WITT SPACES: A GEOMETRIC CYCLE THEORY FOR *KO*-HOMOLOGY AT ODD PRIMES

by P. H. SIEGEL

Introduction. This paper studies a class of stratified piecewise-linear pseudomanifolds, which we call Witt spaces, characterized by natural local intersection homology conditions [11]. We compute the cobordism groups by introducing an invariant taking values in the Witt group of symmetric bilinear forms over the rationals, $W(\mathbf{Q})$. These pseudomanifolds solve a problem posed by D. Sullivan [25]: to construct a class of P.L. cycles with signature which represent the connected version of KO homology at odd primes, $ko_* \otimes Z[1/2]$.

The result was one of the first applications of the intersection homology theory of Goresky and MacPherson [11]. Specifically, the rational intersection homology groups of Witt spaces satisfy a Poincaré duality theorem. To each Witt space X, we associate a P.L. invariant $w(X)$ with values in $W(\mathrm{Q})$. This invariant generalizes the signature of manifolds and satisfies cobordism invariance, additivity, and a product formula. Adapting classical surgery (spherical modification) to this setting, we prove that $w(X)$ determines the cobordism class of X and obtain an explicit description of the Witt cobordism groups. The only nontrivial groups occur in dimensions $4k$, and for $k > 0$ they are $W(\mathrm{Q})$.

Sullivan [25,27] has shown that a class of P.L. pseudomanifolds equipped with such an invariant possesses canonical orientations in $ko_* \otimes Z[1/2]$. The orientations induce a natural transformation μ of homology theories, from Witt space bordism to $ko_* \otimes Z[1/2]$. The structure of $W(\mathrm{Q})$ is known [14], and we conclude that μ is an equivalence at odd primes. In this paper, we study the cobordism theory of a class of stratified P.L. pseudomanifolds by means of the recent intersection homology theory of Goresky and MacPherson [11]. Goresky and MacPherson were interested in finding a class of spaces with cobordism invariant signature for the purpose of extending the Hirzebruch L-class to the setting of "manifolds with

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singularity." Their investigation of cycle intersection phenomena in stratified spaces resulted in their beautiful intersection homology theory [11]. They proved that for a stratified P.L. pseudomanifold X of dimension $4k$ with only even codimension strata, the rational intersection homology group $IH_{2k}^{\bar{m}}(X; \mathbf{Q})$ is self-dual, and they proceeded to define a cobordism invariant signature and L-class for the collection of spaces with only even codimension strata.

The cobordism groups of stratified spaces with even codimension strata have not yet been computed. We prove, though, that a cobordism invariant signature exists and cobordism calculations can be carried out for the class of spaces obtained by imposing an intersection homological link condition on odd codimension strata.

Definition. X is a *Witt-space* if whenever L^{2l} is the link of an odd codimension intrinsic stratum of X, $IH_l^{\bar{m}}(L; \mathbf{Q}) = 0$.

Three observations confirm that this definition is reasonable.

Observation 1. Atiyah in [4] studied examples of smooth fibrations $M^{2l} \to X \to N^{2k}$. He showed that if $H_l(M; \mathbf{Q}) = 0$, then $sign(X) = 0$. The fibration is P.L. [21], and Atiyah's result says that if the mapping cylinder is a Witt cobordism of X to 0 then $sign(X) = 0$.

Observation 2. If X^{4k} is a Witt space without boundary, then $IH_{2k}^{\bar{m}}(X; \mathbf{Q})$ is self-dual (I.3.4). The signature $sign(X)$ is the signature of this rational inner product space. It is a Witt cobordism invariant and extends the signature defined in [11].

Observation 3. The signature is additive for Witt spaces, and the proof, unlike the standard proof in the smooth case [5, p. 588], is entirely geometric: The "pinch cobordism," which roughly is the mapping cylinder of the map collapsing Z in $X \cup_Z -Y$ to a point, is a Witt cobordism of $X \cup_Z -Y$ to $X \cup_Z \text{cone}(Z) - Y \cup_Z \text{cone}(Z)$.

We then adapt classical surgery to Witt spaces and compute the Witt cobordism groups.

THEOREM. *Let Ω_*^{Witt} denote the bordism theory based on Witt spaces. Then:*

$$\Omega_0^{Witt}(pt) = \mathbf{Z}$$

$$\Omega_q^{Witt}(pt) = 0, \qquad q \not\equiv 0 \pmod 4.$$

$$\Omega_q^{Witt}(pt) = W(\mathbf{Q}), \qquad q > 0 \quad \textit{and} \quad q \equiv 0 \pmod 4)$$

Here $W(\mathbf{Q})$ is the Witt ring of $\mathbf{Q}$, and the isomorphism associates a Witt space X^{4k} to the Witt class of $IH_{2k}^{\bar{m}}(X; \mathbf{Q})$.

Remark. For discussion and application of a special case of the Witt class invariant occurring in the theorem, see [3], [10].

We show that these pseudomanifolds provide a geometric description of connected KO homology at odd primes, $ko_* \otimes \mathbf{Z}[1/2]$, thereby solving a problem posed by D. Sullivan [25]. To put the result in perspective, we give some of the historical background to the problem.

Dennis Sullivan [25] discovered the Conner-Floyd-type theorem [9] relating smooth oriented bordism, the signature, and connected KO theory at odd primes:

THEOREM (Sullivan). *For compact* P.L. *pairs, there is a canonical isomorphism (equivalence) of homology theories*:

$$\Omega_*^{SO}(X, A) \otimes_{\Omega_*^{SO}(pt)} \mathbf{Z}\left[\frac{1}{2}\right] \xrightarrow{\cong} ko_*(X, A) \otimes \mathbf{Z}\left[\frac{1}{2}\right],$$

where the theories are regarded as $\mathbf{Z}/4\mathbf{Z}$-graded.

He found (at least) two interesting applications: (1) A geometric description of connected KO homology at odd primes, $ko_* \otimes \mathbf{Z}[1/2]$, as a bordism theory with join-like singularities obtained by geometrically "killing" generators of smooth bordism. See [6], [8], [26]. (2) A construction of a canonical $KO^* \otimes \mathbf{Z}[1/2]$ orientation for a P.L. block bundle over a finite complex, and, in particular, a canonical $KO_* \otimes \mathbf{Z}[1/2]$ orientation for P.L. manifolds via Alexander duality [25]. If the collection of cycles of a geometric homology theory is closed under the operations of taking cartesian product and transversal intersections with P.L. manifolds, and if it possesses a "nice" signature invariant extending the classical signature, the construction shows that the cycles have canonical $KO_* \otimes \mathbf{Z}[1/2]$ orientations [27]. See Chapter IV and the Appendix.

With Witt spaces, we achieve a natural synthesis of the themes of these two applications by solving the following.

PROBLEM. *Construct a geometric cycle theory for $ko_* \otimes \mathbf{Z}[1/2]$ built from a class of* P.L. *cycles $\mathfrak{F}$ with "nice" signature invariant, such that the equivalence (at odd primes)*

$$\mu : \Omega_*^{\mathfrak{F}} \to ko_* \otimes \mathbf{Z}\left[\frac{1}{2}\right]$$

is induced by the orientations described in (2) above.

Using results of D. Sullivan, we construct a natural transformation of homology theories:

$$\mu^{\text{Witt}}: \Omega_*^{\text{Witt}} \to ko_* \otimes \mathrm{Z}\left[\frac{1}{2}\right]$$

which reduces to the signature homomorphism on coefficient groups when $q \equiv 0 \pmod 4$. Tensoring with Z[1/2], we conclude:

THEOREM. *The natural transformation*

$$\mu^{\text{Witt}} \otimes \mathrm{Z}\left[\frac{1}{2}\right]: \Omega_*^{\text{Witt}} \otimes \mathrm{Z}\left[\frac{1}{2}\right] \to ko_* \otimes \mathrm{Z}\left[\frac{1}{2}\right]$$

is an equivalence of homology theories.

We note here that J. Morgan (unpublished) has constructed a geometric cycle theory for the homology associated to G/PL using different techniques.

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Chapter I. Witt Spaces and Generalized Poincare Duality

1. Introduction. We prove (3.4) that if X^q is a Witt space without boundary, the homomorphism $IH_*^{\bar{m}}(X; \mathrm{Q}) \to IH_*^{\bar{n}}(X; \mathrm{Q})$ induced by inclusion of chain complexes is an isomorphism. (The perversities $\bar{m}$ and $\bar{n}$ are as in [11].) It follows that the augmented intersection pairing on $IH_*^{\bar{m}}(X; \mathrm{Q})$ is nondegenerate. We then define (4.1) the Witt class of X, $w(X)$, a P.L. invariant taking values in $W(\mathrm{Q})$.

2. Witt Spaces. Let X^q be a q-dimensional P.L. pseudomanifold with (possibly empty) collared boundary ∂X. Let $x \in X - \partial X$. The link of x, $lk(x, X)$, is unique up to P.L. homeomorphism [22]. Let $d(x)$ be the intrinsic dimension of X at x. Then there is a P.L. homeomorphism: $lk(x, X) \approx S^{d(x)-1} * L(x)$, the join of $S^{d(x)-1}$ and $L(x)$. The space $L(x)$ is a pseudomanifold of dimension $l(x) = q - d(x) - 1$, unique up to P.L. homeomorphism [1, pp. 420 and 424].

Definition 2.1. Let X^q be a q-dimensional P.L. pseudomanifold as above. We say X is a Witt space if

$$IH^{\bar{m}}_{l(x)/2}(L(x); \mathbf{Q}) = 0,$$

for all $x \in X - \partial X$ such that $l(x) \equiv 0 \pmod 2$.

For applications, we must translate the Witt condition in 2.1 into a statement about arbitrary stratifications of X. To this end, we prove a few lemmas which culminate in Proposition 2.5.

LEMMA 2.2. *Let L be a closed* P.L. *pseudomanifold of dimension $2l$. Suppose $L = S^0 * K$ for some* P.L. *pseudomanifold K of dimension $2l - 1$. Then $IH^{\bar{m}}_l(L; \mathbf{Z}) = 0$.*

Proof. The 0-sphere S^0 is the discrete set $\{a, b\}$. Let T be a triangulation of K, and $\mathfrak{F}$ the associated stratification: $\mathfrak{F} = (T_{2l-1} \supset \Sigma_T = T_{2l-3} \supset \cdots \supset T_0)$. Let S be the triangulation of L induced by suspension, and let $\mathcal{S}$ be the stratification on L similarly induced.

Consider a cycle $z \in IC^{\bar{m}}_l(L)$, where perversity is defined with respect to $\mathcal{S}$. Then, $|z| \cap S^0 = \phi$. Choose a triangulation S_1 of L subordinate to S such that z is a sum of l-simplices with coefficients. Each vertex v of $|z|$ lies in the interior of a unique simplex σ of L. If $\mathrm{Int}(\sigma) \subset K$, define $f(\sigma) = \hat{\sigma}$, the barycenter of σ. If $v \in \mathrm{Int}(a * \tau)$ or $v \in \mathrm{Int}(b * \tau)$, where $\mathrm{Int}(\tau) \subset K$, then define $f(\sigma) = \hat{\tau}$. Extend the map f linearly to simplices of $|z|$. By [22, p. 17], f determines a P.L. geometric chain which we denote $f(z)$ in $C_l(L)$, supported in $|K|$. One easily checks that $f(z)$ is a cycle and, in fact, $|f(z)|$ is $(\bar{m}, l)$-allowable. We claim that $f(z)$ represents the same homology class as z in $IH^{\bar{m}}_l(L)$. The triangulation of z can be used to construct a triangulation of $|z| \times [0, 1]$ having as its vertices the set $\{(v, 0), (v, 1) | v \in |z|\}$ and subordinate to the product cell structure on $|z| \times [0, 1]$. See, for instance, [11]. Define $H\colon |z| \times [0, 1] \to L$ by $H(v, 0) = v$ and $H(v, 1) = f(v)$, and extend linearly over simplices of $|z| \times [0, 1]$. Appealing again to [22, p. 17], we obtain a P.L. chain $H(z \times [0, 1])$ in $C_{l+1}(L)$, with $\partial H(z \times [0, 1]) = f(z) - z$. Moreover, it is easy to see that $|H(z \times [0, 1])|$ is $(\bar{m}, l + 1)$-allowable. This proves the claim.

