ratio of the number of degree-one check nodes to the number of edges in the original graph. A critical point is defined as a point where both $r_1(y)$ and its derivative, with respect to y, vanish. In the rest of this paper, we will concentrate on LDPC codes with one nontrivial critical point[1].

III. Computation of the Bit Erasure rate: A Scaling Approach

A. Preliminaries

The peeling algorithm can be considered as a first-order Markov process. However, deriving the error probability directly from the Markov model is computationally intractable [11].

Amraoui *et. al.*, in their papers [7], [8], and [12], present an abstract setting which allows them to approximate the scaled process of the number of degree-one check nodes in the peeling algorithm by a Brownian motion with a parabolic drift. Consequently, we will also exploit the same setting.

Denote each step of the decoding process with a discrete time $t \in \mathbb{N}$. Let $X_t^{(0)}$ be the number of degree-one check nodes in the residual graph at time t. Define the first passage time of the process $\{X_t^{(0)}\}$ as follows:

$$\tau \triangleq \sup_t \left\{ t \middle| X_\gamma^{(0)} \geq 0 \ \ \forall \gamma \leq t \right\}, \tag{1}$$

where $X_\gamma^{(0)}$ represents the number of edges attached to degree-one check nodes at time γ. Moreover, define $P_E^{(d_v)}$ as the erasure probability for degree-d_v variable nodes. Consequently,

$$P_E^{(d_v)} \quad = \quad \frac{E\{L_{d_v}(\tau)\}}{d_v n \Lambda_{d_v}}, \tag{2}$$

where $L_{d_v}(\tau)$ represents the number of edges connected to degree-d_v variable nodes at the first passage time. Note that the original graph contains $n\Lambda_{d_v}$ degree-d_v variable nodes and $d_v n \Lambda_{d_v}$ edges connected to degree-d_v variable nodes. By Bayes' rule, (2) can be rewritten as

$$\begin{aligned} P_E^{(d_v)} = \quad & Prob\{\text{decoding failure}\}.\frac{E\{L_{d_v}(\tau)\big|\text{decoding failure}\}}{d_v n \Lambda_{d_v}} + \\ & Prob\{\text{successful decoding}\}.\frac{E\{L_{d_v}(\tau)\big|\text{successful decoding}\}}{d_v n \Lambda_{d_v}}. \end{aligned} \tag{3}$$

[1]Note that $r_1(y)$ and its derivative always vanish at $y = 0$.

When the decoding is successful, the residual graph is empty, i.e., $L_{d_v}(\tau) = 0$. As a result, $E\left\{L_{d_v}(\tau)\big|\text{successful decoding}\right\} = 0$, and (3) can be reformulated as follows:

$$P_E^{(d_v)} \quad = \quad Prob\{\text{decoding failure}\}.\frac{E\{L_{d_v}(\tau)\big|\text{decoding failure}\}}{d_v n \Lambda_{d_v}}. \tag{4}$$

The problem of the computation of $Prob\{\text{decoding failure}\}$ has been addressed in [13]. Hence, to derive $P_E^{(d_v)}$, we will focus on computation of $E\{L_{d_v}(\tau)\big|\text{decoding failure}\}$. Let's first consider Lemma 1:

Lemma 1: There exist positive constants Ω_1, Ω_2, δ_0, and a function $n_0(\delta)$, such that, for any $\delta > \delta_0$, and $n > n_0(\delta)$,

$$Prob\{|\tau - t^*| \geq \delta n^{6/7}\} \leq \Omega_1 e^{-\Omega_2 \delta^2}, \tag{5}$$

where t^* is the asymptotic critical time of the process $\{X_t^{(0)}\}$, i.e., the time when both $r_1(y)$ and its derivative vanish.

The proof of this lemma is deferred to the Appendix.

