Windowed Decoding of Spatially Coupled Codes

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***Abstract*—We study windowed decoding of spatially coupled codes when the transmission occurs over the binary erasure channel. We characterize the performance of this scheme by defining thresholds on erasure rates that guarantee a target erasure rate. We give analytical lower bounds on these thresholds and show that the performance approaches that of belief propagation exponentially fast in the window size. We give numerical results including the thresholds computed using density evolution and the erasure rate curves for finite-length spatially coupled codes.**

I. Introduction

Spatial coupling of sparse graph codes has been of interest recently after it was shown to produce *threshold saturation* over the binary erasure channel (BEC) [1]. Although the BP thresholds for low-density parity-check (LDPC) convolutional codes [2] were observed to be close to the MAP threshold of the underlying regular LDPC ensemble by others [3], it was suggested in [1] that threshold saturation was more generally true. Subsequently, evidence for similar results over general BMS channels [4], erasure channels with memory [5], and multiple access channels [6] has been presented. Performance improvements through coupling have been reported in systems based on other graphical models, e.g., the random K-SAT, Q-COL problems from computation theory, Curie-Weiss model from statistical mechanics [7], LDGM code ensembles [8], and in compressive sensing [9]. Non-binary LDPC codes obtained through coupling have also been investigated [10].

The good performance of spatially coupled codes is apparent when both the blocklength of individual codes and the coupling length becomes large. However, as either of these parameters becomes large, belief propagation (BP) decoding becomes complex. We therefore consider a *windowed decoder* (WD) that exploits the structure of the coupled codes to reduce the decoding complexity while maintaining the advantages of the BP decoder in terms of performance. An additional advantage of the windowed decoder is the reduced latency of decoding. The windowed decoding scheme studied here was previously used to decode protograph-based codes over erasure channels with and without memory [11]–[13]. The main result of this paper is that the windowed decoding thresholds approach the BP thresholds exponentially in the size of the window W. As a consequence of threshold saturation, WD achieves close-to-ML performance.

The rest of the paper is organized as follows. Section II gives a brief introduction of spatially coupled codes and revisits some known results for BP decoding. In Section III we discuss the windowed decoding scheme. We state here the main result of the paper, provide a proof sketch in Section IV. We give some finite-length results in Section V and conclude in Section VI. Much of the terminology and notation used in the paper is reminiscent of those in [1].

II. Spatially Coupled Codes

We describe the (d_l, d_r) spatially coupled ensemble in terms of its Tanner graph. There are M variable nodes at each position in $[L] \triangleq \{1, 2, \cdots, L\}$. We will assume that there are $M\frac{d_l}{d_r}$ check nodes at every integer position, but only some of these interact with the variable nodes. The variable (check) nodes at position i constitute the i^{th} *section* of variable (check, resp.) nodes in the code. The L sections of variables are together referred to as the *chain* and L is called the *chain length*. For each of the d_l edges incident on a variable at position i, we first choose a section uniformly at random from the set $\{i, i+1, \cdots, i+\gamma-1\}$, then choose a check uniformly at random from the $M\frac{d_l}{d_r}$ checks in the chosen section, and connect the variable to this check, provided the degree of this check is not already d_r. We refer to the parameter γ as the *coupling length*. It can be shown that this procedure amounts roughly to choosing each of the d_r connections of a check node at position i uniformly and independently from the set $\{i-\gamma+1, i-\gamma+2, \cdots, i\}$. Since we are interested in coupled ensembles, we will assume that $\gamma > 1$. Further, we will typically be concerned with this ensemble when $L \gg \gamma$, in which case the *design rate* [1]

$$R(d_l, d_r, \gamma, L) = 1 - \frac{d_l}{d_r}\Big(1 + O(\frac{\gamma}{L})\Big)$$

is close to $1 - \frac{d_l}{d_r}$.