Let W be the obvious P.L. chain in $C_{l+1}(L)$ with $|W| = a * |f(z)|$ and $\partial W = f(z)$. Then $W \in IC^{\bar{m}}_{l+1}(L)$, implying that the homology class of $f(z)$ in $IH^{\bar{m}}_l(L)$ is trivial. Therefore z represents 0 in $IH^{\bar{m}}_l(L)$. Since z was arbitrary, we conclude $IH^{\bar{m}}_l(L) = 0$. □

COROLLARY 2.3. *Let L be a closed* P.L. *pseudomanifold of dimension $2l$. Suppose $L = S^j * K$ for some* P.L. *pseudomanifold K, where $0 \leq j \leq 2l$. Then, $IH_l^{\bar{m}}(L) = 0$.*

Proof. Rewrite L as $S^0 * (S^{j-1} * K)$, and apply 2.2. □

COROLLARY 2.4. *Let L be as in 2.3. Then $IH_l^{\bar{m}}(L; \mathbf{Q}) = 0$.*

Proof. The result follows from 2.3 and the isomorphism of groups: $IH_l^{\bar{m}}(L; \mathbf{Q}) \cong IH_l^{\bar{m}}(L) \otimes \mathbf{Q}$, induced by the definition: $IC_*^{\bar{m}}(L; \mathbf{Q}) = IC_*^{\bar{m}}(L) \otimes \mathbf{Q}$. □

PROPOSITION 2.5. *Let X^q be a stratified* P.L. *pseudomanifold, with (possibly empty) boundary ∂X and stratification $X = (X_q \supset X_{q-1} = X_{q-2} \supset \cdots \supset X_0)$. Let $L(\chi_i; x)$ be the link of $\chi_i = X_i - X_{i-1}$ at x, where $x \in (X - \partial X) \cap \chi_i$, and $\chi_i \neq \phi$. Then, X is a Witt space if and only if*

$$IH_l^{\bar{m}}(L(\chi_i; x); \mathbf{Q}) = 0$$

for $i = q - (2l + 1)$, whenever $l \geq 1$.

Proof. Suppose X is Witt. There are P.L. homeomorphisms:

$$lk(x, X) \approx S^{d(x)} * L(x)$$

$$lk(x, X) \approx S^i * L(\chi_i; x).$$

Let $j = d(x) - i$. Then $j \geq 0$, and

$$L(\chi_i; x) \approx S^{j-1} * L(x) \quad \text{by} \quad [1, \text{p. } 423].$$

Case 1. If $j - 1 = -1$, then $L(\chi_i; x) \approx L(x)$, so the result follows from the fact that X is a Witt space.

Case 2. If $j - 1 \geq 0$, the result follows from 2.4.

The converse follows from the fact that any P.L. stratification of X is subordinate to the intrinsic stratification. □

3. The Intersection Homology Spectral Sequence and Generalized Poincaré Duality. Assume X is a q-dimensional P.L. pseudomanifold with triangulation T, and associated stratification $\mathfrak{J}$. Let r be the largest integer satisfying $2r + 1 \leq q$ (assume $q \geq 2$). *Assume rational coefficients unless otherwise specified.*

Definition 3.1. Let $\bar{p}_k$ be the perversity defined by:

$$\bar{p}_k(c) = \begin{cases} \bar{m}(c) = \left[\dfrac{c-2}{2}\right] \text{ for } c \leq k \\ \\ \bar{n}(c) = \left[\dfrac{c-1}{2}\right] \text{ for } c > k \end{cases}$$

where $1 \leq k \leq q$ and $2 \leq c \leq q$.

Since $\bar{m}(c) = \bar{n}(c)$ for c even, we may assume k odd. Note that $\bar{p}_{2r+1} = \bar{m}$ and $\bar{p}_1 = \bar{n}$.

There is a filtration F on the chain complex $W_*^{\bar{n}}(x)$ [see 11]:

$$F_s W_*^{\bar{n}}(x) = \begin{cases} 0 & \text{for } \; s < 0 \\ W_*^{\bar{p}_{(2r+1)-2s}}(x) & \text{for } \; 0 \leq s \leq r \\ W_*^{\bar{n}}(x) & \text{for } \; s > r \end{cases}$$

The filtration is induced by the inclusions of complexes: $0 \subset W_*^{\bar{m}} = W_*^{\bar{p}_{2r+1}} \subset W_*^{\bar{p}_{2r-1}} \subset \cdots \subset W_*^{\bar{p}_3} \subset W_*^{\bar{p}_1} = W_*^{\bar{n}}$.

Following [24, p. 469], there is a convergent E^1-spectral sequence with

$$E^1_{s,t} = H_{s+t}(F_s W_*^{\bar{n}}(X)/F_{s-1} W_*^{\bar{n}}(X))$$

and E^∞ isomorphic to the bigraded module $G(IH_*^{\bar{n}}(X))$, the associated graded to the filtration

$$F_s(IH_*^{\bar{n}}(X)) = Im(IH_*^{\bar{p}_{(2r+1)-2s}}(X) \to IH_*^{\bar{n}}(X)).$$

THEOREM 3.2. *Assume X is a Witt space. Then*

(1) $$E^\infty_{0,t} = Im(IH_t^{\bar{m}}(X) \to IH_t^{\bar{n}}(X))$$

(2) $$E^\infty_{s,t} = 0 \quad \text{for} \quad s > 0.$$

Proof. Siegel [23] gives a proof based on a geometric cycle-lifting argument, in the classical obstruction theoretic vein. Another proof, using the sheaf-theoretic formulation of intersection homology is given in [12], Section 5.6.1.

Remark 3.3. If the Witt condition on links is satisfied only for odd codimension strata of codimension less than or equal to $2k + 1$, then the spectral sequence collapses partially:

$$E^{\infty}_{s,t} = 0 \quad \text{for} \quad s > (r - k). \qquad \square$$

Finally, we prove generalized Poincaré duality for Witt spaces.

THEOREM 3.4. *Let X^q be a Witt space. There is a nondegenerate rational pairing*

$$IH^{\bar{m}}_i(X; Q) \times IH^{\bar{m}}_j(X; Q) \to Q$$

for $i + j = q$; $i, j \geq 0$.

The pairing is given by augmented intersection product.

Proof. For $q = 0$ or 1, the result claimed is obvious. For $q \geq 2$, Theorem 3.2 implies:

$$Im(IH^{\bar{m}}_*(X; Q) \to IH^{\bar{n}}_*(X; Q)) = IH^{\bar{n}}_*(X; Q) \tag{3}$$

Therefore:

$$\dim_Q IH^{\bar{m}}_j(X; Q) \geq \dim_Q IH^{\bar{n}}_j(X; Q) \quad \text{for} \quad 0 \leq j \leq q. \tag{4}$$

The intersection pairing theorem [11] implies:

$$\dim_Q IH^{\bar{m}}_j(X; Q) = \dim_Q IH^{\bar{n}}_{q-j}(X; Q) \quad \text{for} \quad 0 \leq j \leq q. \tag{5}$$

Combined, (4) and (5) yield:

$$\dim_Q IH^{\bar{m}}_j(X; Q) = \dim_Q IH^{\bar{n}}_j(X; Q) \quad \text{for} \quad 0 \leq j \leq q. \tag{6}$$

It follows that inclusion of chain complexes induces an isomorphism:

$$i_*: IH^{\bar{m}}_*(X; Q) \to IH^{\bar{n}}_*(X; Q). \tag{7}$$

Then, [11] provides a nondegenerate rational pairing

$$IH^{\bar{m}}_i(X; Q) \times IH^{\bar{n}}_j(X; Q) \to Q, \qquad \text{for} \quad i + j = q; i, j \geq 0.$$

Intersection product of cycles is compatible with the isomorphism in (7) by [11]. The theorem follows. □

4. The Witt Class $w(X)$. Let X be a Witt space of dimension $4k$, $k > 0$. Theorem 3.4 proves that $IH^{\bar{m}}_{2k}(X; \mathbf{Q})$ is a symmetric inner product space.

Definition 4.1. (1) Let X^q be a Witt space of dimension q, $q \geq 0$.

If $q = 4k$, $k > 0$, the *Witt class of* X, denoted $w(X)$, is the equivalence class of the inner product space $IH^{\bar{m}}_{2k}(X; \mathbf{Q})$ in $W(\mathbf{Q})$, the Witt group of the rationals.

If $q = 0$, $w(X) = \operatorname{rank}(H_0(X; \mathbf{Q}))\cdot\langle 1\rangle$ in $W(\mathbf{Q})$, where $\langle 1\rangle$ is the one-dimensional inner product space with matrix 1.

If $q \neq 4k$, set $w(X) = 0$ in $W(\mathbf{Q})$.

(2) If $(X, \partial X)$ is Witt, set $w(X) = w(\hat{X})$, where $\hat{X} = X \cup \operatorname{cone}(\partial X)$.

Let $sign_{\mathbf{Q}}: W(\mathbf{Q}) \to \mathbf{Z}$ be the signature homomorphism [14].

Definition 4.2. The *signature of* X, *sign*(X), is the integer $sign_{\mathbf{Q}}(w(X))$.

Chapter II. Properties of the Witt Class $w(X)$.

1. Introduction. In this chapter, we study properties of the Witt class, $w(x)$.

(1) Cobordism invariance:
If $X^{4k} = \partial W^{4k+1}$, then $w(X) = 0$.

(2) Additivity:
If $Z^{4k-1} = \partial X^{4k}$ and $-Z^{4k-1} = \partial Y^{4k}$, then $w(X \cup_Z Y) = w(X) + w(Y)$.

(3) Multiplicativity with respect to signature of closed manifolds:
If X is a Witt space, and M is a closed P.L. manifold,

$$w(M \times X) = sign(M)\cdot w(X).$$

In Theorem 2.1, we give a proof of Witt cobordism invariance of the Witt class.

In Theorem 3.1, we give a geometrical proof of additivity. By means of the "pinch cobordism," we deduce additivity easily from cobordism invariance. Incidentally this result implies the Novikov additivity theorem for the signature of manifolds [cf. 5, p. 588], [15].

Section 4 contains a Kunneth formula for intersection homology of a product $M \times X$, where M is a closed manifold and X a closed Witt space (4.1). From this, we deduce multiplicativity, generalizing a similar formula stated in [10] for the case where X is the quotient space of a prime order diffeomorphism on a smooth manifold.

2. Cobordism Invariance.

THEOREM 2.1. *Let $(Y, \partial Y)$ be a $(4k + 1)$-dimensional Witt space with boundary. Then $w(\partial Y) = 0$.*

Proof. The argument resembles that in [11], which in turn follows [13].

Let $\hat{Y} = Y \cup \text{cone}(\partial Y)$ be the space obtained from Y by adjoining the cone on the boundary. Since Y is a Witt space, there is a commutative diagram of rational vector spaces (assume rational coefficients):

$$\begin{tikzcd}
IH_{2k+1}^{\bar{m}+}(\hat{Y}) \arrow[r, "i"] \arrow[d, "\alpha"] & IH_{2k}^{\bar{m}}(\partial Y) \arrow[r] \arrow[d, "\beta"] & IH_{2k}^{\bar{m}}(\hat{Y}) \arrow[d, "\gamma"] \\
IH_{2k+1}^{\bar{n}}(\hat{Y}) \arrow[r] & IH_{2k}^{\bar{n}}(\partial Y) \arrow[r] & IH_{2k}^{\bar{n}-}(\hat{Y})
\end{tikzcd}$$

Here $\bar{m} + (c) = \bar{m}(c)$ for $2 \le c \le 4k$, $\bar{m} + (4k + 1) = \bar{n}(4k + 1) = 2k$, and $\bar{n} - (c) = \bar{n}(c)$ for $2 \le c \le 4k$, $\bar{n} - (4k + 1) = \bar{m}(4k + 1) = 2k - 1$. Note $(\bar{m}+) + (\bar{n}-) = \bar{t}$.

