

# On Parity-Preserving Variable-Length Constrained Coding

Ron M. Roth

Computer Science Department  
Technion, Haifa 320003, Israel  
ronny@cs.technion.ac.il

Paul H. Siegel

ECE Department and CMRR  
UC San Diego, La Jolla, CA 92023, USA  
psiegel@ucsd.edu

**Abstract**—Previous work by the authors on parity-preserving fixed-length constrained encoders is extended to the variable-length case. Parity-preserving variable-length encoders are formally defined, and a necessary and sufficient condition is presented for the existence of deterministic parity-preserving variable-length encoders for a given constraint. Examples are provided that show that there are coding ratios where parity-preserving variable-length encoders exist, while fixed-length encoders do not.

## I. INTRODUCTION

In mass storage platforms, such as magnetic and optical disks, user data is mapped (encoded) to binary sequences that satisfy certain combinatorial constraints. One common example of such a constraint is the  $(d, k)$ -runlength-limited (RLL) constraint, where the runs of 0's in a sequence are limited to have lengths at least  $d$  (to avoid inter-symbol interference) and at most  $k$  (to allow clock resynchronization) [7]. In virtually all applications, the encoder takes the form of a finite state machine, where user data is broken into binary blocks, and each block is mapped, in a state-dependent manner, into a binary *codeword*, so that the concatenation of the generated codewords satisfies the RLL constraint. In the case of *fixed-length encoders*, the input blocks all have the same length  $p$ , and the codewords all have the same length  $q$ , for prescribed positive integers  $p$  and  $q$ . The coding rate is then  $p : q$ .

In the mentioned storage applications, there is also a need to control the DC content of the recorded modulated sequence. One commonly used strategy to achieve DC control is allowing input blocks to be mapped to more than one codeword, and the encoder then selects the codeword that yields a better DC suppression [9, p. 29]. In the Blu-ray standard, this strategy is applied through the use of *parity-preserving encoders*: such encoders map each input block to a codeword that has the same parity (of the number of 1s), and DC control is achieved by reserving one bit in the input block to be set to a value that minimizes the DC contents [7, §11.4.3], [8], [10]–[12], [15].

Most constructions of parity-preserving encoders that were proposed for commercial use were obtained by ad-hoc methods. In [13], we initiated a study of bi-modal encoders (which

include parity-preserving encoders as a special case), focusing on fixed-length encoders; we will summarize the concepts that pertain to the fixed-length case, along with the main results of [13], as part of the background that we provide in Section II below. On the other hand, the existing ad-hoc parity-preserving constructions typically have *variable length*, where the length  $p$  of the input block and the length  $q$  of the respective codeword may depend on the encoder state, as well as on the input sequence (the *coding ratio*,  $p/q$ , nevertheless, is still fixed).

In this work, we present several results on parity-preserving variable-length encoders (in short, parity-preserving VLEs), focusing on deterministic encoders. To put our results into perspective, we mention that even in the ordinary setting (where parity preservation is not required), the known tools for analyzing and synthesizing VLEs are much less developed, compared to the fixed-length case. A summary of relevant (and mostly known) results on (ordinary) VLEs is provided in Section III. In Section IV we turn to the parity-preserving setting. Much of the discussion in that section deals in fact with the definition of parity-preserving VLEs, as it entails a (nontrivial) extension of the known Kraft conditions on variable-length coding to the parity-preserving case. Our main result is a necessary and sufficient condition for the existence of parity-preserving VLEs that are deterministic. We present several examples that demonstrate the advantages that parity-preserving VLEs may have over their fixed-length counterparts, in terms of the attainable coding ratios and encoding–decoding complexity.

## II. FIXED-LENGTH GRAPHS AND ENCODERS

In this section, we extract from [9, Chapters 2–5] several basic definitions and properties pertaining to ordinary (namely, fixed-length) graphs and fixed-length encoders. We then quote the main result of [13], which applies, in particular, to parity-preserving fixed-length encoders.

A (finite labeled directed ordinary) graph is a graph  $G = (V, E, L)$  where  $V$  is a nonempty finite set of states,  $E$  is a finite set of edges, and  $L : E \rightarrow \Sigma$  is an edge labeling. A graph  $G$  is *deterministic* if no two outgoing edges from the same state in  $G$  have the same labels, and it is *lossless* if no two paths with the same initial state and the same terminal state generate the same word.

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A *constraint*  $S$  over an alphabet  $\Sigma$  is the set of all words that are generated by paths in a graph  $G$ ; we then say that  $G$  *presents*  $S$  and write  $S = S(G)$ . Every constraint  $S$  can be presented by a deterministic graph. The *capacity* of  $S$  is defined by  $\text{cap}(S) = \lim_{\ell \rightarrow \infty} (1/\ell) \log_2 |S \cap \Sigma^\ell|$ . It is known that  $\text{cap}(S) = \log_2 \lambda(A_G)$  where  $\lambda(A_G)$  denotes the spectral radius (Perron eigenvalue) of the adjacency matrix  $A_G$  of any lossless (in particular, deterministic) presentation  $G$  of  $S$ .