BP Performance

The BP performance of the (d_l, d_r, γ, L) spatially coupled ensemble when $M \to \infty$ can be evaluated using *density evolution*. Denote the average erasure probability of a message from a variable node at position i as x_i. We refer to the vector $\underline{x} = (x_1, x_2, \cdots, x_L)$ as the *constellation*. We can write the forward density evolution (DE) equation, for transmission over a BEC with erasure rate ϵ as follows. Set the initial constellation to be $\underline{x}^{(0)} = (1, 1, \cdots, 1)$ and evaluate the constellations $\{\underline{x}^{(\ell)}\}_{\ell=1}^{\infty}$ according to

$$x_i^{(\ell)} = \begin{cases} 0, \text{if } i \notin [L] \ \forall \ \ell \\ \epsilon\Big(1 - \frac{1}{\gamma}\sum_{j=0}^{\gamma-1}(1 - \frac{1}{\gamma}\sum_{k=0}^{\gamma-1} x_{i+j-k}^{(\ell-1)})^{d_r-1}\Big)^{d_l-1}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{else} \end{cases} \quad (1)$$

For ease of notation, we will write

$$g(x_{i-\gamma+1}, \cdots, x_{i+\gamma-1}) = \Big(1 - \frac{1}{\gamma}\sum_{j=0}^{\gamma-1}(1 - \frac{1}{\gamma}\sum_{k=0}^{\gamma-1} x_{i+j-k})^{d_r-1}\Big)^{d_l-1}. \quad (2)$$

It is clear that the function $g(\cdot)$ is monotonic in each of its arguments. It follows from this monotonicity that the sequence of constellations $\{\underline{x}^{(\ell)}\}_{\ell=0}^{\infty}$ are ordered as $\underline{x}^{(\ell)} \succeq \underline{x}^{(\ell+1)} \ \forall \ \ell \geq 0$, i.e., $x_i^{(\ell)} \geq x_i^{(\ell+1)} \ \forall \ \ell \geq 0, i \in [L]$. Since the constellations are all lower bounded by the all-zero constellation $\underline{0}$, the sequence converges pointwise to a limiting constellation $\underline{x}^{(\infty)}$, called the *fixed point* (FP) of the forward DE. The *BP threshold* $\epsilon^{\mathrm{BP}}(d_l, d_r, \gamma, L)$ is defined as the supremum of the channel erasure rates $\epsilon \in [0,1]$ for which the FP of forward DE is the all-zero constellation, i.e., $\underline{x}^{(\infty)} = \underline{0}$.

Table I gives the BP thresholds evaluated from forward density evolution for the $(d_l = 3, d_r = 6)$ coupled ensemble for a few values of γ and L, rounded to the sixth decimal place. The MAP threshold of the underlying (d_l, d_r)-regular

$L \backslash \gamma$	2	3	4
16	0.487983	0.488207	0.489805
32	0.487656	0.487923	0.488044
64	0.487014	0.487514	0.487733

TABLE I
BP THRESHOLDS $\epsilon^{\mathrm{BP}}(d_l = 3, d_r = 6, \gamma, L)$.

ensemble is $\epsilon^{\mathrm{MAP}}(d_l = 3, d_r = 6) \approx 0.488151$. We see from the table that the BP thresholds for (d_l, d_r) spatially coupled codes are close to the MAP threshold of the (d_l, d_r)-regular unstructured code ensemble even for small γ when L is large enough.

It was shown in [1] that the BP thresholds satisfy

$$\lim_{\gamma\to\infty}\lim_{L\to\infty} \epsilon^{\mathrm{BP}}(d_l, d_r, \gamma, L) = \lim_{L\to\infty} \epsilon^{\mathrm{MAP}}(d_l, d_r, \gamma, L) = \epsilon^{\mathrm{MAP}}(d_l, d_r).$$

This means that the BP threshold *saturates* to the MAP threshold, and we can obtain MAP performance with the reduced complexity of the BP decoder. In order to analyse the windowed decoder, we will keep the coupling length γ finite and hence will consider the quantity

$$\epsilon^{\mathrm{BP}}(d_l, d_r, \gamma) \triangleq \lim_{L\to\infty} \epsilon^{\mathrm{BP}}(d_l, d_r, \gamma, L) \quad (3)$$

a measure of the performance of the BP decoder. It immediately follows from [1, Theorem 12] that

$$\epsilon^{\mathrm{BP}}(d_l, d_r, \gamma) \leq \epsilon^{\mathrm{MAP}}(d_l, d_r).$$