In the diagram, the row sequences are exact and, in fact, dual to each other (over Q). The vertical map α is surjective (by I.3.3) and β is the isomorphism induced by inclusion. A standard argument in linear algebra [13] implies that $i(IH_{2k}^{\bar{m}+}(\hat{Y}))$ is self-annihilating in $IH_{2k}^{\bar{m}}(\partial Y)$, under the pairing of I.3.4, and $\dim_Q i(IH_{2k+1}^{\bar{m}+}(\hat{Y})) = (1/2)\dim_Q IH_{2k}^{\bar{m}}(\partial Y)$. It follows that $w(\partial Y) = 0$.

3. Additivity. The cobordism invariance of the Witt class $w(X)$ suggests a simple geometric proof of the additivity of $w(X)$. "Additivity" refers to the property described in the following proposition.

PROPOSITION 3.1. *Let Y be an oriented $4k$-dimensional Witt space, and Z, X_1, X_2 subspaces of Y such that:*

(1) $Y = X_1 \cup X_2$
(2) $X_1 \cap X_2 = Z$, *and Z is bicollared in Y*

(3) (X_1, Z) *and* $(X_2, -Z)$ *are oriented* $4k$-dimensional *Witt spaces with boundary, and orientations compatible with that of* Y.

Then, $w(Y) = w(X_1) + w(X_2)$.

Proof. Let K_1, K_2 be triangulations of X_1, X_2, respectively. Denote by K_1^+, K_2^+ the complexes with isomorphic simplicial collars added on the outside [22]. The induced triangulation of Z (on the outside edge of the collars) will be denoted L. Then $|K| = |K_1^+ \cup_L K_2^+|$ is P.L. homeomorphic to Y, as in [21, p. 24]. The "pinched space" is the polyhedron underlying the complex J defined by:

$$J = [K_1 \cup \operatorname{cone}(L_1)] \underset{v}{\cup} [K_2 \cup \operatorname{cone}(L_2)]$$

where L_1, L_2 are the triangulations of Z in X_1, X_2, and v is the common cone vertex.

Now, define a continuous map: $p: |K| \to |J|$ by collapsing $|L|$ to v in $|J|$. Orient $|J|$ so that the induced map p_* on homology carries the orientation of $|K|$ to that of $|J|$. By constructing a triangulation explicitly, we see that the mapping cylinder C_p is triangulable (see Figure 1). Attach a collar $|J| \times I$ to C_p, forming the space P. It is not difficult to see that P is a pseudomanifold with collared boundary. It can be oriented so that $\partial P = |K| - |J|$. We call P the *pinch cobordism*. See Figure 2.

To check that P is a Witt space, it is enough to check that $IH_{2k}^{\bar{m}}(lk(v, P); Q) = 0$. But $lk(v; P)$ is P.L. homeomorphic to $Z \times [-1, 1] \cup \operatorname{cone}(\partial(Z \times [-1, 1]))$. This is homeomorphic to the suspension of Z with suspension points identified. Therefore, $S^0 * Z$ has the same normalization as $lk(v, P)$ [11], implying

$$IH_{2k}^{\bar{m}}(lk(v, P); Q) \cong IH_{2k}^{\bar{m}}(S^0 * Z; Q) \cong 0,$$

with the latter isomorphism from I.2.4.

By cobordism invariance of the Witt class, we have

$$\begin{aligned} w(Y) = w(|J|) &= w(\hat{X}_1) + w(\hat{X}_2) \\ &= w(X_1) + w(X_2) \quad \text{(by definition I.4.1)} \qquad \square \end{aligned}$$

4. Multiplicativity with Respect to Signature of Manifolds. Suppose M is a P.L. manifold, and X is a Witt space, both without boundary.

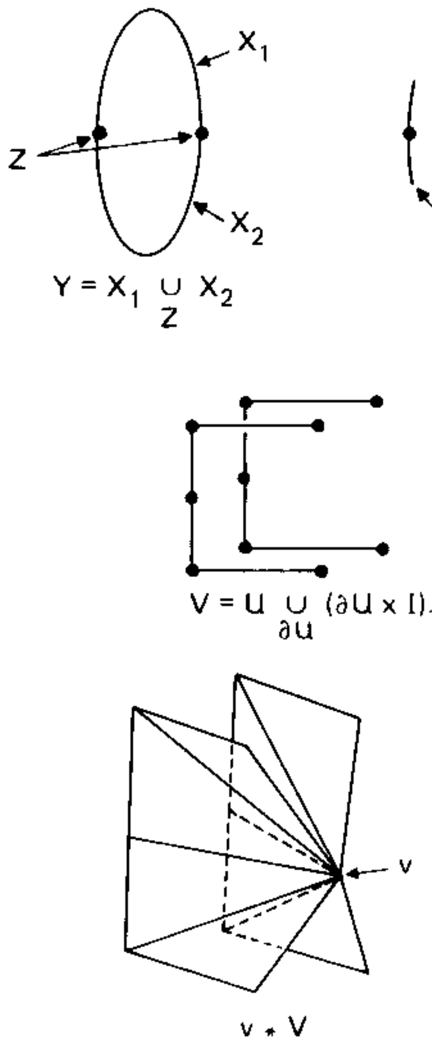

Figure 1. Triangulating a pinch cobordism.

Then $M \times X$ is a Witt space without boundary. We prove the product formula

$$w(M \times X) = sign(M) \cdot w(X) \tag{1}$$

where $sign(M) \cdot w(X)$ is $sign(M)$ times the element $w(X)$ in the abelian group $W(\mathbf{Q})$.

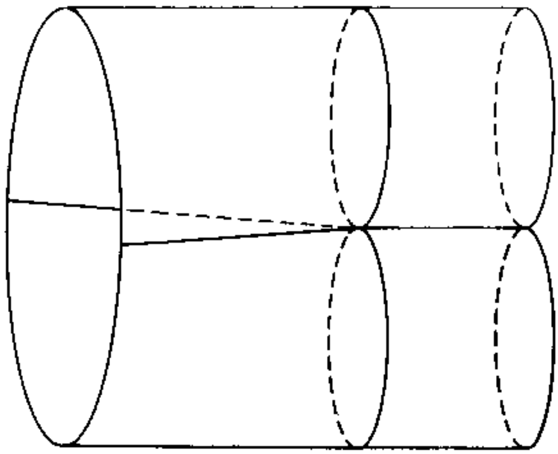

Figure 2. Sketch of a pinch cobordism.

The proof proceeds as follows. For closed manifold M and arbitrary pseudomanifold X, we construct an isomorphism

$$i_*^{\bar{p}}: H_*(M; \mathbf{Q}) \otimes IH_*^{\bar{p}}(X; \mathbf{Q}) \to IH_*^{\bar{p}}(M \times X; \mathbf{Q}),$$

for every perversity $\bar{p}$. This is done in Theorem 4.1.

Assume X is a Witt space, $\dim(M \times X) = 4k$, and specialize to the case $\bar{p} = \bar{m}$. We have an isomorphism

$$\sum_{i+j=2k} H_i(M; \mathbf{Q}) \otimes IH_j^{\bar{m}}(X; \mathbf{Q}) \to IH_{2k}^{\bar{m}}(M \times X; \mathbf{Q})$$

which can be regarded as an isomorphism of rational inner product spaces. The inner product on the right-hand side is that given by Theorem I.3.4, while that on the left side is uniquely determined by:

$$\langle a \otimes b, c \otimes d \rangle$$

$$= \begin{cases} (-1)^{\dim b \dim c} \langle a, c \rangle_M \cdot \langle b, d \rangle_X & \text{if} \quad \dim a + \dim c = \dim M \\ & \qquad \dim b + \dim d = \dim X \\ 0 & \qquad \text{otherwise} \end{cases}$$

where $\langle\ ,\ \rangle_M$ and $\langle\ ,\ \rangle_X$ are the intersection pairings on $H_*(M; \mathbf{Q})$ and $IH_*^{\bar{m}}(X; \mathbf{Q})$. It follows at once that $w(M \times X) = 0$ if $\dim M \equiv 1 \pmod 2$ or dim $M \equiv 2 \pmod 4$ and $w(M \times X) = sign(M)\cdot w(X)$ if dim $M \equiv 0 \pmod 4$.

Now we formulate and prove Theorem 4.1. First, define a chain homomorphism

$$i^{\bar{p}}: C_*(M) \otimes IC_*^{\bar{p}}(X) \to IC_*^{\bar{p}}(M \times X)$$

as follows. Given $c_i \otimes d_j \in C_*(M) \otimes IC_*^{\bar{p}}(X)$, let $\tilde{c}_i$ (resp. $\tilde{d}_j$) be the homology class in $H_i(|c_i|, |\partial c_i|)$ (resp. $H_j(|d_j|, |\partial d_j|)$) corresponding to c_i (resp. d_j). Denote by $c_i \times d_j$ the chain in $C_{i+j}(|c_i| \times |d_j|) \subset C_{i+j}(M \times X)$ corresponding to the ordinary exterior cross product $\tilde{c}_i \times \tilde{d}_j$ in $H_{i+j}(|c_i| \times |d_j|, |\partial c_i| \times |d_j| \cup |c_i| \times |\partial d_j|)$. With respect to the product stratification on $M \times X$, $c_i \times d_j$ lies in $IC_{i+j}^{\bar{p}}(M \times X)$. Set $i^{\bar{p}}(c_i \otimes d_j) = c_i \times d_j$. Clearly $i^{\bar{p}}$ is a chain map. Let $i_*^{\bar{p}}$ be the homomorphism induced on rational homology:

$$I_*^{\bar{p}}: H_*(C_*(M; \mathrm{Q}) \otimes IC_*^{\bar{p}}(X; \mathrm{Q})) \to IH_*^{\bar{p}}(M \times X; \mathrm{Q}).$$

THEOREM 4.1. *$i_*^{\bar{p}}$ is an isomorphism.*

Proof. We assume rational coefficients throughout, when not specified.

LEMMA 4.2. *$i_*^{\bar{p}}$ is injective.*

By the classical Kunneth theorem, there is a canonical isomorphism

$$H_*(C_*(M; \mathrm{Q}) \otimes IC_*^{\bar{p}}(X; \mathrm{Q})) \cong H_*(M; \mathrm{Q}) \otimes IH_*^{\bar{p}}(X; \mathrm{Q}).$$

Let $\bar{q}$ be the perversity such that $\bar{p} + \bar{q} = \bar{t}$, and consider the homomorphism:

$$i_*^{\bar{q}}: H_*(M) \otimes IH_*^{\bar{q}}(X) \to IH_*^{\bar{q}}(M \times X).$$

The dual homomorphism is:

$$(i_*^{\bar{q}})^*: (IH_*^{\bar{q}}(M \times X))^* \to (H_*(M) \otimes IH_*^{\bar{q}}(X))^*$$

There are canonical isomorphisms

$$(IH_*^{\bar{q}}(M \times X))^* \cong IH_*^{\bar{p}}(M \times X) \tag{2}$$

$$(H_*(M) \otimes IH_*^{\bar{q}}(X))^* \cong (H_*(M))^* \otimes (IH^{\bar{q}}(X))^* \cong H_*(M) \otimes IH_*^{\bar{p}}(X) \tag{3}$$

by I.3.4.

So, we regard $(i_*^{\bar{q}})^*$ as a homomorphism

$$(i_*^{\bar{q}})^* : IH_*^{\bar{p}}(M \times X) \to H_*(M) \otimes IH_*^{\bar{p}}(X).$$

Injectivity of $i_*^{\bar{p}}$ follows from the following claim.

Claim. The composition $(i_*^{\bar{q}})^* \circ i_*^{\bar{p}}$ is the identity homomorphism.