Considering the fact that $|\tau - t^*|$ is small on the scale of n, while $|\tau - t^*|$ is large on the scale $O(1)$ of a single step, we will compute the probability density function of the first passage time with a 'continuum' approach. Let's define the rescaled trajectory $\omega(\cdot) \in \mathbb{R}$ as follows (similar to [13]):

$$\omega\left(\Gamma_1(t - t^*)\right) \triangleq \Gamma_2 X_t^{(0)}, \tag{6}$$

with

$$\begin{aligned} \Gamma_1 &= f_*^{(00)} \beta^{-2} n^{-2/3} \Lambda'(1)^{-2} \left(\frac{\partial r_1}{\partial \epsilon}\bigg|_* \right)^{-2} \\ \Gamma_2 &= \beta^{-1} n^{-1/3} \Lambda'(1)^{-1} \left| \left(\frac{\partial r_1}{\partial \epsilon}\bigg|_* \right)^{-1} \right|, \end{aligned}$$

where

$$\begin{aligned} f_*^{(00)} &= \left(\frac{y^{*2}\lambda''(y^*)}{\lambda(y^*)} - \frac{y^{*2}\lambda'(y^*)^2}{\lambda(y^*)^2} \right) \frac{r_2^{*2}}{e^{*2}} + \frac{y^*\lambda'(y^*)}{\lambda(y^*)} \frac{r_2^*}{e^*} \\ \beta &= \left[\frac{\epsilon^{*4} r_2^{*2} (\epsilon^*\lambda'(y^*)^2 r_2^* - x^*(\lambda''(y^*) r_2^* + \lambda'(y^*) x^*))^2}{\Lambda'(1)^2 \rho'(\bar{x}^*)^3 x^{*10} (2\epsilon^*\lambda'(y^*)^2 r_3^* - \lambda''(y^*) r_2^* x^*)} \right]^{1/3}. \end{aligned} \tag{7}$$

Note that $e^* = \epsilon^* y^* \lambda(y^*)$, y^* denotes the fractional rate of the erasures in check-to-variable messages at the critical point, and $\frac{\partial r_1}{\partial \epsilon}\big|_*$ is the partial derivative of r_1 with respect to the erasure rate at the critical point. Also,

$$\begin{aligned} x^* &= \epsilon^*\lambda(y^*) \\ \bar{x}^* &= 1 - x^* \\ r_i^* &= \begin{cases} \epsilon^*\lambda(y^*)\left[y^* - 1 + \rho(1 - \epsilon^*\lambda(y^*))\right] & \text{if } i = 1 \\ \displaystyle\sum_{m \geq j \geq i} (-1)^{i+j} \binom{j-1}{i-1} \binom{m-1}{j-1} \rho_m (\epsilon^*\lambda(y^*))^j & \text{otherwise} \end{cases} \end{aligned} \tag{8}$$

B. Scaling Approximations

It can be shown that $\omega(\cdot)$ can be described as a two-sided Brownian motion with a parabolic drift [13]:

$$\omega(\bar{\theta}) = \omega(0) + B(\bar{\theta}) + \frac{\bar{\theta}^2}{2}, \tag{9}$$

where $B(\bar{\theta})$ is a two-sided Brownian motion, with $B(0) = 0$. Let's denote the first passage time of the rescaled trajectory by $\bar{\theta}_\tau$; i.e.,

$$\bar{\theta}_\tau \triangleq \sup_{\bar{\theta}} \{\bar{\theta} | \omega(\gamma) \geq 0 \ \ \forall \gamma \leq \bar{\theta}\}. \tag{10}$$

Clearly,

$$\bar{\theta}_\tau = \Gamma_1(\tau - t^*). \tag{11}$$

It can be shown that [13]

$$L_{d_v}(t) = L_{d_v}(t^*) - \frac{d_v l^*_{d_v}}{e^*}(t - t^*), \tag{12}$$

where $l^*_{d_v} = \epsilon^* \lambda_{d_v} y^{*d_v}$. As a result,

$$\begin{aligned} L_{d_v}(\tau) &= L_{d_v}(t^*) - \tfrac{d_v l^*_{d_v}}{e^*}(\tau - t^*) \\ &= L_{d_v}(t^*) - \tfrac{d_v l^*_{d_v}}{e^* \Gamma_1}\bar{\theta}_\tau. \end{aligned} \tag{13}$$

Consequently, one can show that [11]

$$\begin{aligned} E\{L_{d_v}(\tau) | \text{decoding failure}\} \;\; = \;\; & d_v n \Lambda_{d_v} \epsilon^* y^{*d_v} - \\ & \tfrac{d_v l^*_{d_v}}{e^* \Gamma_1} E\{\bar{\theta}_\tau | \text{decoding failure}\}. \end{aligned} \tag{14}$$