A graph  $G$  is *irreducible* if it is strongly connected, namely, for any two states  $u$  and  $v$  in  $G$  there is a path from  $u$  to  $v$ . A constraint  $S$  is irreducible if it can be presented by a deterministic irreducible graph. For irreducible constraints, there is a unique deterministic graph presentation that has the smallest number of states; such a presentation is called the *Shannon cover* of  $S$ .

**Example 1.** Let  $S$  be the constraint over the alphabet  $\Sigma = \{a, b, c, d\}$  which is presented by the graph  $G$  in Figure 1. The graph  $G$  is the Shannon cover of  $S$ , and  $\text{cap}(S) = \log_2 \lambda(A_G) = \log_2 2 = 1$ .  $\square$

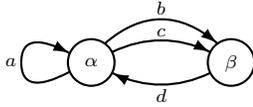


Fig. 1. Graph  $G$  for Example 1.

The power  $G^t$  of a graph  $G = (V, E, L)$  is the graph with the same set of states  $V$  and edges that are the paths of length  $t$  in  $G$ ; the label of an edge in  $G^t$  is the length- $t$  word generated by the path. For  $S = S(G)$  the power  $S^t$  is defined as  $S(G^t)$ .

Given a constraint  $S$  and a positive integer  $n$ , a (*fixed-length*)  $(S, n)$ -encoder is a lossless graph  $\mathcal{E}$  such that  $S(\mathcal{E}) \subseteq S$  and each state has out-degree  $n$ . An  $(S, n)$ -encoder exists iff  $\log_2 n \leq \text{cap}(S)$ . In a *tagged*  $(S, n)$ -encoder, each edge is assigned an input tag from a finite alphabet  $\Upsilon$  of size  $n$ , such that edges outgoing from the same state have distinct tags.

A (*tagged*) *rate*  $p : q$  encoder for a constraint  $S$  is a tagged  $(S^q, 2^p)$ -encoder (the tag alphabet  $\Upsilon$  is then assumed to be  $\{0, 1\}^p$ ); such an encoder exists iff  $p/q \leq \text{cap}(S)$ .

Let  $\Sigma$  be an alphabet and fix a partition  $\{\Sigma_0, \Sigma_1\}$  of  $\Sigma$ . The symbols in  $\Sigma_0$  (resp.,  $\Sigma_1$ ) will be referred to as the *even* (resp., *odd*) symbols of  $\Sigma$ . Extending the definition of parity to words, we say that a word  $w$  over  $\Sigma$  is even (resp., odd) if  $w$  contains an even (resp., odd) number of symbols from  $\Sigma_1$ . The set of even (resp., odd) words in  $\Sigma^t$  will be denoted by  $(\Sigma^t)_0$  (resp.,  $(\Sigma^t)_1$ ). In the practical scenario where  $\Sigma$  is the binary alphabet (with  $\Sigma_0 = \{0\}$  and  $\Sigma_1 = \{1\}$ ), a parity of a word coincides with the ordinary meaning of this term.

Given a graph  $H$  with labeling in  $\Sigma$ , for  $b \in \{0, 1\}$ , we denote by  $H_b$  the subgraph of  $H$  containing only the edges with labels in  $\Sigma_b$ .

Let  $S$  be a constraint over an alphabet  $\Sigma$ , fix a partition  $\{\Sigma_0, \Sigma_1\}$  of  $\Sigma$ , and let  $n_0$  and  $n_1$  be positive integers. A (*fixed-length*)  $(S, n_0, n_1)$ -encoder  $\mathcal{E}$  is an  $(S, n_0 + n_1)$ -encoder such that for each  $b \in \{0, 1\}$ , the subgraph  $\mathcal{E}_b$  is an  $(S, n_b)$ -encoder. A *rate*  $p : q$  *parity-preserving* (*fixed-length*) encoder

for  $S$  is a tagged  $(S^q, 2^{p-1}, 2^{p-1})$ -encoder in which the tag (in  $\{0, 1\}^p$ ) that is assigned to each edge has the same parity as the edge label (when seen as a word in  $\Sigma^q$ ).

The next theorem follows from the results of [13] (see Theorem 1, Corollary 5, and §III-A therein). For a square nonnegative integer matrix  $A$  and a positive integer  $n$ , denote by  $\mathcal{X}(A, n)$  the set of all nonnegative nonzero integer vectors  $x$  that satisfy the inequality  $Ax \geq nx$  componentwise.