To verify the claim, observe that

$$\langle i_*^{\bar{p}}(\Sigma\, \alpha_i \otimes \beta_j), i_*^{\bar{q}}(\Sigma\, \gamma_k \otimes \delta_l) \rangle_{M \times X} = \langle \Sigma\, \alpha_i \otimes \beta_j, \Sigma\, \gamma_k \otimes \delta_l \rangle_{\otimes},$$

where $\langle\ ,\ \rangle_{\otimes}$ is the pairing of $H_*(M) \otimes IH_*^{\bar{p}}(X)$ with $H_*(M) \otimes IH_*^{\bar{q}}(X)$, and that, by I.3.4, intersection numbers determine the isomorphisms (2) and (3) above. □

LEMMA 4.3. *$i_*^{\bar{p}}$ is surjective.*

Proof. A geometric proof is given in Siegel [23]. The main idea is to use Lemma 4.4 below to construct homologies by deformation of cycles. We include this lemma for its independent interest. An axiomatic proof based on the sheaf theoretic formulation of intersection homology may be found in [12], Section 6.2.

The following technical lemma provides the deformation retractions used in the proof of 4.3. We thank Javier Bracho for his assistance on this argument.

LEMMA 4.4. *Let S be a triangulation of M, with skeleta S_j and coskeleta $S^{(j)}$, for $0 \leq j \leq m$. Let T be a triangulation of X, with skeleta T_i, for $0 \leq i \leq x$. Let $\bar{p}$ and $\bar{q}$ be perversities with $\bar{p} + \bar{q} = \bar{t}$. Let $Q_i^{\bar{p}}$ and $Q_i^{\bar{q}}$ be the associated basic sets* [11]. *Given integers i, j with $1 \leq i \leq x$ and $0 \leq j \leq m$, there is an embedding*:

$$M \times X \to (S^{(j)} \times [Q_i^{\bar{q}} \cap T_{x-2}]) * [(S_{m-j-1} \times X) \cup (M \times Q_{x-i+1}^{\bar{p}})]$$

which is the identity on

$$S^{(j)} \times [Q_i^{\bar{q}} \cap T_{x-2}]$$

and

$$(S_{m-j-1} \times X) \cup (M \times Q^{\bar{p}}_{x-i+1})$$

and which takes $M \times X$ to a union of join lines.

Proof. It suffices to define compatible embeddings on products of simplices in S' and T':

$$\Delta^p \times \Delta^q \to (\Delta^a \times \Delta^c) * ((\Delta^b \times \Delta^q) \cup (\Delta^p \times \Delta^d))$$

where $\Delta^p \in S'$, $\Delta^a = \Delta^p \cap S^{(j)}$, $\Delta^b = \Delta^p \cap S_{m-j-1}$, and

$$\Delta^q \in T', \Delta^c = \Delta^q \cap [Q^{\bar{q}}_i \cap T_{x-2}], \Delta^d = \Delta^q \cap Q^{\bar{p}}_{x-i+1}.$$

Let $\{v^a_i\}$, $\{v^b_i\}$, $\{v^c_i\}$, $\{v^d_i\}$ be vertices of the simplices in these decompositions:

$$\Delta^p = \Delta^a * \Delta^b$$

$$\Delta^q = \Delta^c * \Delta^d.$$

Any points $x \in \Delta^p$ and $y \in \Delta^q$ have unique expressions in terms of barycentric coordinates:

$$x = \Sigma\, a_i v^a_i + \Sigma\, b_i v^b_i, \text{ with } \Sigma\, a_i + \Sigma\, b_i = 1, a_i, b_i \geq 0$$

and

$$y = \Sigma\, c_i v^c_i + \Sigma\, d_i v^d_i, \text{ with } \Sigma\, c_i + \Sigma\, d_i = 1, c_i, d_i \geq 0.$$

Let $I = [0, 1]$. Define

$$t : \Delta^p \to I \quad \text{by} \quad t(x) = \Sigma\, b_i$$

and

$$s : \Delta^q \to I \quad \text{by} \quad s(y) = \Sigma\, d_i.$$

Both are P.L. maps.

Consider the closed convex subspaces of $\Delta^p \times \Delta^q$:

$$A = \{(x, y) \mid t(x) \le s(y)\}$$

$$B = \{(x, y) \mid t(x) \ge s(y)\}$$

$$C = \{(x, y) \mid t(x) = s(y)\}.$$

See Figure 3. We define the embedding separately on these subspaces. It is easy to check compatibility on intersections:

Define $h_C: C \to (\Delta^a \times \Delta^c) * (\Delta^b \times \Delta^d) \subset (\Delta^a \times \Delta^c) * ((\Delta^b \times \Delta^q) \cup (\Delta^p \times \Delta^d))$ by

$$(x, y) \to \begin{cases} \left(\dfrac{\Sigma a_i v_i^a}{\Sigma a_i}, \dfrac{\Sigma c_i v_i^c}{\Sigma c_i}\right)(1 - t) + \left(\dfrac{\Sigma b_i v_i^b}{\Sigma b_i}, \dfrac{\Sigma d_i v_i^d}{\Sigma d_i}\right)t, \\ \qquad\qquad \text{if } t = t(x) \neq 0, 1 \\ (x, y) \qquad\qquad \text{if } t = t(x) = 0,1. \end{cases}$$

Define $h_A: A \to (\Delta^a \times \Delta^c) * (\Delta^p \times \Delta^d) \subset (\Delta^a \times \Delta^c) * ((\Delta^b \times \Delta^q) \cup (\Delta^p \times \Delta^d))$ by

$$(x, y) \to$$

$$\begin{cases} \left(\dfrac{\Sigma a_i v_i^a}{1 - t}, \dfrac{\Sigma c_i v_i^c}{1 - s}\right)(1 - s) + \left(\dfrac{\Sigma a_1 v_1^a}{1 - t}\left(1 - \dfrac{t}{s}\right) + \dfrac{\Sigma b_i v_i^b}{t}\left(\dfrac{t}{s}\right), \dfrac{\Sigma d_i v_i^d}{s}\right)s \\ \qquad\qquad \text{if } t = t(x) \neq 1 \text{ and } s = s(y) \neq 0 \\ (x, y) \qquad\qquad \text{if } s(y) = 0 \text{ or } t(x) = 1. \end{cases}$$

Define $h_B: B \to (\Delta^a \times \Delta^c) * (\Delta^b \times \Delta^q) \subset (\Delta^a \times \Delta^c) * ((\Delta^b \times \Delta^q) \cup (\Delta^p \times \Delta^d))$ similarly:

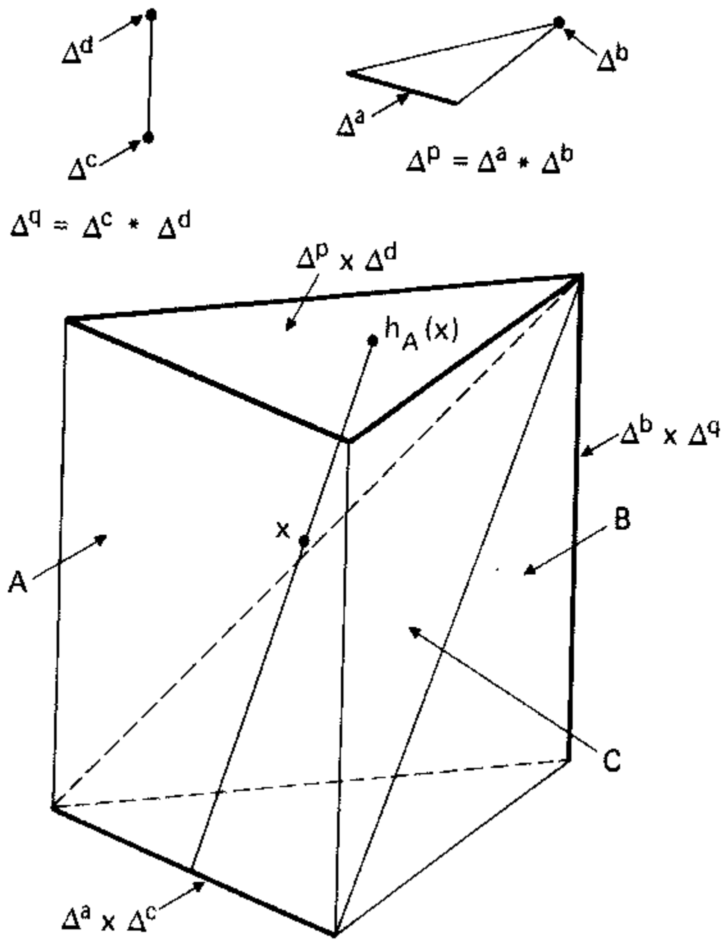

Figure 3. Example of a production deformation.

$$(x, y) \to$$

$$\begin{cases} \left(\left(\dfrac{\Sigma a_i v_i^a}{1-t}, \dfrac{\Sigma c_i v_i^c}{1-s}\right)(1-t) + \left(\dfrac{\Sigma b_i v_i^b}{t}, \dfrac{\Sigma c_i v_i^c}{1-s}\left(1-\dfrac{s}{t}\right) + \dfrac{\Sigma d_i v_i^d}{s}\left(\dfrac{s}{t}\right)\right)t \right. \\ \qquad\qquad\qquad \text{if } t = t(x) \neq 0 \text{ and } s = s(y) \neq 1 \\ (x, y) \qquad\qquad \text{if } t(x) = 0 \text{ or } s(y) = 1. \end{cases}$$

Continuity is checked using barycentric coordinates, and the definitions agree on overlaps. The image is easily seen to be a union of join lines. Since $\Delta^p \times \Delta^q$ is compact Hausdorff and the map is injective, we have an embedding as claimed. □

Chapter III. Rational Surgery on Witt Spaces.

1. Description of Program. In this chapter, we develop a technique for performing rational surgery on Witt spaces. We use it to prove that the Witt class $w(X)$ determines the cobordism class of X. In particular, if X is a closed Witt space of dimension $4k$, and $w(X) = 0$, we find a "surgery basis" $\{(\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)\}$ for $IH_{2k}^{\bar{m}}(X; \mathbf{Q})$ with respect to which the inner product has matrix $\operatorname{diag}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]$. By collapsing a regular neighborhood of an irreducible representative for α_1 (see Section 2), we obtain a Witt space X_1 with

$$w(X_1) = 0 \tag{1}$$

and

$$\operatorname{rank} IH_{2k}^{\bar{m}}(X_1, \mathbf{Q}) = \operatorname{rank} IH_{2k}^{\bar{m}}(X; \mathbf{Q}) - 2. \tag{2}$$

Roughly speaking, the homology classes $\{(\alpha_1, \beta_1)\}$ are killed. Moreover, the mapping cylinder of the collapse (with a collar added) gives a Witt cobordism between X and X_1. We say that X_1 is obtained from X by an elementary surgery, and call the cobordism the trace of the surgery.

Iterating this procedure n times, we construct a Witt space X_n with $IH_{2k}^{\bar{m}}(X_n; \mathbf{Q}) = 0$ and a Witt cobordism Y between X and X_n. Coning off the copy of X_n in ∂Y produces a Witt cobordism of X to zero.

A similar procedure works when the dimension of X is $4k + 2$. In that case, we find a symplectic basis $\{(\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)\}$ with respect to which the skew-symmetric form on $IH_{2k+1}^{\bar{m}}(X; \mathbf{Q})$ has matrix $\operatorname{diag}\left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right]$. Then proceed as above. The contents of the chapter are as follows. We define and prove existence of irreducible representative cycles (2.1, 2.2). Then we study the intersection homology of the regular neighborhood of an irreducible cycle (3.1) and show that the mapping cylinder of a collapse as described above is a Witt cobordism (4.1, 4.2). Finally, we construct a surgery exact diagram (4.3) very much in the classical vein [18] to compute the effect of surgery on intersection homology, verifying (1) and (2) above.