Let's define the conditional cumulative probability density function of $\bar{\theta}_\tau$ as

$$F_{\bar{\theta}_\tau}(\gamma | \text{decoding failure}) \triangleq Prob\{\bar{\theta}_\tau \leq \gamma | \text{decoding failure}\}. \tag{15}$$

It is not hard to see that

$$\begin{aligned} E\{\bar{\theta}_\tau | \text{decoding failure}\} = \;\; & \textstyle\int_0^\infty 1 - F_{\bar{\theta}_\tau}(\gamma | \text{decoding failure}) d\gamma - \\ & \textstyle\int_{-\infty}^0 F_{\bar{\theta}_\tau}(\gamma | \text{decoding failure}) d\gamma. \end{aligned} \tag{16}$$

Define

$$g(\gamma) \;\; \triangleq Prob\{\omega(\bar{\theta}) > 0 \ \ \forall \bar{\theta} \leq \gamma\}. \tag{17}$$

From the definition of the conditional cumulative density function, we have

$$\begin{aligned} F_{\bar{\theta}_\tau}(\gamma | \text{decoding failure}) \;\; &= \tfrac{Prob\{\bar{\theta}_\tau \leq \gamma, \text{ decoding failure}\}}{Prob\{\text{decoding failure}\}} \\ &= \tfrac{1 - g(\gamma)}{Prob\{\text{decoding failure}\}}. \end{aligned} \tag{18}$$

We will proceed by dividing the problem of computation of $g(\gamma)$ into two cases. The first case corresponds to scenarios where $\gamma \leq 0$, and the second case corresponds to scenarios where $\gamma > 0$.

Case 1 ($\gamma \leq 0$). By the definition of $g(\gamma)$,

$$\begin{aligned} g(\gamma) &= Prob\{\omega(\bar{\theta}) > 0 \ \ \forall \bar{\theta} \leq \gamma\} \\ &= \textstyle\int_0^\infty Prob\{\omega(\bar{\theta}) > 0 \forall \bar{\theta} < \gamma | \omega(\gamma) = \xi\} P\{\omega(\gamma) = \xi\} d\xi, \end{aligned} \tag{19}$$

where $\omega(\gamma)$ has a Gaussian distribution [13]. We denote the mean and the variance of $\omega(\gamma)$ as follows:

$$\begin{aligned} \mu(\gamma) \;\; &\triangleq E\{\omega(\gamma)\} \\ \sigma^2(\gamma) \;\; &\triangleq E\{\omega^2(\gamma)\} - E^2\{\omega(\gamma)\}. \end{aligned} \tag{20}$$

As a result,

$$P\{\omega(\gamma) = \xi\} = \frac{1}{\sqrt{2\pi}\sigma(\gamma)} e^{-\frac{(\xi - \mu(\gamma))^2}{2\sigma^2(\gamma)}}. \tag{21}$$

One can show that

$$\begin{aligned} \mu(\gamma) \;\; &= \mu(0) + \tfrac{\gamma^2}{2} \\ \sigma^2(\gamma) \;\; &= \sigma^2(0) + |\gamma|. \end{aligned} \tag{22}$$

Define a process $\bar{\omega}(\cdot)$ as follows:

$$\begin{aligned} \bar{\omega}(\bar{\theta}) \;\; &\triangleq -\omega(\bar{\theta}) \\ &= -\omega(0) - \tfrac{\bar{\theta}^2}{2} - B(\bar{\theta}). \end{aligned} \tag{23}$$

It is not hard to see that $-B(\bar{\theta})$ is also a double-sided Brownian motion process, so we will define $\bar{B}(\bar{\theta}) \triangleq -B(\bar{\theta})$, where $\bar{B}(\bar{\theta})$ is a double-sided Brownian motion process starting at $\bar{B}(0) = 0$. We can show that

$$\begin{aligned} & Prob\{\omega(\bar{\theta}) > 0 \ \ \forall \bar{\theta} < \gamma | \omega(\gamma) = \xi\} = \\ & \qquad Prob\{\bar{\omega}(\bar{\theta}) < 0 \ \ \forall \bar{\theta} > -\gamma | \bar{\omega}(-\gamma) = -\xi\}. \end{aligned} \tag{24}$$