**Theorem 1** ([13]). *Let  $S$  be an irreducible constraint, presented by an irreducible deterministic graph  $G$ , and let  $n_0$  and  $n_1$  be positive integers. Then the following holds.*

- (a) *There exists an  $(S, n_0, n_1)$ -encoder, if and only if  $\mathcal{X}(A_{G_0}, n_0) \cap \mathcal{X}(A_{G_1}, n_1) \neq \emptyset$ .*
- (b) *There exists a deterministic  $(S, n_0, n_1)$ -encoder, if and only if  $\mathcal{X}(A_{G_0}, n_0) \cap \mathcal{X}(A_{G_1}, n_1)$  contains a 0–1 vector.*

### III. VARIABLE-LENGTH GRAPHS AND ENCODERS

In this section, we summarize several definitions and properties relating to variable-length graphs and variable-length encoders (see also [9, §6.4]).

#### A. Variable-length graphs

In a variable-length graph (in short, VLG), the labels of the edges may be words of any positive (finite) length over the label alphabet  $\Sigma$ ; the length of the edge is then defined as the length of its label. Given a VLG  $H$ , the constraint  $S(H)$  that is presented by  $H$  is defined as the set of all (consecutive) *sub-words* of words obtained by concatenating the labels that are read along finite paths in  $H$ . Equivalently,  $S(H)$  is the constraint presented by the (ordinary) graph  $G$  obtained from  $H$  by replacing each length- $\ell$  edge  $e$  in  $H$  by a path of  $\ell$  length-1 edges (connected through newly introduced dummy states) which generates the label of  $e$ . The *follower set* of a state  $u$  in  $H$  is the set of all *prefixes* of words that are generated by finite paths that start at  $u$ .

A VLG  $H$  is called *deterministic* if the labels of the outgoing edges from each state in  $H$  form a *prefix-free* list, namely, no label is a prefix of any other label. The notions of losslessness and irreducibility carry over from ordinary graphs.

**Example 2.** Letting  $G$  and  $S$  be as in Example 1, the VLG  $H$  in Figure 2 is irreducible and deterministic, and it presents  $S$ , i.e.,  $S(H) = S(G) = S$ .  $\square$

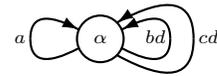


Fig. 2. VLG  $H$  for Example 2.

#### B. Variable-length encoders

Let  $\Upsilon$  be a finite alphabet<sup>1</sup> and let  $\mathcal{L}$  be a finite list of nonempty finite words over  $\Upsilon$ . We say that  $\mathcal{L}$  is *exhaustive* if every word over  $\Upsilon$  either has a prefix in  $\mathcal{L}$  or is a prefix of some word in  $\mathcal{L}$ . The next result is well known [4, p. 298].

<sup>1</sup>We use here the notation  $\Upsilon$  for an alphabet (instead of  $\Sigma$ ) since in the context of variable-length encoders, that alphabet will be the alphabet of tags.

**Proposition 2.** Given an alphabet  $\Upsilon$  and a nonnegative integer sequence  $\mu = (\mu_\ell)_{\ell \geq 1}$  with finite support, there exists an exhaustive prefix-free list  $\mathcal{L}$  over  $\Upsilon$  such that

$$\mu_\ell = |\mathcal{L} \cap \Upsilon^\ell|, \quad \ell = 1, 2, 3, \dots,$$

iff  $\mu$  satisfies the Kraft inequality with equality, namely:

$$\sum_{\ell \geq 1} \frac{\mu_\ell}{|\Upsilon|^\ell} = 1. \quad (1)$$

Let  $S$  be a constraint over an alphabet  $\Sigma$  and let  $n$  be a positive integer. Also, let  $\mathcal{E} = (V, E, L)$  be a VLG, and for every  $u \in V$  and  $\ell \geq 1$ , denote by  $\mu_\ell(u)$  the number of edges of length  $\ell$  outgoing from  $u$  in  $\mathcal{E}$ . We say that  $\mathcal{E}$  is a *variable-length  $(S, n)$ -encoder* (in short, an  $(S, n)$ -VLE) if the following conditions hold.

- (E1)  $\mathcal{E}$  is lossless,
- (E2)  $S(\mathcal{E}) \subseteq S$ , and—
- (E3) for every  $u \in V$ :

$$\sum_{\ell \geq 1} \frac{\mu_\ell(u)}{n^\ell} = 1.$$

(This definition reduces to that of a fixed-length  $(S, n)$ -encoder when  $\mu_\ell(u) = 0$  for every  $u \in V$  and  $\ell > 1$ .)

Extending now the notion of tagging to the variable-length case, let  $\Upsilon$  be a (base tag) alphabet of size  $|\Upsilon| = n$ . A tagging of an  $(S, n)$ -VLE  $\mathcal{E}$  is an assignment of input tags—namely, words over  $\Upsilon$ —to the edges of  $\mathcal{E}$ , such that:

- (T1) the length of each input tag equals the length of (the label of) the edge, and—
- (T2) the input tags of the outgoing edges from each state in  $\mathcal{E}$  form an exhaustive prefix-free list over  $\Upsilon$ .