III. WINDOWED DECODING

The *windowed decoder* (WD) exploits the structure of the spatially coupled codes to break down the BP decoding scheme into a series of sub-optimal decoding steps. When decoding with a window of size W, the WD performs BP over the subcode consisting of the first W sections of the variable nodes and their neighboring check nodes in an attempt to decode a subset of symbols (those in the first section) within the window. The symbols to be decoded within a window are referred to as the *targeted symbols*. Upon successful decoding of the targeted symbols (or when a maximum number of iterations have been performed, or when the decoder is stuck in stopping sets) the window slides over one section to the right and performs BP attempting to decode the targeted symbols in the window in the new position.

More formally, let $\underline{x}$ be the constellation representing the average erasure probability of messages from variables in each of the sections 1 through L. Initially, the window consists only of the first W sections in the chain. We will refer to this as the *first window configuration*, and as the window slides to the right, we will increment the window configuration. The c^{th} *window constellation*, denoted $\underline{y}_{\{c\}}$, is the average erasure probability of the variables in the c^{th} window configuration. Thus,

$$\underline{y}_{\{c\}} = (y_{1,\{c\}}, \cdots, y_{W,\{c\}}) = (x_c, \cdots, x_{c+W-1})$$

for $c \in [L]$, where we assume that $x_c = 0 \ \forall \ c > L$.

Remark 1 (Notation): When the constellation after a particular number of iterations ℓ of density evolution is to be specified, we write $\underline{y}^{(\ell)} = (y_1^{(\ell)}, y_2^{(\ell)}, \cdots, y_W^{(\ell)})$. Although $\underline{y}_{\{c\}}^{(\ell)}$ would be the most general way of specifying the window constellation for the c^{th} window configuration after ℓ iterations of density evolution, for notational convenience we will write as few of these parameters as possible based on the context.

A. Complexity and Latency

For the BP decoder, the number of iterations required to decode all the symbols in a (d_l, d_r, γ, L) spatially coupled code when $\epsilon \in (\epsilon^{\mathrm{BP}}(d_l, d_r), \epsilon^{\mathrm{BP}}(d_l, d_r, \gamma, L)]$ scales as $O(L)$ [14]. Therefore, in the waterfall region, the complexity of the BP decoder scales as $O(ML^2)$. For the WD of size W, if we let the number of iterations performed scale as $O(W)$, the overall complexity is of the order $O(MW^2L)$. Thus, for small window sizes $W < \sqrt{L}$, we see that the complexity of the decoder can be reduced. A larger reduction in the complexity is possible if we fix the number of iterations performed within each window. In latency-constrained applications, the WD can work with a latency that is a fraction $\frac{W}{L}$ that of the BP decoder.

B. Asymptotic Performance

We will consider the performance of the ensemble with $M \to \infty$ when the transmission happens over a BEC with channel erasure rate $\epsilon \in [0,1]$. Further we will assume that for each window configuration, infinite rounds of message passing are performed.

Definition 1 (WD Forward Density Evolution): Consider the WD of a (d_l, d_r, γ, L) spatially coupled code over a BEC with channel erasure rate ϵ with a window of size W. We can write the forward DE equation as follows. Set $\underline{x}_{\{0\}}$ as

$$x_{i,\{0\}} = \begin{cases} 1, & i \in [L] \\ 0, & i \notin [L]. \end{cases}$$

For every window configuration $c = 1, 2, \cdots, L$, let

$$\underline{y}^{(0)}_{\{c\}} = (x_{c,\{c-1\}}, x_{c+1,\{c-1\}}, \cdots, x_{c+W-1,\{c-1\}})$$

and evaluate the sequence of window constellations $\{\underline{y}^{(\ell)}_{\{c\}}\}_{\ell=1}^{\infty}$ using the update rules