The righthand side of Eq. (24) corresponds to the probability that the maximum of a two-sided Brownian motion with a parabolic drift is less than zero. This problem has been studied in [14]. Adapting the results from [14] to our situation, we have

$$\begin{aligned} & Prob\{\omega(\bar{\theta}) > 0 \ \ \forall \bar{\theta} < \gamma | \omega(\gamma) = \xi\} = 2^{-\frac{2}{3}} e^{\frac{-\gamma^3 + 6\gamma\xi}{6}} \\ & \quad \textstyle\int_{-\infty}^{\infty} e^{i\nu\gamma} \frac{\mathrm{Ai}(i2^{1/3}\nu)\mathrm{Bi}(i2^{1/3}\nu + 2^{1/3}\xi) - \mathrm{Bi}(i2^{1/3}\nu)\mathrm{Ai}(i2^{1/3}\nu + 2^{1/3}\xi)}{\mathrm{Ai}(i2^{1/3}\nu)} d\nu. \end{aligned} \tag{25}$$

where $i = \sqrt{-1}$, and $\mathrm{Ai}(\cdot)$, and $\mathrm{Bi}(\cdot)$, are the Airy functions defined on page 446, of [15]. Putting everything together, for $\gamma \leq 0$, we have

$$\begin{aligned} & g(\gamma) = \tfrac{2^{-\frac{2}{3}}}{\sqrt{2\pi}\sigma(\gamma)} \textstyle\int_0^\infty e^{-\frac{(\xi - \mu(\gamma))^2}{2\sigma^2(\gamma)} + \frac{6\gamma\xi - \gamma^3}{6}} \\ & \quad \textstyle\int_{-\infty}^{\infty} e^{i\nu\gamma} \frac{\mathrm{Ai}(i2^{1/3}\nu)\mathrm{Bi}(i2^{1/3}\nu + 2^{1/3}\xi) - \mathrm{Bi}(i2^{1/3}\nu)\mathrm{Ai}(i2^{1/3}\nu + 2^{1/3}\xi)}{\mathrm{Ai}(i2^{1/3}\nu)} d\nu d\xi. \end{aligned} \tag{26}$$

Now let's focus on the second case, i.e., when $\gamma \geq 0$.

Case 2 ($\gamma > 0$). Since $B(0) = 0$, computation of $g(\gamma)$ when $\gamma > 0$ requires a slightly different approach. It's not hard to see that

$$\begin{aligned} g(\gamma) = \textstyle\int_0^\infty \quad & P\{\omega(0) = \xi\} Prob\{\omega(\bar{\theta}) > 0 \ \ \forall \bar{\theta} < 0 | \omega(0) = \xi\} \\ & Prob\{\omega(\bar{\theta}) > 0 \ \ \forall 0 < \bar{\theta} < \gamma | \omega(0) = \xi\} d\xi. \end{aligned} \tag{27}$$

Pursuing steps similar to those used in the calculation of $g(\gamma)$ in the first case, one can easily show that

$$\begin{aligned} & Prob\{\omega(\bar{\theta}) > 0 \ \ \forall \bar{\theta} < 0 | \omega(0) = \xi\} = 2^{-\frac{2}{3}} \\ & \quad \textstyle\int_{-\infty}^{\infty} \frac{\mathrm{Ai}(i2^{1/3}\nu)\mathrm{Bi}(i2^{1/3}\nu + 2^{1/3}\xi) - \mathrm{Bi}(i2^{1/3}\nu)\mathrm{Ai}(i2^{1/3}\nu + 2^{1/3}\xi)}{\mathrm{Ai}(i2^{1/3}\nu)} d\nu, \end{aligned} \tag{28}$$

and

$$P\{\omega(0) = \xi\} = \frac{1}{\sqrt{2\pi}\sigma(0)} e^{-\frac{(\xi - \mu(0))^2}{2\sigma^2(0)}}. \tag{29}$$

One can further see that

$$\begin{aligned} & Prob\{\omega(\bar{\theta}) > 0 \ \ \forall 0 < \bar{\theta} < \gamma | \omega(0) = \xi\} = \\ & \qquad Prob\{\bar{\omega}(\bar{\theta}) < 0 \ \ \forall 0 < \bar{\theta} < \gamma | \bar{\omega}(0) = -\xi\}. \end{aligned} \tag{30}$$