Proposition 2 and condition (E3) guarantee that every  $(S, n)$ -VLE can be tagged consistently with conditions (T1)–(T2). Condition (T1) means that the *coding ratio* is fixed to be 1 at all edges, regardless of their length (by grouping symbols and tags into nonoverlapping blocks, any fixed coding ratio can be reduced to the case of a coding ratio of 1). We note that this is the variable-length encoding model assumed in [1], [2], [6].

**Example 3.** Letting  $\Sigma$  and  $S$  be as in Example 1, the graph  $H$  in Figure 2 is a deterministic  $(S, 2)$ -VLE. Taking  $\Upsilon = \{0, 1\}$ , one possible tag assignment to (the labels of) the edges of  $H$  is given by  $0 \leftrightarrow a$ ,  $10 \leftrightarrow bd$ , and  $11 \leftrightarrow cd$ . The coding ratio at each state is 1, so this encoder is capacity-achieving. Note that this tag assignment is parity-preserving w.r.t. the following partition of  $\Sigma$ :

$$\Sigma_0 = \{a, b\} \quad \text{and} \quad \Sigma_1 = \{c, d\}. \quad (2)$$

In contrast, using Theorem 1(a), it was shown in [13] that for this partition, a coding rate of  $t : t$  cannot be achieved by any parity-preserving *fixed-length* encoder for  $S$  for any positive integer  $t$ .  $\square$

**Example 4.** Letting  $\Sigma$  and  $S$  be as in Example 1, the graph  $\mathcal{E}$  in Figure 3 presents another  $(S, 2)$ -VLE. The coding

ratio at each state is 1, making  $\mathcal{E}$  capacity-achieving. However,  $\mathcal{E}$  is not deterministic.

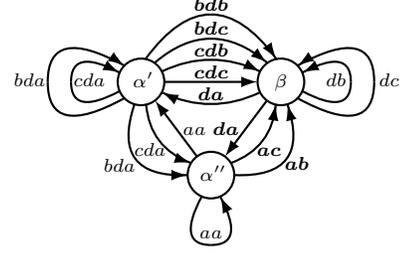


Fig. 3. VLE  $\mathcal{E}$  for the constraint presented by Figure 1.

Consider now the following partition  $\{\Sigma_0, \Sigma_1\}$  of  $\Sigma$ :

$$\Sigma_0 = \{a\} \quad \text{and} \quad \Sigma_1 = \{b, c, d\} \quad (3)$$

(the odd labels w.r.t. this partition are marked in boldface in Figure 3). Taking the tag alphabet  $\Upsilon = \{0, 1\}$ , one possible parity-preserving tag assignment to the edges of  $\mathcal{E}$  is shown in Table I. Similarly to the partition (2), it was shown in [13]

TABLE I  
POSSIBLE TAG ASSIGNMENT FOR THE ENCODER IN FIGURE 3.

State $\alpha'$	State $\alpha''$	State $\beta$
000, 011 $\leftrightarrow$ <i>bda</i>	00, 11 $\leftrightarrow$ <i>aa</i>	01, 10 $\leftrightarrow$ <b>da</b>
101, 110 $\leftrightarrow$ <i>cda</i>	01 $\leftrightarrow$ <b>ac</b>	00 $\leftrightarrow$ <i>db</i>
001 $\leftrightarrow$ <b>bdb</b>	10 $\leftrightarrow$ <b>ab</b>	11 $\leftrightarrow$ <i>dc</i>
010 $\leftrightarrow$ <b>bdc</b>		
100 $\leftrightarrow$ <b>cdb</b>		
111 $\leftrightarrow$ <b>cdc</b>		

that for the partition (3), too, one cannot achieve a coding ratio of 1 by any parity-preserving fixed-length encoder for  $S$ . Yet unlike the partition (2), for (3), it can be shown (details omitted) that there is no deterministic VLE that has coding ratio 1 and a parity-preserving assignment.  $\square$

**Example 5.** Let  $S$  be the  $(2, \infty)$ -RLL constraint, whose Shannon cover is given by the graph  $G$  in Figure 4. The

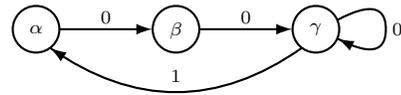


Fig. 4. Shannon cover  $G$  of the  $(2, \infty)$ -RLL constraint.

capacity of  $S$  is approximately 0.5515, so there exists a rate  $1 : 2$  fixed-length encoder for  $S$  (namely, an  $(S^2, 2)$ -encoder); such a (tagged) encoder  $\mathcal{E}$  is shown in Figure 5. This encoder

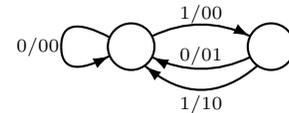


Fig. 5. Rate  $1 : 2$  fixed-length encoder  $\mathcal{E}$  for the  $(2, \infty)$ -RLL constraint.

is not deterministic; in fact, the smallest integer  $p$  for which there exists a rate  $p : 2p$  deterministic fixed-length encoder for  $S$  is  $p = 7$ , as this is the smallest integer for which the set  $\mathcal{X}(A_G^{2p}, 2^p)$  contains a 0–1 vector (see [9, Theorem 7.15]).