2. Irreducible P.L. Geometric Cycles.

Definition 2.1. Let X be a P.L. pseudomanifold. A representative cycle z for $\alpha \in IH_j^{\bar{p}}(X; G)$, $G = \mathbf{Z}$ or $\mathbf{Q}$, is *irreducible* if

(1) $H_j(|z|; \mathbf{Z}) = \mathbf{Z}$

(2) the generator of $H_j(|z|; \mathbf{Z})$ has coefficient ± 1 on every j-simplex of $|z|$, in some triangulation of $|z|$.

Let X^{2k} be a P.L. pseudomanifold of dimension $2k$ with irreducible fundamental class $[X]$. By abuse of language, we say X is irreducible.

LEMMA 2.2. *Let X^{2k} be an irreducible* P.L. *pseudomanifold. Given $a \in IH_j^{\bar{p}}(X; \mathbf{Q})$, there exists a representative cycle z which is irreducible, if $0 \leq j < 2k - 1$.*

Proof. The proof proceeds in 2 steps. Step 1 constructs, by a general position argument, a representative cycle v such that $|v|$ supports a simplicial cycle with coefficient ± 1 on every j-simplex. Step 2 uses the technique of piping [22, p. 67] to produce a cycle z which, additionally, satisfies the condition $H_j(|z|, \mathbf{Z}) = \mathbf{Z}$. If J is a j-cycle, Σ_J denotes the complement of the intrinsic open j-stratum.

Step 1. In some triangulation T of X, choose a representative simplicial cycle for α. By reorienting j-simplices if necessary, we may assume all non-zero coefficients of j-simplices are positive. Clear denominators so as to obtain an integral cycle $y = \Sigma_{\sigma \in T_\sigma} n(\sigma)\sigma$ such that all distinct non-zero coefficients share no common factor.

If $n(\sigma) = 1$ whenever $n(\sigma) \neq 0$, then set $v = y$ and Step 1 will be completed.

If not, let $E_1, \ldots, E_n$ be the connected components of the intrinsic open j-stratum of $Y = |y|$. To each component we can associate a unique coefficient n_i, the coefficient of simplices in that component. Let $\tilde{W}$ be the disjoint union of n_i copies of $\bar{E}_i$, $i = 1, \ldots, n$, and $f: \tilde{W} \to Y$ the obvious P.L. map. Let W be the quotient space of $\tilde{W}$ obtained by identifying x to x' if $x, x' \in f^{-1}(\Sigma_Y)$ and $f(x) = f(x')$. Then W supports a simplicial j-cycle w having coefficient $+1$ on all j-simplices, and such that $f_*(w) = y$, where f_* denotes the induced map on chains.

Note that $f(W - \Sigma_W) \subset X - \Sigma_X$. Perform a finite sequence of shifts to the map f, bringing f into general position with respect to Σ_W [1, p. 418], [22, p. 61]. We move $f(\hat{\tau})$ only if $\mathrm{Int}(\tau) \subset W - \Sigma_W$. Let $g: W \to X$ be the resulting map. The linear trace of the shifts yields a P.L. homotopy $H: W$

$\times [0, 1] \to X$ with $H_0 = f$ and $H_1 = g$. Then $g_*(w)$ is a P.L. geometric cycle which, when triangulated, has coefficient $+1$ on each j-simplex. Its support is also $(\bar{p}, j)$-allowable.

Give $W \times [0, 1]$ the orientation such that $\partial(w \times [0, 1]) = w \times 1 - w \times 0$. Then $H_*(w \times [0, 1]) = g_*(w) - f_*(w) = g_*(w) - y$. Since $H(x, t) = H(x, 0) = f(x)$ when $x \in \Sigma_W$, for all t, and $H(x, t) \in X - \Sigma_X$ when $x \in W - \Sigma_W$, for all t, it follows that $|H_*(w \times [0, 1])|$ is $(\bar{p}, j + 1)$-allowable. We conclude that $g_*(w)$ represents the same homology class as y in $IH_j^{\bar{p}}(X; \mathbf{Q})$. Let v be the multiple of $g_*(w)$ such that $[v] = \alpha \in IH_j^{\bar{p}}(X; \mathbf{Q})$. Then v is the desired representative, and Step 1 is complete.

Step 2. Let $\{F_i\}_{i=1,\ldots,k}$ be the connected components of the intrinsic open j-stratum of $|v|$. Since $\dim(\alpha) < 2k - 1$, we can construct an orientation respecting pipe P_i from F_i to F_{i+1}, for $1 \leq i \leq k - 1$, with distinct pipes disjoint [22, p. 67]. Note that since $|v|$ is a subcomplex of some triangulation of X, the pair $(X - \Sigma_X, |v| \cap (X - \Sigma_X))$ is locally unknotted at all points in the interiors of j-dimensional simplices of $|v|$. Therefore the piping construction can always be carried out.

Let Z be the P.L. subspace of X we obtain after the piping is done. Then Z supports a simplicial cycle which has coefficient 1 on every simplex in Z. Since $Z - \Sigma_Z$ is connected, this cycle generates $H_j(Z; \mathbf{Z})$. Clearly, its support is $(\bar{p}, j)$-allowable and, by "filling in the pipes," we see that it represents the same homology class as $g_*(w)$ in $IH_j^{\bar{p}}(X; \mathbf{Q})$. Let z be the multiple of this cycle such that $[z] = [v] \in IH_j^{\bar{p}}(X; \mathbf{Q})$. Then z satisfies the conditions (1) of Definition 2.1. □

Remark 2.3. If X^2 is an irreducible P.L. pseudomanifold of dimension 2, any class $\alpha \in IH_1^{\bar{p}}(X; \mathbf{Q})$ is represented by a cycle whose support is a P.L. embedded S^1. For example, in $S^1 \times S^1$, with standard generators α, β for $H_1(S^1 \times S^1; \mathbf{Z})$, the class $a\alpha + b\beta$, where $a, b \in \mathbf{Z}$, is represented by an embedded S^1 iff g.c.d.$(a, b) = 1$.

3. Regular Neighborhoods. Let X^{2k} be an irreducible pseudomanifold. Let z be an irreducible representative for $\alpha \in IH_k^{\bar{m}}(X; \mathbf{Q})$, with U a (closed) regular neighborhood of $|z|$ in X. Then U is a pseudomanifold with collared boundary ∂U, and intrinsic stratification induced from that of X [1, p. 437]. Let $\hat{U} = U \cup \operatorname{cone}(\partial U)$ be U with the cone on ∂U adjoined. The stratification of U induces a stratification on $\hat{U}$. Since $|z|$ is $(\bar{m}, k)$-allowable, the cone point v_0 constitutes the entire 0-stratum of the stratification of $\hat{U}$.

PROPOSITION 3.1. *If X is a Witt space and* $\langle \alpha, \alpha \rangle = 0$, *then*:

$$IH_k^{\bar{m}}(\hat{U}; \mathbf{Q}) = 0.$$

To prove this, we prove a series of preliminary lemmas. Let $i: IC_*^{\bar{m}-}(\hat{U}; \mathbf{Z}) \to IC_*^{\bar{m}}(\hat{U}; \mathbf{Z})$ be the inclusion of chain complexes, where $\bar{m}-$ is the perversity: $\bar{m}-(c) = \bar{m}(c)$ for $2 \le c \le 2k - 1$ and $\bar{m}-(2k) = \bar{m}(2k) - 1 = k - 2$. An argument identical to that given in the proof of II.2.1 proves

LEMMA 3.2. *There is an exact sequence of integral intersection homology groups.*

$$IH_{k+1}^{\bar{m}}(\hat{U}) \xrightarrow{r_*} IH_k^{\bar{m}}(\partial U) \xrightarrow{\partial_*} IH_k^{\bar{m}-}(\hat{U}) \to IH_k^{\bar{m}}(\hat{U}) \to 0. \qquad \square$$

We made no use of the irreducibility of z in Lemma 3.2, but it is crucial in the next lemma.

LEMMA 3.3. *Let u be a generator of* $H_k(|z|; \mathbf{Z}) = Z_k(|z|, \mathbf{Z})$. *Then* $IH_k^{\bar{m}-}(\hat{U}; \mathbf{Z}) = \mathbf{Z}$ *and the homology class of u is a generator.*

Proof of 3.3. Let N denote a collar of ∂U in U. There is a P.L. homeomorphism of $N \cup \text{cone}(\partial U)$ with $\text{cone}(\partial U) = v_0 * \partial U$. Let $U_0 = \text{Cl}(U - N)$. Suppose $\xi \in IH_k^{\bar{m}-}(\hat{U}; \mathbf{Z})$, with representative cycle $y \in IC_k^{\bar{m}-}(\hat{U}; \mathbf{Z})$. Then $|y| \cap v_0 = \phi$. By pseudoradial projection, we can assume $|y| \subset U_0$.

Give U the structure of a derived neighborhood [22, p. 33]. Then every $\sigma \in U$ satisfies:

$$|\sigma| = (|\sigma| \cap |z|) * (|\sigma| \cap \partial U)$$

where $A * \phi = \phi * A = A$, by the usual convention.

Triangulate the cycle y so that each simplex lies in a unique simplex with interior in U. If a vertex $v \in |y|$ lies in $\text{Int}(\sigma)$, then $|\sigma| \cap |z| \neq \phi$, and we let τ denote the simplex in $|z|$ with $|\tau| = |\sigma| \cap |z|$.

Define $f(v) = \hat{\tau}$. Extend this map on vertices of $|y|$ linearly over simplices, obtaining a P.L. map $f: |y| \to |z| \subset \hat{U}$. Then f determines a geometric cycle $f_*(y)$ in $C_k(\hat{U}; \mathbf{Z})$. Since the support of $f_*(y)$ lies in $|z|$, $f_*(y) \in IC_k^{\bar{m}-}(\hat{U}; \mathbf{Z})$. Proceed as in I.2.2 to show that $f_*(y)$ represents ξ. But, $f_*(y)$ is a k-cycle supported in $|z|$, and $H_k(|z|; \mathbf{Z}) = \mathbf{Z}$ by hypothesis. Therefore $f_*(y) = j \cdot u$ for some $j \in \mathbf{Z}$, and so $\xi = j \cdot [u]$.

It is easy to check that $[u]$ is nontrivial. $\square$

By the intersection pairing theorem [11], there is a nondegenerate pairing induced by intersection product:

$$IH_k^{\bar{m}-}(\hat{U}; \mathbb{Q}) \times IH_k^{\bar{n}+}(\hat{U}; \mathbb{Q}) \xrightarrow{\langle\,,\,\rangle} \mathbb{Q}.$$

Here $\bar{n}+$ is the perversity satisfying $(\bar{m}-) + (\bar{n}+) = \bar{t}$. Choose $\beta \in IH_k^{\bar{n}+}(U; \mathbb{Q})$ a generator such that

$$\langle \alpha, \beta \rangle = 1.$$

LEMMA 3.4. *The following diagram commutes and rows are exact.*

$$\begin{array}{ccccc} IH_k^{\bar{m}-}(\hat{U}; \mathbb{Q}) & \xrightarrow{h} & IH_k^{\bar{m}}(\hat{U}; \mathbb{Q}) & \rightarrow & 0 \\ \downarrow g & & \downarrow i & & \\ IH_k^{\bar{n}+}(\hat{U}; \mathbb{Q}) & \xleftarrow[j]{} & IH_k^{\bar{n}}(\hat{U}; \mathbb{Q}) & \leftarrow & 0 \end{array}$$

All maps are induced by inclusion of chain complexes. The map g is given by:

$$g(\xi) = \langle \xi, \alpha \rangle \cdot \beta.$$

Proof. Since $IC_{k+1}^{\bar{n}}(\hat{U}; \mathbb{Q}) = IC^{\bar{n}+}_{k+1}(\hat{U}; \mathbb{Q})$, the map j is injective. Finally, the description of g is immediate from the definition of β, and commutativity now follows easily. □

We now prove Proposition 3.1.