Again using results from [14], we have

$$\begin{aligned} & Prob\{\omega(\bar{\theta}) > 0 \ \ \forall 0 < \bar{\theta} < \gamma | \omega(0) = \xi\} = \\ & \quad 1 - \textstyle\int_0^\gamma e^{-\frac{\vartheta^3}{6}} h_{\frac{1}{2},\xi}(\vartheta) d\vartheta, \end{aligned} \tag{31}$$

where the function $h_{\frac{1}{2},\xi}(\vartheta)$ has the Laplace transform

$$\begin{aligned} H_{\frac{1}{2},\xi}(\omega) &= \int_0^\infty e^{-\omega\vartheta} h_{\frac{1}{2},\xi}(\vartheta) d\vartheta \\ &= Ai\left(2^{1/3}(\xi+\omega)\right) / Ai\left(2^{1/3}\omega\right). \end{aligned} \tag{32}$$

Putting everything together, we deduce that, when $\gamma > 0$,

$$g(\gamma) = \frac{2^{-\frac{2}{3}}}{\sqrt{2\pi}\sigma(0)} \int_0^\infty e^{-\frac{(\xi-\mu(0))^2}{2\sigma^2(0)}} \left(1 - \int_0^\gamma e^{-\frac{\vartheta^3}{6}} h_{\frac{1}{2},\xi}(\vartheta) d\vartheta\right) \int_{-\infty}^{\infty} \frac{\mathrm{Ai}(i2^{1/3}\nu)\mathrm{Bi}(i2^{1/3}\nu + 2^{1/3}\xi) - \mathrm{Bi}(i2^{1/3}\nu)\mathrm{Ai}(i2^{1/3}\nu + 2^{1/3}\xi)}{\mathrm{Ai}(i2^{1/3}\nu)} d\nu d\xi. \tag{33}$$

C. Bounds on the Bit Erasure Rate

Due to the complexity of the numerical evaluation of $g(\gamma)$, we further derived upper and lower bounds on $g(\gamma)$. These bounds can be employed in the computation of upper and lower bounds on the scaling approximation to the bit erasure rate. It can be shown that, for $\gamma \leq 0$ [11],

$$\begin{aligned} &1 - Q\left(\frac{\hat{\mu}_\gamma(0)}{\hat{\sigma}_\gamma(0)}\right) - \frac{1}{\sqrt{2\pi}\hat{\sigma}_\gamma(0)} e^{-\frac{\hat{\mu}_\gamma^2(0)}{2\hat{\sigma}_\gamma^2(0)}} \leq g(\gamma) \leq \\ &\min_{M>0} \left[\left(1 - Q\left(\frac{\tilde{\mu}_\gamma(0,M)}{\tilde{\sigma}_\gamma(0)}\right) - \frac{(1+M)^{-1/3}}{\sqrt{2\pi}\tilde{\sigma}_\gamma(0)} e^{-\frac{\tilde{\mu}_\gamma^2(0,M)}{2\tilde{\sigma}_\gamma^2(0)}} \right)^{\frac{1}{2}} \left(1 - Q\left(\frac{\tilde{\mu}_\gamma(0,M)}{\tilde{\sigma}_\gamma(0)}\right)\right)^{\frac{1}{2}} \right], \end{aligned} \tag{34}$$

with

$$\begin{aligned} \hat{\mu}_\gamma(0) &= \Gamma_2(\epsilon^* - \epsilon) \frac{\partial R_1}{\partial \epsilon}|_* + \frac{1}{2}\gamma^2 \\ \hat{\sigma}_\gamma^2(0) &= \frac{\alpha^2 \Gamma_2^2}{n} \left(\frac{\partial R_1}{\partial \epsilon}|_*\right)^2 + |\gamma| \\ \tilde{\mu}_\gamma(0,M) &= \Gamma_2(\epsilon^* - \epsilon) \frac{\partial R_1}{\partial \epsilon}|_* + \frac{\gamma^2}{2}\left(1 + \frac{1}{M}\right) \\ \tilde{\sigma}_\gamma^2(0) &= \frac{\alpha^2 \Gamma_2^2}{n} \left(\frac{\partial R_1}{\partial \epsilon}|_*\right)^2 + |\gamma|. \end{aligned} \tag{35}$$