On the other hand, the graph in Figure 6, with the tagging  $0 \leftrightarrow 00$ ,  $10 \leftrightarrow 01.00$ , and  $11 \leftrightarrow 10.00$ , is a deterministic VLE for  $S$  with a coding ratio of  $1/2$  (see [6]; since the alphabet of  $S^2$  consists of pairs of bits, we have used dots to delimit the symbols within each label). Note, however, that such a tag assignment is *not* parity-preserving; we will return to this example in Examples 6 and 7 below.  $\square$

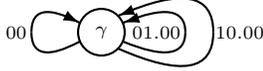


Fig. 6. VLE for the  $(2, \infty)$ -RLL constraint.

### C. Deterministic variable-length encoders

In this section, we focus on VLEs which are deterministic, and quote a necessary and sufficient condition for having such encoders.

Let  $H = (V, E, L)$  be a VLG whose labels are over a finite alphabet  $\Sigma$  and let  $n$  be a positive integer. Fix some nonempty subset  $V' \subseteq V$ , and let  $H' = (V', E', L')$  be the subgraph of  $H$  that is induced by  $V'$  (namely,  $E'$  consists of all the edges in  $H$  both of whose endpoints are in  $V'$ ). For every  $u \in V'$  and  $\ell \geq 1$ , denote by  $\mu_\ell(u|V')$  the number of outgoing edges of length  $\ell$  from  $u$  in  $H'$ . We say that  $V'$  is a set of *principal states in  $H$  w.r.t.  $n$*  if for every  $u \in V'$ :

$$\sum_{\ell \geq 1} \frac{\mu_\ell(u|V')}{n^\ell} \geq 1.$$

It readily follows from this definition that  $V'$  is a set of principal states in a VLG  $H$  w.r.t.  $n$ , iff it is also so in the subgraph  $H'$  of  $H$  that is induced by  $V'$ .

The following result is essentially known (see [2], [5], [6]).

**Theorem 3.** *Let  $S$  be an irreducible constraint and let  $n$  and  $r$  be positive integers. There exists a deterministic  $(S, n)$ -VLE whose edges all have length at most  $r$ , iff  $S$  is presented by an irreducible deterministic VLG  $H = (V, E, L)$  whose edges all have length at most  $r$ , and  $V$  contains a subset of principal states w.r.t.  $n$ .<sup>2</sup>*

Given an ordinary irreducible deterministic graph  $G$  (with length-1 edges) and positive integers  $n$  and  $r$ , Franaszek described in [6] a polynomial-time algorithm for testing whether  $S(G)$  can be presented by a VLG  $H$  that satisfies the conditions of Theorem 3 (see also [2], [3]). His algorithm, which is based on dynamic programming, effectively finds a set of principal states  $V'$  (which is a subset of the states of  $G$ ) and a subgraph  $H'$  of  $H$  that is induced by  $V'$  (the graph  $H$  itself is not explicitly constructed in [6]).

**Example 6.** Let  $G$  and  $S$  be as in Example 5, and take  $n = 2$ . Since there are no deterministic  $(S^2, 2)$ -encoders, we cannot have any principal states when  $r = 1$ . Selecting  $r = 2$ , an application of Franaszek's algorithm from [6] to  $G^2$  yields

<sup>2</sup>Moreover, the graph  $H$  can be assumed to be *reduced*, namely, the follower sets of its states are distinct. For the case where all the edge lengths are 1, the graph  $H$  is the Shannon cover of  $S$ .

a (unique) set of principal states  $V'$  consisting only of state  $\gamma$ . Since w.l.o.g.  $H$  is reduced, that implies a unique subgraph  $H'$  that is induced by  $V'$ , which is the graph in Figure 6 (see [6, §V]).  $\square$

## IV. PARITY-PRESERVING VARIABLE-LENGTH ENCODERS

In this section, we provide a formal definition of a parity-preserving variable-length encoder. A key ingredient in that definition will be an adaptation of Proposition 2 to the parity-preserving case, which we do in Section IV-A; that adaptation may be of independent interest, beyond its use in this work. We then state a necessary and sufficient condition for having a parity-preserving VLE which is deterministic.

### A. Parity-preserving Kraft conditions

Let  $\Upsilon$  be a finite alphabet and assume a partition  $\{\Upsilon_0, \Upsilon_1\}$  of  $\Upsilon$ . Given a finite list  $\mathcal{L}$  of nonempty words over  $\Upsilon$ , the (*parity-preserving*) *length distribution* of  $\mathcal{L}$  is a pair of nonnegative integer sequences  $(\boldsymbol{\eta} = (\eta_\ell)_{\ell \geq 1}, \boldsymbol{\omega} = (\omega_\ell)_{\ell \geq 1})$ , where  $\eta_\ell = |\mathcal{L} \cap (\Upsilon^\ell)_0|$  and  $\omega_\ell = |\mathcal{L} \cap (\Upsilon^\ell)_1|$ ,  $\ell = 1, 2, 3, \dots$ .