$$y^{(\ell)}_{i,\{c\}} = \begin{cases} x_{c+i-1,\{c-1\}}, & i \notin [W] \ \forall\ \ell \\ \epsilon g(y^{(\ell-1)}_{i-\gamma+1,\{c\}}, \cdots, y^{(\ell-1)}_{i+\gamma-1,\{c\}}), & i \in [W] \end{cases}$$

and set $\underline{x}_{\{c\}}$ as

$$x_{i,\{c\}} = \begin{cases} x_{i,\{c-1\}}, & i \neq c \\ y^{(\infty)}_{1,\{c\}}, & i = c. \end{cases}$$

Discussion The constellation $\underline{x}_{\{c\}}$ keeps track of the erasure probabilities of targeted symbols of all window configurations upto the c^{th}. As defined, $\underline{x}_{\{c\}}$ discards all information obtained by running the WD in its c^{th} configuration apart from the targeted symbols.

Definition 1 implicitly assumes that the limiting window constellations $\underline{y}^{(\infty)}_{\{c\}}$ exist. The following guarantees that the updates for $x_{i,\{c\}}$ are well-defined.

Definition 2 (c^{th} Window Configuration FP of FDE): Consider the WD forward DE (FDE) of a (d_l, d_r, γ, L) spatially coupled code over a BEC with erasure rate ϵ with a window of size W. Then the limiting window constellation $\underline{y}^{(\infty)}_{\{c\}}$ exists for each $c \in [L]$. We refer to this constellation as the c^{th} window configuration FP of forward DE.

Discussion As noted earlier, $\underline{y}^{(0)}_{\{c\}} = \underline{1}\ \forall\ c \in [L]$, and $\underline{y}^{(0)}_{\{c\}} = \underline{1} \succeq \underline{\epsilon} \succeq \underline{y}^{(1)}_{\{c\}}$. By induction, from the monotonicity of $g(\cdot)$, this implies that $\underline{y}^{(\ell)}_{\{c\}} \succeq \underline{y}^{(\ell+1)}_{\{c\}}\ \forall\ \ell \geq 0$. Since these constellations are lower bounded by $\underline{0}$, the c^{th} window configuration FP of forward DE $\underline{y}^{(\infty)}_{\{c\}}$ exists for every $c \in [L]$.

The c^{th} window configuration FP of forward DE therefore satisfies

$$y^{(\infty)}_{i,\{c\}} = \begin{cases} x_{c+i-1,\{c-1\}}, & i \notin [W] \\ \epsilon g(y^{(\infty)}_{i-\gamma+1,\{c\}}, \cdots, y^{(\infty)}_{i+\gamma-1,\{c\}}), & i \in [W] \end{cases} \quad (4)$$

for every $c \in [L]$. It is clear that $\underline{0}$ cannot be the c^{th} window configuration FP of forward DE. Therefore, $\underline{y}^{(\infty)}_{\{c\}} \succ \underline{0}\ \forall\ c \in [L]$. This means that WD can never reduce the erasure probability of the symbols of a spatially coupled code to zero, although it can be made arbitrarily small by using a large enough window. Therefore, an acceptable *target erasure rate* δ forms a part of the description of the WD. We say that the WD is *successful* when $\underline{x}_{\{L\}} \preceq \underline{\delta}$.

Lemma 3 (Maximum of $\underline{x}_{\{L\}}$): The vector $\underline{x}_{\{L\}}$ obtained at the end of WD forward DE satisfies $x_{i-1,\{L\}} \leq x_{i,\{L\}}\ \forall\ i \in [L-\gamma+1]$. Moreover, $\exists\ \hat{x} \in [0,1]$ independent of L such that $x_{i,\{L\}} \leq \hat{x}\ \forall\ i$.

The monotonicity in $\underline{x}_{\{L\}}$ follows from the monotonicity of $g(\cdot)$. The second claim follows by bounding from above the entries of this vector by the largest value when $L = \infty$.