where $\frac{\partial R_1}{\partial \epsilon}|_*$ represents the partial derivative of the number of degree-one check nodes with respect to the erasure rate at the critical point, and α is the scaling factor defined in [13]:

$$\alpha = \left(\frac{\rho^2(\bar{x^*}) - \rho(\bar{x^*}^2) + \rho'(\bar{x^*})(1 - 2x^*\rho(\bar{x^*})) - \bar{x^*}^2 \rho'(\bar{x^*}^2)}{\Lambda'(1)\lambda^2(y^*)\rho'(\bar{x^*})^2} + \frac{(\epsilon^*)^2\lambda^2(y^*) - (\epsilon^*)^2\lambda((y^*)^2) - (y^*)^2(\epsilon^*)^2\lambda'((y^*)^2)}{\Lambda'(1)\lambda^2(y^*)} \right)^{\frac{1}{2}}. \tag{36}$$

For $\gamma \geq 0$, it can be shown that [11],

$$\begin{aligned} &Prob\{\text{successful decoding}\} \leq g(\gamma) \leq \\ &Prob\{\text{successful decoding}\} \frac{\left(1 - Q\left(\frac{\mu(\gamma)}{\sigma(\gamma)}\right)\right)}{1 - Q\left(\frac{\hat{\mu}_\gamma(0)}{\hat{\sigma}_\gamma(0)}\right) - \frac{1}{\sqrt{2\pi}\hat{\sigma}_\gamma(0)} e^{-\frac{\hat{\mu}_\gamma^2(0)}{2\hat{\sigma}_\gamma^2(0)}}}, \end{aligned} \tag{37}$$

where the problem of the computation of $Prob\{\text{successful decoding}\}$ has been addressed in [13].

IV. Numerical Analysis

To empirically investigate the accuracy of the scaling approximations, we performed Monte Carlo simulations of random LDPC code ensembles of various lengths and degree distributions over BECs.

In Figures 1 and 2, we compare the simulated bit erasure rates of degree-3 and degree-9 variable nodes with the results

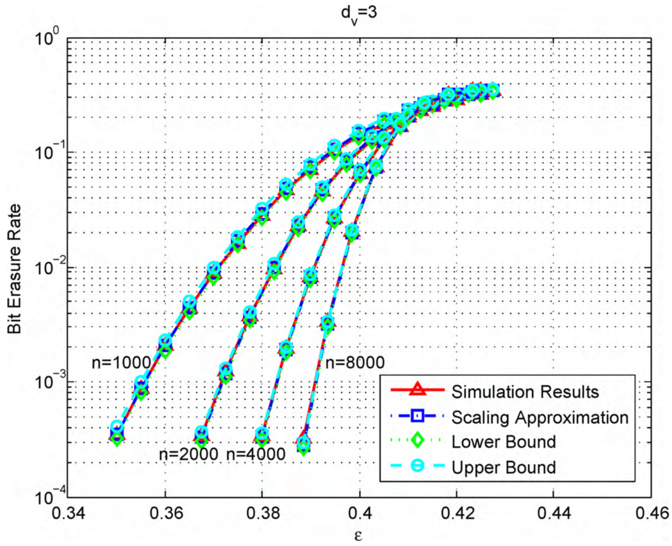

Fig. 1. Bit erasure rate curves of degree-3 variable nodes of LDPC($n, \lambda(x) = \frac{2}{5}x^2 + \frac{12}{25}x^3 + \frac{3}{25}x^8, \rho(x) = \frac{7}{15}x^6 + \frac{8}{15}x^7$) codes when used over binary erasure channel of erasure probability ϵ.