Proof of 3.1. Refer to the diagram of 3.4. Under the hypothesis of the proposition, g is the zero map. Injectivity of j and commutativity imply that $i \circ h$ is the zero map also. $\hat{U}$ inherits a Witt space structure from X, so I.3.4 implies that i is an isomorphism, so h must in fact be the zero map. The proposition now follows from surjectivity of h. □

4. The Surgery Exact Diagram. Let X be an irreducible Witt space of dimension $2k$. Assume there is a decomposition of $IH_k^{\bar{m}}(X; \mathbb{Q})$,

$$IH_k^{\bar{m}}(X; \mathbb{Q}) = V_0 \oplus V_1$$

where $V_0 = (\gamma_1, \ldots, \gamma_l)$, $V_1 = (\alpha_1, \beta_1)$ and with respect to this basis the intersection pairing satisfies

$$\langle \alpha_1, \gamma_i \rangle = \langle \beta_1, \gamma_i \rangle = 0, \qquad i = 1, \ldots, l$$

$$\langle \alpha_1, \alpha_1 \rangle = \langle \beta_1, \beta_1 \rangle = 0$$

$$\langle \beta_1, \alpha_1 \rangle = \pm 1.$$

Let U be a regular neighborhood of an irreducible representative cycle z for α_1. We may assume that ∂U is bicollared in X [1, p. 437]. Form the space

$$X_1 = (X - \operatorname{Int}(U)) \cup \operatorname{cone}(\partial U).$$

The passage from X to X_1 is called an *elementary surgery*. Note X_1 is (topologically) homeomorphic to the space obtained from X by collapsing U to a point.

The following sequence of propositions, whose proofs will be deferred briefly, partially describe the effect of the elementary surgery.

PROPOSITION 4.1. *X_1 is an irreducible Witt space.*

PROPOSITION 4.2. *Let Y be the mapping cylinder of the collapse $X \to \tilde{X}$, with a collar $\tilde{X} \times I$ attached. Then Y can be given the structure of a Witt cobordism between X and X_1. (We call Y the trace of the elementary surgery on X.)*

PROPOSITION 4.3. *There is an isomorphism*

$$IH_k^{\bar{m}}(X_1; \mathrm{Q}) \cong V_0$$

which preserves the intersection form.

Repeated application of these propositions yields the result alluded to in the introduction.

THEOREM 4.4. *Let X be an irreducible Witt space of dimension $2k$, $k \geq 1$. If $w(X) = 0$, then X is Witt cobordant to zero.*

Proof. Suppose $\dim X \equiv 0 \pmod 4$. Then $w(X) = 0$ implies that the inner product space $IH_k^{\bar{m}}(X; \mathrm{Q})$ is split [14]. Choose a basis $\{\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_n, \beta_n\}$ with respect to which the matrix of the form is $\operatorname{diag}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]$. Let $V_0 = (\alpha_2, \beta_2, \ldots, \alpha_n, \beta_n)$ and $V_1 = (\alpha_1, \beta_1)$, and apply 4.1–4.3. We obtain X_1, Witt cobordant to X with $IH_k^{\bar{m}}(X_1; \mathrm{Q}) = (\alpha_2', \beta_2', \ldots, \alpha_n', \beta_n')$, isomorphic to V_0 as rational inner product space. Now decompose $IH_k^{\bar m}(X_1, \mathbf{Q})$, setting $V_0' = (\alpha_3', \beta_3', \ldots, \alpha_n', \beta_n')$ and $V_f' = (\alpha_2', \beta_2')$, and repeat the procedure. Continue until we obtain X_n, cobordant to X, with $IH_k^{\bar m}(X_n; \mathbf{Q}) = 0$. The cobordism to X is obtained by pasting the traces of the elementary surgeries along their boundaries. Now attach cone(X_n) to complete the cobordism of X to zero.

If $\dim X \equiv 2 \pmod 4$, then the intersection form on $IH_k^{\bar m}(X; \mathbf{Q})$ is skew-symmetric. Choose a symplectic basis $\{\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n\}$ with respect to which the matrix of the form is $\operatorname{diag}\left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right]$. The same reasoning as above shows that a sequence of n surgeries will yield a cobordism of X to zero. □

We now supply the proofs of 4.1–4.3.

Proof of 4.1. The fact that X is a Witt space and ∂U is bicollared in X immediately implies that X_1 is a Witt space. If $k > 1$, general position proves that the nonsingular part of $X - \operatorname{Int}(U)$ is path connected. Therefore X_1 is irreducible.

If $k = 1$, then X has at most point singularities. Therefore z is an embedded S^1 and U is homeomorphic to $S^1 \times [-1, 1]$. The assumption that $[z] \in IH_1^{\bar m}(X; \mathbf{Q})$ is nontrivial implies the nonsingular part of $X - \operatorname{Int}(U)$ is path connected, so X_1 is irreducible.

Proof of 4.2. The trace of the elementary surgery is homeomorphic to the following polyhedron, defined analogously to the trace of a spherical modification [18]. To $(X - \operatorname{Int}(U)) \times [0, 1]$, attach U along $\partial U \times 0$ and cone(∂U) along $\partial U \times 1$. There is an embedding of $\hat U$ in the resulting space as $\partial U \times I$ with U and cone(∂U) attached as above. Now attach cone$(\hat U) = v * \hat U$ along $\hat U$ via this embedding, yielding the trace of the surgery Y. Y is a pseudomanifold with collared boundary, and it can be oriented so that $\partial Y = X_1 - X$. To prove that Y is a Witt space, it suffices to check that $lk(v, Y)$ satisfies

$$IH_k^{\bar m}(lk(v, Y); \mathbf{Q}) = 0.$$

But $lk(v, Y) \approx \hat U$, so Proposition 3.1 completes the proof. □

Proof of 4.3.

Step 1. Transverse chains

Let U denote (by abuse of notation) the fundamental cycle of X restricted to the regular neighborhood U, and ∂U its boundary chain. Let $TC_*^{\bar m}(X)$ be the subcomplex of $IC_*^{\bar m}(X)$ consisting of chains c for which c

and ∂c are dimensionally transverse to ∂U. To each element $c \in TC_i^{\bar{m}}(X)$, we can associate the intersection chain $c \cap U$, and, by the Lefschetz boundary formula:

$$\partial(c \cap U) = \partial c \cap U + (-1)^{2k-i} c \cap \partial U.$$

This defines a homomorphism

$$t : TC_i^{\bar{m}}(X) \to IC_i^{\bar{m}}(\hat{U}) \quad \text{for each } i$$

by $t(c) = c \cap U$.

Define $IC_i^{\bar{m}}(X, X - U)$ to be the image of t in $IC_i^{\bar{m}}(\hat{U})$. Give the graded group $IC_*^{\bar{m}}(X, X - U)$ the structure of a chain complex, defining

$$\delta : IC_i^{\bar{m}}(X, X - U) \to IC_{i-1}^{\bar{m}}(X, X - U)$$

by

$$\delta(t(c)) = \partial c \cap U = t(\partial c).$$

Note that δ is well-defined: if $t(c) = 0$, then

$$\partial(t(c)) = \partial(c \cap U) = \partial c \cap U + (-1)^{2k-i} c \cap \partial U = 0.$$

Dimensional transversality implies that

$$\dim(|\partial c \cap U| \cap |c \cap \partial U|) \leq i - 2$$

so

$$\partial c \cap U = 0$$

and

$$c \cap \partial U = 0.$$

In particular, $\delta(t(c)) = \partial c \cap U = 0$.

It is easy to check that $\delta \circ \delta = 0$. We conclude that $t : TC_i^{\bar{m}}(X) \to IC_i^{\bar{m}}(X, X - U)$ is a chain homomorphism. Let $K_*^{\bar{m}}$ denote the kernel of t, with the induced chain complex structure. There is a short exact sequence of complexes:

$$0 \to K_*^{\bar{m}} \to TC_*^{\bar{m}}(X) \to IC_*^{\bar{m}}(X, X - U) \to 0,$$

and an associated long exact sequence in homology:

$$\to H_{k+1}(IC_*^{\bar{m}}(X, X - U)) \to H_k(K_*^{\bar{m}})$$

$$\to H_k(TC_*^{\bar{m}}(X)) \to H_k(IC_*^{\bar{m}}(X, X - U)) \to$$

The notation $IC_*^{\bar{m}}(X, X - U)$ is justified by the following fact. If $IC_*^{\bar{m}}(X - U)$ is defined as the subcomplex of $IC_*^{\bar{m}}(X)$ with chains supported on $X - \mathrm{Int}(U)$, then the homology of the quotient complex of the inclusion $IC_*^{\bar{m}}(X - U) \to IC_*^{\bar{m}}(X)$ is isomorphic to the homology of $IC_*^{\bar{m}}(X, X - U)$. This is proved using arguments like those referred to in Step 2. In Steps 2–4 we analyze the homology groups occurring in this sequence.

Step 2. $H_*(TC_*^{\bar{m}}(X)) \cong IH_*^{\bar{m}}(X)$
The inclusion of chain complexes

$$i: TC_*^{\bar{m}}(X) \to IC_*^{\bar{m}}(X)$$

induces an isomorphism on homology. In other words, $H_*(TC_*^{\bar{m}}(X)) \cong IH_*^{\bar{m}}(X)$.

Proof. For surjectivity, apply stratified general position [17] to a cycle $z \in IC_i^{\bar{m}}(X)$. Injectivity follows from the relative general position lemma.

Step 3. The complex $IC_*^{\bar{m}+}(\hat{U})$
Let $\bar{m}+$ denote the sequence with

$$\bar{m} + (c) = \bar{m}(c) \quad \text{for} \quad 2 \le c \le 2k - 1$$

$$\bar{m} + (2k) = \bar{m}(2k) + 1 = k.$$

Although $\bar{m}+$ is not a perversity, we may define $(\bar{m}+, j)$-allowability of the support of a geometric chain with respect to a stratification. Consider the complex $IC_*^{\bar{m}+}(\hat{U})$ consisting of chains c of dimension j such that $|c|$ is

$(\bar{m}+, j)$-allowable and $|\partial c|$ is $(\bar{m}+, j-1)$-allowable with respect to the stratification on $\hat{U}$.

The inclusion $IC_*^{\bar{m}}(\hat{U}) \to IC_*^{\bar{m}+}(\hat{U})$ induces a short exact sequence of complexes. We can analyze the homology of the quotient and obtain an isomorphism

$$IH_{k+1}^{\bar{m}}(\hat{U}) \cong IH^{\bar{m}+}_{k+1}(\hat{U})$$

and an exact sequence

$$0 \to IH_k^{\bar{m}}(\hat{U}) \to IH_k^{\bar{m}+}(\hat{U}) \to IH_{k-1}^{\bar{m}}(\partial U) \to IH_{k-1}^{\bar{m}}(\hat{U}) \to \cdots.$$

We get another long exact sequence from the short exact sequence induced by the inclusion $IC_*^{\bar{n}}(\hat{U}) \to IC_*^{\bar{n}+}(\hat{U})$, where $\bar{n}+$ is the perversity:

$$\bar{n}+(c) = \bar{n}(c) \quad \text{for} \quad 2 \le c \le 2k-1$$

$$\bar{n}+(2k) = \bar{n}(2k) + 1 = k.$$

There is a commutative ladder of integral intersection homology groups:

$$\begin{array}{ccccccccc}
0 \to & IH_k^{\bar{m}}(\hat{U}) & \to & IH_k^{\bar{m}+}(\hat{U}) & \to & IH_{k-1}^{\bar{m}}(\partial U) & \to & IH_{k-1}^{\bar{m}}(\hat{U}) & \to \\
& \downarrow & & \downarrow{\scriptstyle \phi} & & \downarrow & & \downarrow & \\
0 \to & IH_k^{\bar{n}}(\hat{U}) & \to & IH_k^{\bar{n}+}(\hat{U}) & \to & IH_{k-1}^{\bar{n}}(\partial U) & \to & IH_{k-1}^{\bar{n}}(\hat{U}) & \to
\end{array}$$

All vertical maps are induced by chain complex inclusions. Since X is Witt, $\hat{U}$ and ∂U are also Witt. Proposition I.3.4 and the 5-lemma imply that $\phi \otimes \mathbb{Q}$ is an isomorphism.