As a consequence of Lemma 3, we can say that the WD is successful when $\hat{x} \leq \delta$. This definition of the success of WD allows us to compare the performance of WD to that of the BP decoder through the thresholds defined in Equation (3).

Definition 4 (WD Thresholds): Consider the WD of a (d_l, d_r, γ, L) spatially coupled code over a BEC of erasure rate ϵ with a window of size W. The WD threshold $\epsilon^{\text{WD}}(d_l, d_r, \gamma, W, \delta)$ is defined as the supremum of channel erasure rates ϵ for which $\hat{x} \leq \delta$.

We will now state the main result in this paper and provide a proof sketch in the following section. A full version of the paper with all the proofs will be shortly published [15].

Theorem 5 (WD Threshold Bound): Consider windowed decoding of the (d_l, d_r, γ, L) spatially coupled ensemble over the binary erasure channel. Then for a target erasure rate $\delta < \delta_*$, there exists a positive integer $W_{\min}(\delta)$ such that when the window size $W \geq W_{\min}(\delta)$ the WD threshold satisfies

$$\epsilon^{\text{WD}}(d_l, d_r, \gamma, W, \delta) \geq \left(1 - \frac{d_l d_r}{2} \delta^{\frac{d_l-2}{d_l-1}}\right) \times \left(\epsilon^{\text{BP}}(d_l, d_r, \gamma) - e^{-\frac{1}{\mathsf{B}}\left(\frac{W}{\gamma-1} - \mathsf{A} \ln \ln \frac{\mathsf{D}}{\delta} - \mathsf{C}\right)}\right). \quad (5)$$

Here $\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}$ and δ_* are strictly positive constants that depend only on the ensemble parameters d_l, d_r and γ.

Theorem 5 says that the WD thresholds approach the BP threshold $\epsilon^{\text{BP}}(d_l, d_r, \gamma)$ defined in Equation (3) at least exponentially fast in the ratio of the size of the window W to the coupling length γ for a fixed target erasure probability $\delta < \delta_*$. The requirement that $W \geq W_{\min}(\delta)$ is necessary to keep the term within parentheses in the exponent non-negative. Therefore, the minimum required window size $W_{\min}(\delta)$ also depends on the constants A, C and D, and in turn, on the ensemble parameters d_l, d_r and γ.

The bound guaranteed by Theorem 5 turns out to be fairly loose. Numerical results suggest that the minimum window size $W_{\min}(\delta)$ is actually much smaller than the one obtained from analysis. Density evolution also reveals that for a fixed window size, the WD thresholds are much closer to the BP threshold than the bound obtained from Theorem 5. We note here that the gap between analytical results and numerical experiments is mainly due to the reliance on bounding the density evolution function in Equation (2) using the counterpart for regular unstructured LDPC ensembles, which proves to be easier to handle than the multivariate Equation (2).

Table II gives the WD thresholds obtained through forward DE for the $(d_l = 3, d_r = 6, \gamma = 3, L)$ spatially coupled ensemble for different target erasure rates δ and different window sizes W.

$W\backslash\delta$	10^{-6}	10^{-12}	10^{-18}
4	0.068403	0.000772	0.000008
8	0.472992	0.390749	0.254339
16	0.487504	0.487504	0.487504

TABLE II
WD THRESHOLDS $\epsilon^{\mathrm{WD}}(d_l = 3, d_r = 6, \gamma = 3, W, \delta)$.

IV. PROOF SKETCH

A. First Window Configuration

From Definition 1, forward DE for the first window configuration amounts to the following. Set $\underline{y}^{(0)}_{\{1\}} = \underline{1}$ and evaluate the sequence of window constellations $\{\underline{y}^{(\ell)}_{\{1\}}\}_{\ell=1}^{\infty}$ according to

$$y^{(\ell)}_{i,\{1\}} = \begin{cases} 0, & i \leq 0 \\ \epsilon g(y^{(\ell-1)}_{i-\gamma+1,\{1\}}, \cdots, y^{(\ell-1)}_{i+\gamma-1,\{1\}}), & i \in [W] \\ 1, & i > W. \end{cases} \quad (6)$$

Since $\underline{y}^{(0)}_{\{1\}}$ is non-decreasing, i.e. $y^{(0)}_{i,\{1\}} \leq y^{(0)}_{i+1,\{1\}}\ \forall\ i$, so is the first window configuration FP, $\underline{y}^{(\infty)}_{\{1\}}$, by induction and monotonicity of $g(\cdot)$.