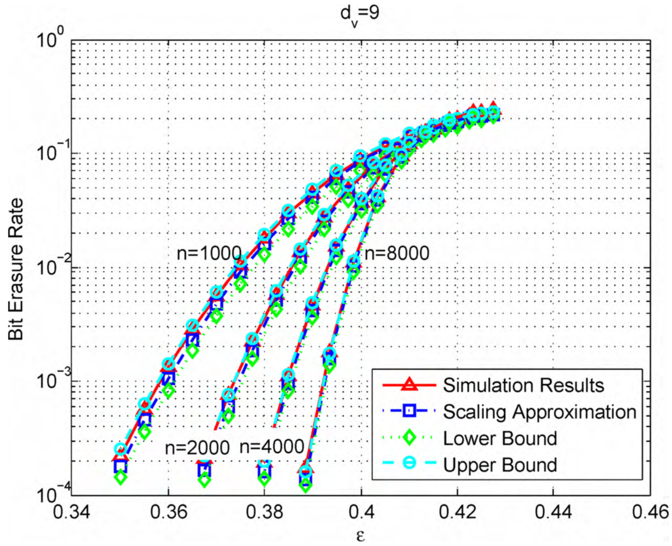

Fig. 2. Bit erasure rate curves of degree-9 variable nodes of LDPC($n, \lambda(x) = \frac{2}{5}x^2 + \frac{12}{25}x^3 + \frac{3}{25}x^8, \rho(x) = \frac{7}{15}x^6 + \frac{8}{15}x^7$) codes when used over binary erasure channel of erasure probability ϵ.

obtained from our scaling approximation, and both upper and lower bounds for LDPC code ensembles with parameters $\lambda(x) = \frac{2}{5}x^2 + \frac{12}{25}x^3 + \frac{3}{25}x^8$ and $\rho(x) = \frac{7}{15}x^6 + \frac{8}{15}x^7$. In each graph, we present results for codes of length $n = 1000$, 2000, 4000, and 8000. Note that the simulation results are quite close to the scaling approximation. Furthermore, by increasing the code length, the upper and lower bounds also become very close to the simulation results.

V. Concluding Remarks

In this paper, we investigated the UEP properties of finite-length irregular LDPC codes in the waterfall region of the peeling decoder. We introduced a scaling approach to compute the bit erasure rate of variable nodes with a given degree over BECs. Simulation results showed that, for a wide range of code lengths, scaling approximations provide a very close estimate to the bit erasure rate. We further derived upper and lower bounds on the scaling approximation to the bit erasure rate. We showed that these bounds are quite tight, and by increasing the length of the code, they become very close to the Monte Carlo simulation results.

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Appendix
Proof of Lemma 1

In this appendix, we present the proof of Lemma 1. We start by showing that with high probability, $|X_{t^*}^{(0)}| < n^{2/3}$.Then, we show that, for any time t such that $X_t^{(0)} - X_{t^*}^{(0)} < n^{2/3}$, with high probability we have $|t - t^*| < n^{6/7}$. Combining these two results, we deduce that for any time t, such that $|t - t^*| \geq n^{6/7}$, with high probability $X_t^{(0)}$ is bounded away from zero, which concludes the proof of the Lemma.

It can be shown that (see [[13], Lemma 4] for the proof)

$$Prob\left\{|X_{t^*}^{(0)} - \bar{X}_{t^*}^{(0)}| \geq n^{7/12}\right\} \leq 2e^{-\frac{n^{7/6}}{2\Omega_0 t^*}}, \quad (38)$$

where Ω_0 is a positive constant and independent of n. Consider the fact that $\bar{X}_{t^*}^{(0)}$ is of order $O(\sqrt{n})$. Note that this corresponds to the erasure probability ϵ being in a critical window $\epsilon - \epsilon^* = O(n^{-1/2})$ [13]. One can show that, for values of n large enough,

$$Prob\left\{|X_{t^*}^{(0)}| \geq n^{2/3}\right\} \leq 2e^{-\frac{n^{7/6}}{2\Omega_0 t^*}}, \quad (39)$$

which implies the fact that as $n \to \infty$, the absolute value of $X_{t^*}^{(0)}$ with high probability is less than or equal to $n^{2/3}$. Let's define (similar to [13])

$$Y_{t-t^*} \triangleq \frac{1}{\kappa_4}\left(X_t^{(0)} - X_{t^*}^{(0)}\right), \quad (40)$$

where κ_4 is a positive constant. Let $t_l \triangleq 2^l n^{6/7}\delta^{2/3}$, where δ is a positive constant. For the sake of simplicity, here we will focus on the case $t > t^*$. The case $t < t^*$ can be treated similarly. Then,