Given positive integers  $n_0, n_1$ , and  $\ell$  and a pair  $(\boldsymbol{\eta}, \boldsymbol{\omega})$  of nonnegative integer sequences, each with finite support, define  $K_\ell^+ = K_\ell(\boldsymbol{\eta} + \boldsymbol{\omega}, n_0 + n_1)$  and  $K_\ell^- = K_\ell(\boldsymbol{\eta} - \boldsymbol{\omega}, n_0 - n_1)$  by

$$\begin{aligned} K_\ell^\pm &= K_\ell(\boldsymbol{\eta} \pm \boldsymbol{\omega}, n_0 \pm n_1) \\ &= (n_0 \pm n_1)^\ell - \sum_{i=1}^{\ell} (\eta_i \pm \omega_i)(n_0 \pm n_1)^{\ell-i}. \end{aligned}$$

Denoting by  $r = r(\boldsymbol{\eta}, \boldsymbol{\omega})$  the largest index in the union of the supports of  $\boldsymbol{\eta}$  and  $\boldsymbol{\omega}$ , the notation  $K^\pm = K(\boldsymbol{\eta} \pm \boldsymbol{\omega}, n_0 \pm n_1)$  will stand for  $K_r^\pm$ . Thus, (1) becomes

$$K^+ = K^+(\boldsymbol{\eta} + \boldsymbol{\mu}, n_0 + n_1) = 0,$$

where we have taken  $n_0 = |\Upsilon_0|$  and  $n_1 = |\Upsilon_1|$ .

The next proposition provides a necessary and sufficient condition for a pair  $(\boldsymbol{\eta}, \boldsymbol{\omega})$  to be a (parity-preserving) length distribution of an exhaustive prefix-free list.

**Proposition 4.** *Given a partition  $\{\Upsilon_0, \Upsilon_1\}$  of a finite alphabet  $\Upsilon$  with  $|\Upsilon_0| = n_0$  and  $|\Upsilon_1| = n_1$ , let  $(\boldsymbol{\eta}, \boldsymbol{\omega})$  be a pair of nonnegative integer sequences, each with finite support. Then there exists an exhaustive prefix-free list over  $\Upsilon$  with a length distribution  $(\boldsymbol{\eta}, \boldsymbol{\omega})$ , iff the following conditions hold.*

- (a)  $K^+ = 0$ , and—
- (b)  $K_\ell^+ \geq |K_\ell^-|$  for every  $\ell \geq 1$ .

The proof of the proposition can be found in [14].

### B. Definition of parity-preserving variable-length encoders

Let  $S$  be a constraint over an alphabet  $\Sigma$  and assume a partition  $\{\Sigma_0, \Sigma_1\}$  of  $\Sigma$ . Also, let  $\mathcal{E} = (V, E, L)$  be a VLG, and for every  $u \in V$  and  $\ell \geq 1$ , denote by  $\eta_\ell(u)$  (resp.,  $\omega_\ell(u)$ ) the number of edges of length  $\ell$  outgoing from  $u$  in  $\mathcal{E}$  that have even (resp., odd) labels (when the labels are regarded as words over  $\Sigma$ ). Writing

$$\boldsymbol{\eta}(u) = (\eta_\ell(u))_{\ell \geq 1} \quad \text{and} \quad \boldsymbol{\omega}(u) = (\omega_\ell(u))_{\ell \geq 1},$$

the pair  $(\boldsymbol{\eta}(u), \boldsymbol{\omega}(u))$  thus stands for the length distribution of the set of labels of the outgoing edges from  $u$  in  $H$ .

Fix now  $n_0$  and  $n_1$  to be positive integers, and for every  $u \in V$  define

$$K_\ell^\pm(u) = K_\ell(\boldsymbol{\eta}(u) \pm \boldsymbol{\omega}(u), n_0 \pm n_1)$$

and

$$K^\pm(u) = K_r(\boldsymbol{\eta}(u) \pm \boldsymbol{\omega}(u), n_0 \pm n_1),$$

where  $r = r(u) = r(\boldsymbol{\eta}(u), \boldsymbol{\omega}(u))$ . We say that  $\mathcal{E}$  is a (*parity-preserving*)  $(S, n_0, n_1)$ -VLE if for every  $u \in V$  it satisfies the three conditions (E1)–(E3) in Section III-B, as well as the following fourth condition:

$$(E4) \quad K_\ell^+(u) \geq |K_\ell^-(u)| \text{ for every } \ell \geq 1.$$

Now, let  $\Upsilon$  be a base tag alphabet of size  $n_0 + n_1$  that has a partition  $\{\Upsilon_0, \Upsilon_1\}$  with  $|\Upsilon_0| = n_0$  and  $|\Upsilon_1| = n_1$ . A (parity-preserving) tagging of an  $(S, n_0, n_1)$ -VLE is an assignment of input tags to the edges of  $\mathcal{E}$  such that conditions (T1)–(T2) in Section III-B hold, and, in addition:

$$(T3) \text{ at each edge, the parity of the input tag (as a word over } \Upsilon) \text{ is the same as the parity of the label (as a word over } \Sigma).$$

It follows from Proposition 4 and conditions (E3)–(E4) that every  $(S, n_0, n_1)$ -VLE can be tagged consistently with (T3).