We now give some bounds on the FP erasure probabilities of individual sections within a window. The proofs are similar to those used for analysing the BP decoder and are given in the full version of this paper [15].

Lemma 6 (Bounds on FP): Consider the WD of the (d_l, d_r, γ, L) ensemble with a window of size W over a channel with erasure rate ϵ. The first window configuration FP $\underline{y}$ satisfies

$$y_i \geq \Big(\epsilon(\frac{\gamma-1}{2\gamma})^{d_l-1}\Big)^{\frac{(d_l-1)^j-1}{d_l-2}} y_{i+j}^{(d_l-1)^j}$$
$$y_i \leq \epsilon\Big(1-\alpha_k(1-y_{i+k})^{d_r-1}\Big)^{d_l-1}$$

for $i \in [1, W], j \in [0, W+1-i], k \in [0, \gamma-1]$, where $\alpha_k = (1 - \frac{(\gamma-k-1)(\gamma-k)}{2\gamma^2})^{d_r-1}$.
The following shows that once the FP erasure probability of a section within the window is smaller than a certain value, it decays very quickly as we move further to the left in the window.

Lemma 7 (Doubly-Exponential Tail of the FP): Consider WD of the (d_l, d_r, γ, L) ensemble with a window of size W over a channel with erasure rate $\epsilon \in (0, 1)$. Let $d_l \geq 3$ and let $\underline{y}$ be the first window configuration FP of forward DE. If there exists an $i \in [W]$ such that $y_i < \delta_0 \triangleq \Big((d_r-1)^{\frac{d_l-1}{d_l-2}}\Big)^{-1}$, then

$$y_{i-j(\gamma-1)} \leq \Psi e^{-\psi(d_l-1)^j}$$

where $\Psi = \delta_0 \epsilon^{\frac{-1}{d_l-2}}$ and $\psi = \ln(\frac{\Psi}{\delta_0}) = \frac{1}{d_l-2}\ln\frac{1}{\epsilon} > 0$.
The proof uses the fact that, for random LDPC ensembles, below the *breakout value* [16], the erasure probability converges doubly exponentially in the number of iterations, and relates the role of iterations in the context of random LDPC ensembles to the role of spatially separated sections in the present context.

Definition 8 (Transition Width): Consider WD of a (d_l, d_r, γ, L) spatially coupled code over a BEC of erasure rate ϵ. Let $\underline{y}$ be the 1st window configuration FP of forward DE. Then we define the transition width $\tau(\epsilon, \delta)$ of $\underline{y}$ as

$$\tau(\epsilon, \delta) = |\{i \in [W] : \delta < y_i \leq 1\}|.$$

Definition 9 (First Window Threshold): Consider WD of the (d_l, d_r, γ, L) spatially coupled ensemble with a window of size W over a BEC with erasure rate ϵ. The first window threshold $\epsilon^{\mathrm{FW}}(d_l, d_r, \gamma, W, \delta)$ is defined as the supremum of channel erasure rates for which the first window configuration FP of forward DE $\underline{y}$ satisfies $y_1 \leq \delta$.