$$\begin{aligned} &Prob\{\min_{t \geq n^{\frac{6}{7}}\delta^{\frac{2}{3}}} Y_t \leq n^{\frac{2}{3}}\delta^{\frac{4}{3}}\} \leq \\ &\quad \textstyle\sum_{l=0}^{\infty} Prob\{\min_{t_l \leq t < t_{l+1}} Y_t \leq n^{\frac{2}{3}}\delta^{\frac{4}{3}}\}, \end{aligned} \quad (41)$$

or, equivalently,

$$\begin{aligned} &Prob\{\min_{t \geq n^{\frac{6}{7}}\delta^{\frac{2}{3}}} Y_t \leq n^{\frac{2}{3}}\delta^{\frac{4}{3}}\} \leq \\ &\quad \textstyle\sum_{l=0}^{\infty} Prob\{\min_{t_l \leq t < t_{l+1}} Y_t - \frac{1}{n}t^2 + \frac{\kappa_5\delta}{\sqrt{n}}t \leq \\ &\quad\quad n^{\frac{2}{3}}\delta^{\frac{4}{3}} - \frac{1}{n}t_l^2 + \frac{\kappa_5\delta}{\sqrt{n}}t_{l+1}\}, \end{aligned} \quad (42)$$

where κ_5 is a positive constant. Adapting a result on the concentration properties of the $X_t^{(0)}$ from [[13], Eq. (2.27)], we can rewrite (42) as

$$\begin{aligned} &Prob\{\min_{t \geq n^{\frac{6}{7}}\delta^{\frac{2}{3}}} Y_t \leq n^{\frac{2}{3}}\delta^{\frac{4}{3}}\} \leq \\ &\quad \Omega_1 \textstyle\sum_{l=0}^{\infty} \exp\left\{-\frac{\Omega_2}{t_{l+1}}\left(n^{\frac{2}{3}}\delta^{\frac{4}{3}} - \frac{1}{n}t_l^2 + \frac{\kappa_5\delta}{\sqrt{n}}t_{l+1}\right)^2\right\}, \end{aligned} \quad (43)$$

where Ω_1 and Ω_2 are positive constants. After some algebraic manipulations, one can show that

$$\begin{aligned} &Prob\{\min_{t \geq n^{\frac{6}{7}}\delta^{\frac{2}{3}}} Y_t \leq n^{\frac{2}{3}}\delta^{\frac{4}{3}}\} \leq \\ &\quad \Omega_1 \textstyle\sum_{l=0}^{\infty} \exp\left\{-\frac{\Omega_2\delta^2 n^{\frac{4}{7}}}{2^{l+1}}\left(2^{2l} - n^{\frac{-1}{21}} - \kappa_5 2^{l+1} n^{\frac{-5}{14}}\delta^{1/3}\right)^2\right\}. \end{aligned} \quad (44)$$

For $n > \max\left\{1, (2\kappa_5\delta^{1/3})^{14/5}\right\}$,

$$\begin{aligned} &Prob\{\min_{t \geq n^{\frac{6}{7}}\delta^{\frac{2}{3}}} Y_t \leq n^{\frac{2}{3}}\delta^{\frac{4}{3}}\} \leq \\ &\quad \Omega_1 \exp\left\{-\frac{\Omega_2\delta^2 n^{\frac{4}{7}}}{2}\left(1 - n^{\frac{-1}{21}} - 2\kappa_5 n^{\frac{-5}{14}}\delta^{1/3}\right)^2\right\} + \\ &\quad \Omega_1 \textstyle\sum_{l=1}^{\infty} \exp\left\{-\frac{\Omega_2\delta^2 n^{\frac{4}{7}}}{2^{l+1}}\left(2^{2l} - 1 - 2^l\right)^2\right\}. \end{aligned} \quad (45)$$

Now it is not hard to show that there exist two positive constants, Ω_1', and Ω_2', such that [11]

$$Prob\{\min_{t \geq n^{\frac{6}{7}}\delta^{\frac{2}{3}}} Y_t \leq n^{\frac{2}{3}}\delta^{\frac{4}{3}}\} \leq \Omega_1' \exp\{-\Omega_2'\delta^2\}. \quad (46)$$

Note that Ω_1' and Ω_2' are not necessarily the same as Ω_1 and Ω_2. Combining the results from (39), (40), and (46) concludes the proof.

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