We now identify $IH_j^{\bar{m}}(U, \partial U)$ with $IH_j^{\bar{m}+}(\hat{U})$ in dimensions $j \ge k$. Let N be a simplicial collar of ∂U in U. There is a P.L. homeomorphism $\operatorname{cone}(\partial U) \cup N \approx \operatorname{cone}(\partial U)$ preserving cone lines in the cone on the left. In $\hat{U}$, we can triangulate any chain so that its intersection with $\operatorname{cone}(\partial U) \cup N$ will be subordinate to the simplicial structure on $\operatorname{cone}(\partial U) \cup N$. Let p_* denote the chain map induced by pseudoradial projection along cone lines from the cone vertex v to the boundary of $\operatorname{Cl}(U - N)$, $p_*: C_*(\hat{U}) \to C_*(\hat{U})$. Note that $p_*(v) = v$, and $p_* \circ p_* = p_*$.

Define the homomorphisms:

$$\psi_j : IC_j^{\bar{m}}(X, X - U) \to IC_j^{\bar{m}+}(\hat{U}), \qquad j \geq k + 1, \qquad \text{and}$$

$$\psi_k : IZ_k^{\bar{m}}(X, X - U) \to IZ_k^{\bar{m}+}(\hat{U}) \quad \text{(cycles) by}$$

$$\psi_j(t(c)) = p_*(t(c) + (-1)^{j+1}\nu * (c \cap \partial U)).$$

The discussion of t in Step 1 shows that ψ_j is well defined.

For $j \geq k + 1$, one checks that $\psi_{j-1}(\delta t(c)) = \partial\psi_j(t(c))$. Therefore ψ_j induces a homomorphism $(\psi_j)_*$ on homology for $j \geq k$, which is in fact an isomorphism.

Proof of surjectivity. Let $z \in IZ_j^{\bar{m}+}(\hat{U})$. The cycle $p_*(z)$ represents the same class as z in $IH_j^{\bar{m}+}(U)$. The chain $p_*(z) \cap U$ lies in $IC_j^{\bar{m}}(X, X - U)$ and satisfies: $\psi_j(p_*(z) \cap U) = p_*(z)$.

Proof of injectivity. Let $z \in IZ_j^{\bar{m}}(X, X - U)$ and suppose $\psi_j(z) = \partial w$, where $w \in IC^{\bar{m}}{}_{j+1}^{+}(\hat{U})$. Then $p_*(w) \cap U$ lies in $IC_{j+1}^{\bar{m}+}(X, X - U)$ and

$$\begin{aligned}\delta(p_*(w) \cap U) &= \partial p_*(w) \cap U \\ &= p_*(\partial w) \cap U \\ &= p_*\psi_j(z) \cap U \\ &= \psi_j(z) \cap U, \qquad \text{since } p_* \circ p_* = p_*.\end{aligned}$$

The chains $\psi_j(z)$ and $z + (-1)^{j+1}\nu*(z \cap \partial U)$ are homologous in $IC_j^{\bar{m}+}(\hat{U})$, via the chain we denote by y. By the relative stratified general position lemma, we can assume y is dimensionally transverse to ∂U. Then $y \cap U \in IC_{j+1}^{\bar{m}}(X, X - U)$ and $\delta(y \cap U) = (\psi_j(z) \cap U) - z$. Thus $z = \delta(p_*(w) \cap U - y \cap U)$.

Step 4. The complex $K_*^{\bar{m}}$

Let $IC_*^{\bar{m}}(X - U)$ be as in Step 1. Note that there is a natural inclusion $IC_*^{\bar{m}}(X - U) \to IC_*^{\bar{m}}(X \mathbin{\hat{-}} U)$.

Define homomorphisms for all j,

$$\phi_j : K_j^m \to IC_j^{\bar{m}}(X - U)$$

by $\phi_j(c) = c \cap (X - \mathrm{Int}(U))$, where by abuse of notation, $X - \mathrm{Int}(U)$ represents the restriction of the fundamental cycle of X to $X - \mathrm{Int}(U)$. Then ϕ_j is injective, because

$$(1) \qquad c = (c \cap (X - \mathrm{Int}(U))) + (c \cap U) \quad \text{for} \quad c \in TC_j^{\bar{m}}(X)$$

and

$$(2) \qquad t(c) = c \cap U = 0 \quad \text{for} \quad c \in K_j^{\bar{m}}.$$

Clearly the ϕ_j are chain maps and we regard them as an inclusion of chain complexes.

The induced maps on homology $(\phi_j)_*$ are isomorphisms for $j \leq k$.

Proof of surjectivity. Let N be a collar of ∂U in $X - \mathrm{Int}(U)$, and p_* be the chain map on $C_*(X \mathbin{\hat{-}} U)$ induced by pseudoradial projection along cone lines to the inner boundary of N. If $z \in IC_j^{\bar{m}}(X - U)$ for $j \leq k$ and z is a cycle, $p_*(z)$ represents the same homology class in $IH_{j*}^{\bar{m}}(X - U))$. But $p_*(z) \in \phi(K_j^{\bar{m}})$, since its support lies in $X - U$.

Proof of injectivity. Let $\alpha \in H_j(K_*^{\bar{m}})$. If the cycle z represents α, then $p_*(z)$ represents α also. Assume $p_*(z) = \partial w$, where $w \in IC_{j+1}^{m}(X - U)$. If $j \leq k$, then $p_*(w) \in \phi(K_{j+1}^{\bar{m}})$. Moreover,

$$\begin{aligned} \partial p_*(w) &= p_*(\partial w) \\ &= p_* p_*(z) \\ &= p_*(z) \end{aligned}$$

Step 5. The surgery exact diagram

There is a commutative diagram of rational homology groups, with exact rows and column (*assume rational coefficients*):

$$\begin{tikzcd}
IH_{k+1}^{\bar{m}}(U, \partial U) \arrow[r] \arrow[d, "\cong"'] & IH_k^{\bar{m}}(\partial U) \arrow[r] \arrow[d, "j_*"] & IH_k^{\bar{m}}(U) \arrow[r] \arrow[d] & IH_k^{\bar{n}}(U, \partial U) \arrow[d, "\cong"] \\
IH_{k+1}^{\bar{m}}(X, X - U) \arrow[r, "\partial_*"] \arrow[ur, "r_*"] \arrow[d, "\cong"] & IH_k^{\bar{m}}(X - U) \arrow[r] \arrow[d] & IH_k^{\bar{m}}(X) \arrow[r, "t_*"] & IH_k^{\bar{n}}(X, X - U) \arrow[d, leftrightarrow, "\equiv"] \\
IH_{k+1}^{\bar{m}+}(\hat{U}) & IH_k^{\bar{m}}(X \mathbin{\hat{-}} U) \arrow[d] & & IH_k^{\bar{n}+}(\hat{U}) \cong IH_k^{\bar{n}+}(\hat{U}) \\
& 0 & &
\end{tikzcd}$$

The lower horizontal exact sequence is a segment of the exact sequence in Step 1. We have used the isomorphisms proved in Steps 2, 3, and 4 to relabel some of the groups.

The upper horizontal exact sequence is derived from the inclusion $IC_*^{\bar{m}}(\partial U) \to IC_*^{\bar{m}}(U)$, where $IC_*^{\bar{m}}(U)$ is the subcomplex of $IC_*^{\bar{m}}(X)$ with chains supported in U. The homology of the quotient complex is denoted by $IH_*^{\bar{m}}(U, \partial U)$.

The two vertical homomorphisms in the middle of the ladder are induced by the obvious inclusion of chain complexes. The excision isomorphism $IH^{\bar{m}}(U, \partial U) \cong IH^{\bar{m}}(X, X - U)$ follows from an argument very much like the proof for ordinary homology. We do not require this isomorphism here, however.

The exactness of the vertical sequence follows from straightforward analysis of the inclusion $IC_*^{\bar{m}}(X - U) \to IC_*^{\bar{m}}(X \hat{-} U)$.

The homomorphism r_* is taken from lemma 3.2 and commutativity of the triangle follows from chain level definitions of homomorphisms.

From Step 3, the map t_* is given by $t_*(\gamma) = \langle \gamma, \alpha, \rangle \cdot \beta$, where β generates $IH_k^{\bar{n}+}(\hat{U})$ and $\langle\ ,\ \rangle$ is inner product on $IH_k^{\bar{m}}(X)$.

Step 6. Conclusion

Assume rational coefficients. Recall from the beginning of this section the basis $\{\alpha_1, \beta_1, \gamma_1, \ldots, \gamma_l\}$ of $IH_k^{\bar{m}}(X)$ and the decomposition $IH_k^{\bar{m}}(X) \cong V_0 \oplus V_1$. From Step 5, we have

$$t_*(\gamma_i) = 0, \qquad i = 1, \ldots, l$$

and

$$t_*(\alpha_1) = 0.$$

Similarly,

$$t_*(\beta_1) = \beta$$

where $IH_k^{\bar{m}}(X, X - U)$ is identified with $IH_k^{\bar{n}+}(\hat{U})$ by the isomorphism in Step 3. By exactness,

$$IH_k^{\bar{m}}(X - U) = V_0' \oplus (\alpha_1') \oplus \mathrm{Image}(\partial_*)$$

as a rational vector space, and $V_0' \oplus (\alpha_1')$ is isomorphic to $V_0 \oplus (\alpha_1)$ by an isomorphism compatible with intersection product. Now, Proposition 3.1, Lemmas 3.2 and 3.3, and commutativity of the left triangle imply:

$$\mathrm{Image}(j) = (\alpha_1') \oplus \mathrm{Image}(\partial_*)$$

Finally, since the vertical sequence is exact, we conclude that:

$$IH_k^{\bar{m}}(X \mathbin{\hat{-}} U) \cong V_0' \cong V_0.$$

The isomorphism is compatible with intersection product. This completes the proof of 4.3. □

Chapter IV. Witt Space Bordism Theory

1. The bordism theory Ω_*^{Witt}. Let $\mathcal{L}$ be the class of oriented pseudomanifolds L satisfying:

(1) L is a Witt space
(2) $IH^{\bar{m}}(L; \mathrm{Q}) = 0$ if $\dim L = 2l$.

The axioms for a $\mathrm{Z}/2\mathrm{Z}$-bordism theory [2] can be easily modified to give axioms for an oriented bordism theory. It is not difficult to check that $\mathcal{L}$ is a class of (oriented) singularities (use I.2.4). The associated sequence $\{\mathfrak{F}^n\}$ is the class of all oriented Witt spaces. Let Ω_*^{Witt} denote the bordism theory based on the class of singularities $\mathcal{L}$.

PROPOSITION 1.1. *The coefficient groups $\Omega_*^{Witt}(pt)$ are:*

$$\Omega_0^{Witt}(pt) = \mathrm{Z}$$

$$\Omega_q^{Witt}(pt) = 0, \qquad \text{if } q \not\equiv 0 \pmod 4)$$

$$\Omega_q^{Witt}(pt) \cong W(\mathrm{Q}), \qquad \text{if } q > 0 \quad \textit{and} \quad q \equiv 0 \pmod 4).$$

The isomorphisms in dimension $q \geq 0$ are induced by the Witt class.

Proof. The case $q = 0$ is evident. When $q \equiv 1 \pmod 2)$, observe that any odd dimensional Witt space bounds the cone on itself.

The case $q \equiv 2 \pmod 4)$ follows from cobordism invariance (II.2.1) and surgery (III.4.4). In applying III.4.4, we can use the fact that any Witt space X^{2k} which is not irreducible is Witt cobordant to an irreducible Witt space X'. Simply take connected sum of components of the $2k$-stratum of X, say $E_1, \ldots, E_m$, in order, so that the $2k$-stratum of X' is $E_1 \# E_2 \# \cdots \# E_n$.