Proposition 10 (Maximum Transition Width): Consider the first window configuration FP of forward DE $\underline{y}$ for the (d_l, d_r, γ, L) spatially coupled ensemble with a window of size $W < L$ for $\epsilon \in [\frac{\epsilon^{\mathrm{BP}}(d_l,d_r,\gamma)+\epsilon^{\mathrm{BP}}(d_l,d_r)}{2}, \epsilon^{\mathrm{BP}}(d_l, d_r, \gamma)) = \mathscr{E}$. Then,

$$\tau(\epsilon, \delta) \leq (\gamma-1)\Big(\mathsf{A}\ln\ln\frac{\mathsf{D}}{\delta} + \mathsf{B}\ln\frac{1}{\Delta\epsilon} + \hat{\mathsf{C}}\Big) \triangleq \hat{\tau}(\epsilon, \delta)$$

provided $\delta \leq \delta_0$. Here $\Delta\epsilon = \epsilon^{\mathrm{BP}}(d_l, d_r, \gamma) - \epsilon$, and $\mathsf{A}, \mathsf{B}, \hat{\mathsf{C}}$, and D are strictly positive constants that depend only on the ensemble parameters d_l, d_r and γ and δ_0 is as defined in Lemma 7.

From Definitions 8, 9 and Proposition 10, we can see that by ensuring that $W \geq \hat{\tau}(\epsilon, \delta)$, we can bound $\epsilon^{\mathrm{FW}}(d_l, d_r, \gamma, W, \delta) \geq \epsilon$. The proof of Proposition 10 is reserved for the full version of the paper. This result means that the smallest window size that guarantees $y_1 \leq \delta$ for a channel erasure rate $\frac{\epsilon^{\mathrm{BP}}(d_l,d_r,\gamma)+\epsilon^{\mathrm{BP}}(d_l,d_r)}{2}$ is

$$\begin{aligned}\hat{W}_{\min}(\delta) &= (\gamma-1)\Big(\mathsf{A}\ln\ln\frac{\mathsf{D}}{\delta} + \mathsf{B}\ln\frac{1}{\Delta\epsilon_{\max}} + \hat{\mathsf{C}}\Big) \\ &= \hat{\tau}\Big(\frac{\epsilon^{\mathrm{BP}}(d_l,d_r,\gamma)+\epsilon^{\mathrm{BP}}(d_l,d_r)}{2}, \delta\Big)\end{aligned}$$

where $\Delta\epsilon_{\max} = \frac{\epsilon^{\mathrm{BP}}(d_l,d_r,\gamma)-\epsilon^{\mathrm{BP}}(d_l,d_r)}{2}$. When $W \geq \hat{W}_{\min}(\delta)$, we have

$$\begin{aligned}\epsilon^{\mathrm{FW}}(d_l, d_r, \gamma, W, \delta) &\geq \epsilon^{\mathrm{BP}}(d_l, d_r, \gamma) \\ &\quad - e^{-\frac{1}{\mathsf{B}}(\frac{W}{\gamma-1} - \mathsf{A}\ln\ln\frac{\mathsf{D}}{\delta} - \hat{\mathsf{C}})}. \end{aligned} \quad (7)$$

B. c^{th} Window Configuration, $1 < c \leq L$

We use the result from the previous subsection to show that, when you let the left end of the window have a non-zero but small erasure probability, corresponding to sliding the window through the sections of the code, the same results hold with some minor adjustments.

Proposition 11 (WD & FW Thresholds): Consider WD of the (d_l, d_r, γ, L) spatially coupled ensemble with a window of size $W \geq W_{\min}(\delta) = \hat{W}_{\min}(\delta) + \gamma - 1$ over a BEC with erasure rate ϵ. Then, we have

$$\begin{aligned}\epsilon^{\mathrm{WD}}(d_l, d_r, \gamma, W, \delta) \geq &\Big(1 - \frac{d_l d_r}{2}\delta^{\frac{d_l-2}{d_l-1}}\Big) \\ &\times \epsilon^{\mathrm{FW}}(d_l, d_r, \gamma, W-\gamma+1, \delta)\end{aligned}$$

provided $\delta < \delta_* = \left(\frac{2}{d_l d_r}\right)^{\frac{d_l-1}{d_l-2}}$, where $\epsilon^{\text{FW}}(d_l, d_r, \gamma, W, \delta)$ is the first window threshold.