### C. Deterministic parity-preserving variable-length encoders

The main result of this section is Theorem 5 below, which is the parity-preserving counterpart of Theorem 3: it presents a necessary and sufficient condition for having a deterministic parity-preserving VLE.

Let  $\Sigma$  be an alphabet which is partitioned into  $\{\Sigma_0, \Sigma_1\}$  and let  $H = (V, E, L)$  be a VLG whose labels are over  $\Sigma$ . Fix some nonempty subset  $V' \subseteq V$  and positive integers  $n_0$  and  $n_1$ , and for every  $u \in V'$  and  $\ell \geq 1$ , let  $(\boldsymbol{\eta}(u|V'), \boldsymbol{\omega}(u|V'))$  be the length distribution of the set of labels of the outgoing edges from  $u$  in the subgraph  $H' = (V', E', L')$  of  $H$  that is induced by  $V'$ . Also, (re-)define

$$K_\ell^\pm(u) = K_\ell(\boldsymbol{\eta}(u|V') \pm \boldsymbol{\omega}(u|V'), n_0 \pm n_1)$$

and

$$K^\pm(u) = K_r(\boldsymbol{\eta}(u|V') \pm \boldsymbol{\omega}(u|V'), n_0 \pm n_1),$$

where  $r = r(u) = r(\boldsymbol{\eta}(u|V'), \boldsymbol{\omega}(u|V'))$ . We say that  $V'$  is a set of (*parity-preserving*) *principal states* in  $H$  w.r.t.  $(n_0, n_1)$  if for every  $u \in V'$ :

$$K^+(u) \leq -|K^-(u)|$$

and

$$K_\ell^+(u) \geq |K_\ell^-(u)|, \quad \ell = 1, 2, \dots, r(u) - 1.$$

**Theorem 5.** *Let  $S$  be an irreducible constraint over an alphabet  $\Sigma$ , assume a partition  $\{\Sigma_0, \Sigma_1\}$  of  $\Sigma$ , and let  $n_0, n_1$ , and  $r$  be positive integers. There exists a deterministic  $(S, n_0, n_1)$ -VLE whose edges all have length at most  $r$ , iff  $S$  is presented by an irreducible deterministic VLG  $H = (V, E, L)$  whose edges all have length at most  $r$ , and  $V$  contains a subset of principal states w.r.t.  $(n_0, n_1)$ .*

The proof of the theorem builds upon an (alternate) proof of Theorem 3; both proofs can be found in [14].

**Example 7.** Let  $S$  be the  $(2, \infty)$ -RLL constraint, which is presented by the graph  $G$  in Figure 4. Recall from Example 5 that there is no deterministic  $(S^2, 2)$ -encoder in this case and, so, there is no VLG  $H$  that satisfies the conditions of Theorem 3 for  $r = 1$ .

Turning to  $r = 2$ , recall from Example 6 that the VLE in Figure 6 is the unique induced subgraph  $H'$  of any (reduced) VLG  $H$  that satisfies the conditions of Theorem 3. Yet, assuming the ordinary definition of parity of binary words, the set of states  $V' = \{\gamma\}$  of  $H'$  is not a set of (parity-preserving) principal states (in  $H'$  and therefore in  $H$ ) w.r.t.  $(n_0, n_1) = (1, 1)$ . Hence, for  $r = 2$ , there is no deterministic  $(S^2, 1, 1)$ -VLE.

On the other hand, there exists a deterministic  $(S^2, 1, 1)$ -VLE for  $r = 3$ , as shown in Figure 7, along with the tag assignment in Table II.

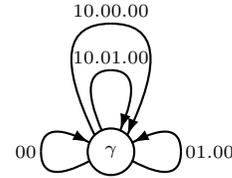


Fig. 7. Parity-preserving VLE for the  $(2, \infty)$ -RLL constraint.

TABLE II  
TAG ASSIGNMENT FOR THE ENCODER IN FIGURE 7.