Finally, in the case $q \equiv 0 \pmod 4$, II.2.1 and III.4.4 show that the homomorphism

$$w : \Omega_{4k}^{\mathrm{Witt}}(pt) \to W(\mathrm{Q})$$

induced by the Witt class is injective. In Section 2, we construct explicit generators for $\Omega_{4k}^{\mathrm{Witt}}(pt)$ and prove that w is also surjective (2.2). □

2. Generators for $\Omega_*^{\mathrm{Witt}}(pt)$.

THEOREM 2.1. [7]. *Let B be an $n \times n$ matrix with integer entries, symmetric, and with even entries on the diagonal. Then, for $k \geq 1$, there is a manifold with boundary $(M, \partial M)$ of dimension $4k$ such that:*

(1) *M is $(2k-1)$-connected, ∂M is $(2k-2)$-connected, and $H_{2k}(M)$ is free abelian;*

(2) *The matrix of intersections $H_{2k}(M) \otimes H_{2k}(M) \to \mathrm{Z}$ is given by B.* □

Given an element $[V, \beta] \in W(\mathrm{Q})$, it is possible to find a basis $\{v_i\}$ for V with respect to which the matrix of the inner product, B, is an integral symmetric matrix with even entries on the diagonal. Theorem 2.1 produces a manifold with boundary $(M, \partial M)$ such that $H_{2k}(M; \mathrm{Q})$ represents the Witt class $[V, \beta]$ in $W(\mathrm{Q})$. In the long exact sequence of rational homology groups (rational coefficients assumed):

$$\to H_{2k}(\partial M) \to H_{2k}(M) \xrightarrow{i_*} H_{2k}(M, \partial M) \to H_{2k-1}(\partial M) \to$$

the transformation i_* has matrix B with respect to the bases $\{v_i\}$ and $\{v_i^*\}$, where $\{v_i^*\}$ is dual to $\{v_i\}$ under the intersection pairing of $H_{2k}(M)$ with $H_{2k}(M, \partial M)$:

$$H_{2k}(M) \times H_{2k}(M, \partial M) \to \mathrm{Q}.$$

Therefore the Novikov group of $(M, \partial M)$ and the isomorphic group $IH_{2k}^{\bar{m}}(\hat{M}; \mathrm{Q})$, where $\hat{M} = M \cup \operatorname{cone}(\partial M)$, represent the class of (V, β). That is,

$$w(\hat{M}) = [V, \beta] \in W(\mathbf{Q}).$$

This proves the result used in Section 1:

PROPOSITION 2.2. *The homomorphisms*

$$w : \Omega_{4k}^{Witt}(pt) \to W(\mathbf{Q}), \qquad k > 0.$$

which are induced by the Witt class, are surjective. □

Since $\hat{M}$ has a natural stratification with only even dimensional strata, we obtain the following concrete result about Ω_*^{ev}.

COROLLARY 2.4. *The homomorphism of graded cobordism groups*

$$\Omega_*^{ev}(pt) \to \Omega_*^{Witt}(pt)$$

is surjective. □

Remark 2.5. Simple examples show that the homomorphism in 2.4 is not injective.

3. Witt spaces: A Geometric Cycle Theory for $ko_* \otimes \mathbf{Z}[1/2]$. Suppose that $\Omega_*^{\mathfrak{F}}$ is a bordism theory based on a class $\mathfrak{F}$ of P.L. spaces satisfying the following properties:

(1) $\mathfrak{F}$ is closed under the operations of taking cartesian product with a P.L. manifold, and intersecting transversely with a closed P.L. manifold in Euclidean space.
(2) $\mathfrak{F}$ has a signature invariant defined on the cycle level which is *cobordism invariant*:

   There is a homomorphism

$$\text{sign} : \Omega_*(pt) \to \mathbf{Z}$$

   which extends the identification

$$\Omega_0(pt) \to \mathbf{Z}.$$

(3) The signature can be extended to relative cycles $(X, \partial X)$, so that it is *additive*:

If X, Y are relative cycles such that $Z = \partial X$, $-Z = \partial Y$, then $sign(X \cup_Z Y) = sign(X, Z) + sign(Y, Z)$.

(4) The signature is *multiplicative with respect to closed manifolds*:

If X is a cycle in $\mathfrak{F}$, and M is a closed manifold, then:

$$sign(M \times X) = sign(M) \cdot sign(X), \qquad \text{where}$$

$sign(M)$ denotes the classical signature of M.

Then, Dennis Sullivan [27] has defined a natural transformation of homology theories:

$$\mu^{\mathfrak{F}} : \Omega_*^{\mathfrak{F}} \to ko_* \otimes \mathrm{Z}\left[\frac{1}{2}\right].$$

[See Appendix for details.]

Moreover, for $[X] \in \Omega_q(pt)$, where $q \equiv 0 \pmod 4$,

$$\mu_{pt}^{\mathfrak{F}}([X]) = \mathrm{sign}(X) \in \mathrm{Z}\left[\frac{1}{2}\right] = ko_q(pt) \otimes \mathrm{Z}\left[\frac{1}{2}\right].$$

In Chapters I and II, we demonstrated that properties (1) through (4) hold for the class of Witt spaces, so we receive a natural transformation:

$$\mu^{\mathrm{Witt}} : \Omega_*^{\mathrm{Witt}} \to ko_* \otimes \mathrm{Z}\left[\frac{1}{2}\right].$$

Now, Proposition 1.1 and the structure of $W(\mathrm{Q})$ [14, 20] yield the main result:

THEOREM 3.1. *The natural transformation* (5) *is an equivalence of homology theories.* □

Appendix. Sullivan's Construction of $ko_* \otimes \mathrm{Z}[1/2]$ Orientations. In Chapter IV, we defined a natural transformation

$$\mu^{\mathrm{Witt}} : \Omega_*^{\mathrm{Witt}} \to ko_* \otimes \mathrm{Z}\left[\frac{1}{2}\right]$$

by assigning to every Witt space $(X^q, \partial X)$ a canonical orientation class in $ko_q(X, \partial X) \otimes Z[1/2]$. What is needed to carry out this assignment is the alchemy of Dennis Sullivan: Given a geometric homology theory $\Omega_*^{\mathfrak{F}}$ based on a class of cycles $\mathfrak{F}$, with signature invariant, and satisfying the properties listed in Section IV.3, Sullivan translates the signature data into canonical $ko_* \otimes Z[1/2]$ orientations for $\mathfrak{F}$-cycles, from which he constructs a natural transformation of homology theories: $\mu^{\mathfrak{F}}: \Omega_*^{\mathfrak{F}} \to ko_* \otimes Z[1/2]$, such that the induced homomorphism on coefficient groups is the signature homomorphism.

The construction is found in [25] and [27]. We briefly sketch it here.

Sullivan proves in [25] that elements of $KO^i(X) \otimes Z[1/2]$ are commutative diagrams:

$$\begin{array}{ccc} \Omega_{i+4*}(X) \otimes Q & \xrightarrow{\sigma} & Q \\ \downarrow i_* & & \downarrow i \\ \Omega^{l}_{i+4*}(X; Q/Z) & \xrightarrow{\tau} & Q/Z \end{array}$$

where σ, τ are homomorphisms satisfying the "periodicity relations":

$$\tau((V \xrightarrow{f} X) \times (M \to pt)) = sign(M) \cdot \tau(V \xrightarrow{f} X)$$

and

$$\sigma((V \xrightarrow{f} X) \times (M \to pt)) = sign(M) \cdot \sigma(V \xrightarrow{f} X)$$

Here $\Omega^{l}_*(X, Q/Z)$ is the odd part of Q/Z-bordism. The elements of $KO^i(X, A) \otimes Z[1/2]$ (that is, in the relative case) are analogously described.

Now, suppose that X is a cycle in the geometric homology theory $\Omega_*^{\mathfrak{F}}$ mentioned earlier. Let $h: X^q \to R^N$ be a P.L. embedding into a large Euclidean space such that X has codimension $4k$, and let U be a regular neighborhood of X. A canonical orientation μ_X in $ko_q(X) \otimes Z[1/2]$ corresponds by Alexander duality to a canonical element of $ko^{4k}(U, \partial U) \otimes Z[1/2]$.

That is, it corresponds to a commutative "diagram":

(1)
$$\begin{tikzcd} \Omega_{4*}(U, \partial U) \otimes Q \arrow[r, "\sigma"] \arrow[d] & Q \arrow[d] \\ \Omega^{l}_{4*}(U, \partial U; Q/Z) \arrow[r, "\tau"] & Q/Z \end{tikzcd}$$

which satisfies the periodicity relation. Sullivan described how to obtain such a diagram from the signature invariant and transversality for P.L. manifolds in [25] and for general $\mathfrak{F}$ in [26].

Given $[(M, \partial M), f]$ in $\Omega_{4*}(U, \partial U)$, the relative block transversality theorem ([8], [16]) implies that we can assume $f^{-1}(X) \subset M$ is a cycle with the same local structure as X, with codimension $4k$, say. That is, $f^{-1}(X)$ is a cycle in $\mathfrak{F}$, and has a signature $sign(f^{-1}(X)) \in Z$. By relative transversality again, and by cobordism invariance of the signature, this procedure associates to $[(M, \partial M), f]$ a well-defined integer $\sigma_X([M, \partial M), F])$. Define σ in (1) to be $\sigma_X \otimes Q$. For τ, it suffices to define a compatible collection of homomorphisms: $\tau_n : \Omega_{4*}(U, \partial U; Z/nZ) \to Z/nZ$ for n odd, compatible with the partial ordering of divisibility.

To do this, recall that $\Omega_*(\cdot; Z/nZ)$ is, geometrically, bordism of Z/nZ-manifolds [19]. Using transversality as above, assign to a singular Z/nZ-cycle $[\bar{M}, g]$ a Z/nZ-cycle $g^{-1}(X)$ in the bordism theory with Z/nZ coefficients associated to $\Omega^{\mathfrak{F}}_*$.

If Y is the relative cycle from which $g^{-1}(X)$ is obtained by identifications on the boundary, we define:

$$sign(g^{-1}(X)) = sign(Y),$$

as in the case of Z/nZ-manifolds. Additivity of the signature invariant $\mathfrak{F}$ guarantees that the correspondence:

$$\tau_n : \Omega_*(U, \partial U; Z/nZ) \to Z/nZ$$

$$[\bar{M}, g] \to sign(g^{-1}(X))$$

is a well-defined homomorphism. Let $\tau = \varprojlim_{n \text{ odd}} \tau_n$.

By multiplicativity of the signature with respect to manifolds, it is clear that σ, τ satisfy the periodicity relations. Finally, the diagram (1) corresponding to σ, τ commutes. The definition of orientations for relative cycles proceeds similarly.

The orientation μ_X satisfies:

$$c_*(\mu_X) = sign(X) \in ko_n(pt) \otimes \mathbf{Z}\left[\frac{1}{2}\right]$$

where $c: X \to pt$ is the collapse map.

The natural transformation $\mu^{\mathfrak{F}}$ is defined using these orientations. Specifically, given $[(X, \partial X), f] \in \Omega_*^{\mathfrak{F}}(A, B)$, we set $\mu^{\mathfrak{F}}[(X, \partial X), f] = f_*[\mu_X] \in ko_*(A, B) \otimes \mathbf{Z}[1/2]$, where f_* is the induced homomorphism in $ko_* \otimes \mathbf{Z}[1/2]$.

Specializing now to the bordism theory associated to the class of Witt spaces, we have the theorem:

THEOREM. *Every Witt space $(X, \partial X)$ of dimension $q \geq 0$ has a canonical orientation class $\mu_X \in ko_q(X, \partial X) \otimes \mathbf{Z}[1/2]$. If $\partial X = \phi$ then the homomorphism*

$$c_*: ko_q(X) \otimes \mathbf{Z}\left[\frac{1}{2}\right] \to ko_q(pt) \otimes \mathbf{Z}\left[\frac{1}{2}\right].$$

carries μ_X to $sign(X)$. There is, therefore, a natural transformation

$$\mu^{Witt}: \Omega_*^{Witt} \to ko_* \otimes \mathbf{Z}\left[\frac{1}{2}\right]$$

which reduces to the signature homomorphism on coefficient groups. □

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