From Proposition 11 and Equation (7), we immediately have that

$$\epsilon^{\text{WD}}(d_l, d_r,\gamma, W, \delta) \geq \left(1 - \frac{d_l d_r}{2}\delta^{\frac{d_l-2}{d_l-1}}\right) \times \left(\epsilon^{\text{BP}}(d_l, d_r, \gamma) - e^{-\frac{1}{\mathsf{B}}\left(\frac{W-\gamma+1}{\gamma-1} - \mathsf{A}\ln\ln\frac{\mathsf{D}}{\delta} - \hat{\mathsf{C}}\right)}\right)$$

provided $W \geq W_{\min}(\delta)$. By making the substitution $\mathsf{C} = \hat{\mathsf{C}} + 1$, we see that this proves Theorem 5.

V. Experimental Results

In this section, we give results obtained by simulating windowed decoding of finite-length spatially coupled codes over the binary erasure channel. The code used for simulation was generated randomly by fixing the parameters $M = 1024$, $d_l = 3, d_r = 6$, with coupling length $\gamma = 3$ and chain length $L = 64$. The blocklength of the code was hence $n = ML = 65,536$ and the rate was $R \approx 0.484375$. From Table I, the BP threshold for the ensemble to which this code belongs is $\epsilon^{\text{BP}}(d_l = 3, d_r = 6, \gamma = 3, L = 64) \approx 0.487514$.

Figure 1 shows the bit erasure rates achieved by using windows of length $W = 4, 6, 8$. From the figure, it is clear

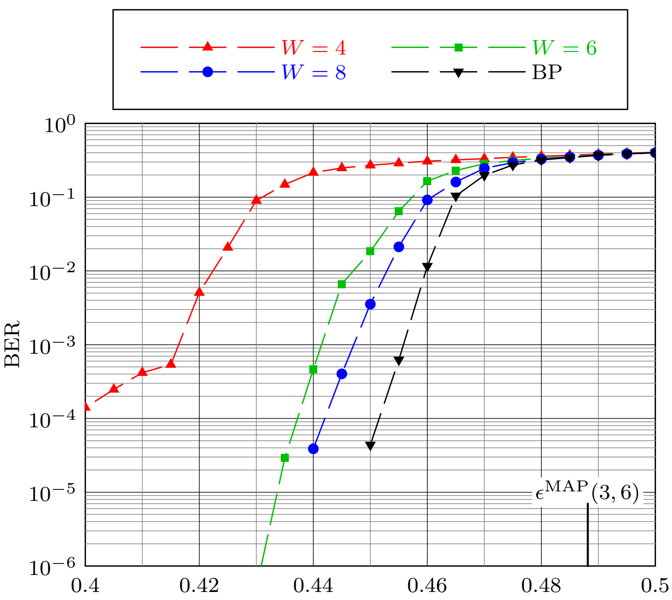

Fig. 1. Bit erasure probability of the $(d_l = 3, d_r = 6, \gamma = 3, L = 64)$ spatially coupled code with $M = 1024$ achieved with a windowed decoder of window sizes $W = 4, 6$ and 8.

that good performance can be obtained for a wide range of channel erasure rates even for small window lengths, e.g., $W = 6, 8$. In performing the simulations above, we let the decoders (BP and WD) run for as many iterations as possible, until the decoder could solve for no further bits. A more in-depth analysis of the complexity of WD is a topic of future research. Although the smaller window sizes have a large reduction in complexity and a decent BER performance, the block erasure rate performance can be fairly bad, e.g., for the window of size 4, the block erasure rate was 1 in the range of erasure rates considered in Figure 1. However, the block erasure rate improves dramatically with increasing window size—it is $\approx 6.3 \times 10^{-4}$ for window size 8 at $\epsilon = 0.44$.

VI. Conclusions

We considered a windowed decoding (WD) scheme for decoding spatially coupled codes that has a smaller complexity and latency compared to the BP decoder. We analysed the asymptotic performance limits of such a scheme by defining WD thresholds for meeting target erasure rates. Through simulations, we showed that WD is a viable scheme for decoding finite-length spatially coupled codes and that even for small window sizes, good performance is attainable for a wide range of channel erasure rates. The exact finite-length performance analysis of the WD scheme and analysis over channels that introduce errors are topics for future research.

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