0	$\leftrightarrow$	00
10	$\leftrightarrow$	01.00
110	$\leftrightarrow$	10.01.00
111	$\leftrightarrow$	10.00.00

Comparing to the fixed-length case, using Theorem 1(a), one can verify that there exists a (not necessarily deterministic)  $(S^{2p}, 2^{p-1}, 2^{p-1})$ -encoder, iff  $p \geq 3$ . For  $p = 3$ , any vector  $\mathbf{x} \in \mathcal{X}(A_{(G^6)_0}, 4) \cap \mathcal{X}(A_{(G^6)_1}, 4)$  satisfies  $\|\mathbf{x}\|_\infty \geq 6$  (and equality is attained only by  $\mathbf{x} = (2 \ 3 \ 6)^\top$ ). By Corollaries 4 and 5 in [13] we then get that any rate  $3 : 6$  parity-preserving fixed-length encoder for  $S$  must have at least six states and decoding look-ahead of at least 2 (measured in 6-bit symbols); in contrast, recall that when there is no requirement for parity preservation, we have the simple encoder in Figure 5. Using Theorem 1(b), one can determine that there exists a rate  $p : 2p$  parity-preserving fixed-length encoder for  $S$  which is deterministic, (if and) only if  $p \geq 8$ .  $\square$

*Remark 1.* Unlike Theorem 3, we do not have (as of yet) an extension of Franaszek's algorithm from [6] to the parity-preserving case; namely, a polynomial-time algorithm is yet to be found for determining whether, for given  $S$ ,  $\{\Sigma_0, \Sigma_1\}$ ,  $n_0$ ,  $n_1$ , and  $r$ , there is a VLG  $H$  that satisfies the conditions of Theorem 5. (The problem, however, is still decidable, since there are only finitely many reduced VLGs  $H$  with edge lengths at most  $r$  such that  $S(H) = S$ .)  $\square$

## REFERENCES

- [1] R.L. Adler, J. Friedman, B. Kitchens, B.H. Marcus, "State splitting for variable-length graphs," *IEEE Trans. Inf. Theory*, 32 (1986), 108–113.
- [2] M.-P. Béal, "The method of poles: A coding method for constrained channels," *IEEE Trans. Inf. Theory*, 36 (1990), 763–772.
- [3] M.-P. Béal, "Extensions of the method of poles for code construction," *IEEE Trans. Inf. Theory*, 49 (2003), 1516–1523.
- [4] R.E. Blahut, *Digital Transmission of Information*, Addison-Wesley, Reading, Massachusetts, 1990.
- [5] P.A. Franaszek, "Sequence-state coding for digital transmission," *Bell Sys. Tech. J.*, 47 (1968), 143–155.
- [6] P.A. Franaszek, "On synchronous variable length coding for discrete noiseless channels," *Inform. Control*, 15 (1969), 155–164.
- [7] K.A.S. Immink, *Codes for Mass Data Storage Systems*, Second Edition, Shannon Foundation Publishers, Eindhoven, The Netherlands, 2004.
- [8] J.A.H.M. Kahlman, K.A.S. Immink, "Device for encoding/decoding  $N$ -bit source words into corresponding  $M$ -bit channel words, and vice versa," US Patent 5,477,222, 1995.
- [9] B.H. Marcus, R.M. Roth, P.H. Siegel, *An Introduction to Coding for Constrained Systems*, Lecture Notes, 2001, available online at: [ronny.cswp.cs.technion.ac.il/wp-content/uploads/sites/54/2016/05/chapters1-9.pdf](http://ronny.cswp.cs.technion.ac.il/wp-content/uploads/sites/54/2016/05/chapters1-9.pdf)
- [10] T. Miyauchi, Y. Shinohara, Y. Iida, T. Watanabe, Y. Urakawa, H. Yamagishi, M. Noda, "Application of turbo codes to high-density optical disc storage using 17PP Code," *Jpn. J. Appl. Phys.*, 44 No. 5B (2005), 3471–3473.
- [11] T. Narahara, S. Kobayashi, M. Hattori, Y. Shimpuku, G.J. van den Enden, J.A.H.M. Kahlman, M. van Dijk, R. van Woudenberg, "Optical disc system for digital video recording," *Jpn. J. Appl. Phys.*, 39 No. 2B (2000), 912–919.
- [12] M. Noda, H. Yamagishi, "An 8-state DC-controllable run-length-limited code for the optical-storage channel," *Jpn. J. Appl. Phys.*, 44 No. 5B (2005), 3462–3466.
- [13] R.M. Roth, P.H. Siegel, "On bi-modal constrained coding," available online at <http://arxiv.org/abs/1912.12748>. An abstract titled "On parity-preserving constrained coding" appeared in *Proc. IEEE Int'l Symp. Inform. Theory (ISIT)*, Vail, Colorado (June 2018), 1804–1808.
- [14] R.M. Roth, P.H. Siegel, "On parity-preserving variable-length constrained coding," available online at <https://arxiv.org/abs/2005.05455>.
- [15] W.Y.H. Wilson, K.A.S. Immink, X.B. Xi, C.T. Chong, "A Comparison of two coding schemes for generating DC-free runlength-limited sequences," *Jpn. J. Appl. Phys.*, 39 No. 2B (2000), 815–818.