A Decoder Example for Analog Error-Correcting Codes

Ziyuan Zhu and Paul H. Siegel
Department of Electrical and Computer Engineering, CMRR
University of California San Diego, La Jolla, CA, USA
Email: {ziz050, psiegel}@ucsd.edu

Changcheng Yuan and Anxiao Jiang
Department of Computer Science and Engineering
Texas A&M University, College Station, TX, USA
Email: {ericycc, ajiang}@tamu.edu

Ron M. Roth
Computer Science Department
Technion, Haifa, Israel
Email: ronny@cs.technion.ac.il

1 Introduction

Analog error-correcting codes over the real field have been proposed for protecting analog in-memory computation against a mixed noise model consisting of small perturbations and sparse outlier errors. The purpose of this note is to give an example illustrating the decoder used in the proof of the error-handling condition in [1, 2]. The example is based on a real-valued repetition code and is intended only to clarify the decoder behavior.

2 Error model and decoder

Let n and k be positive integers with $k \le n$. A real-valued linear $[n, k]$ code is a k-dimensional subspace $\mathcal{C} \subseteq \mathbb{R}^n$. Equivalently, it can be represented by a real $k \times n$ generator matrix $G = [\boldsymbol{g}_0, \ldots, \boldsymbol{g}_{n-1}]$, where $\boldsymbol{g}_i \in \mathbb{R}^k$, and each codeword has the form

$$\boldsymbol{c} = \boldsymbol{u}G, \qquad \boldsymbol{u} \in \mathbb{R}^k.$$

Throughout this note, for a positive integer n, we write

$$[n\rangle = \{0, 1, \ldots, n-1\}.$$

Let δ and Δ be positive real thresholds satisfying $\Delta > \delta > 0$. An error vector $\boldsymbol{\varepsilon} = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{n-1}) \in \mathbb{R}^n$ is referred to as an LME vector if $\varepsilon_i \in [-\delta, \delta]$ for all $i \in \{0, 1, \ldots, n-1\}$. Let $\boldsymbol{e} = (e_0, e_1, \ldots, e_{n-1}) \in \mathbb{R}^n$ denote a UME vector representing outliers. A noisy received word $\boldsymbol{y} = (y_0, \ldots, y_{n-1}) \in \mathbb{R}^n$ is given by $\boldsymbol{y} = \boldsymbol{c} + \boldsymbol{\varepsilon} + \boldsymbol{e}$.

The vector $\boldsymbol{e}$ is assumed to be sparse, since it models outlier faults caused by rare events such as stuck cells or shorted cells in the array. Unlike Gaussian noise, such faults occur infrequently, but when they do occur, they typically have large magnitudes. An outlier is defined by $|e_i| > 0$. Among such outliers, **significant** outliers are those satisfying $|e_i| > \Delta$.

Following [1, 2], we define the support of an outlier vector $\boldsymbol{e}$ as

$$\mathrm{Supp}_0(\boldsymbol{e}) := \{i \in [n\rangle : |e_i| > 0\}.$$

Similarly, define the Δ-support of $\boldsymbol{e}$ as

$$\mathrm{Supp}_\Delta(\boldsymbol{e}) := \{i \in [n\rangle : |e_i| > \Delta\}.$$

The error-handling capability of analog ECC differs conceptually from that of classical error-correcting codes over finite fields because of the nature of the analog error model. For nonnegative integers τ, σ, not both zero, we say that a length-n analog code $\mathcal{C}$ is τ-error locating and $(\tau+\sigma)$-error detecting if there is a decoder

$$\mathcal{D} : \mathbb{R}^n \to \mathcal{C} \,\cup\, \{\text{“e”}\}$$

with the following property.

(i) $\mathrm{Supp}_\Delta(\boldsymbol{e}) \subseteq \mathcal{D}(\boldsymbol{y}) \subseteq \mathrm{Supp}_0(\boldsymbol{e}), \quad \text{if } |\,\mathrm{Supp}_0(\boldsymbol{e})| \leq \tau.$

(ii) $\mathcal{D}(\boldsymbol{y})=\text{“e” or } \mathrm{Supp}_\Delta(\boldsymbol{e}) \subseteq \mathcal{D}(\boldsymbol{y}) \neq \text{“e”}, \quad \text{otherwise.}$

In other words, if $\boldsymbol{e}$ has no more than τ UMEs, the decoder output is a subset of the support of $\boldsymbol{e}$ that contains all significant UMEs. Otherwise, the decoder output is either the error flag “e” or a set of positions that includes all significant outlying errors of $\boldsymbol{e}$. Note that in the latter case, the set may include “false alarms” that are positions outside the support of $\boldsymbol{e}$.

The proof of Theorem 1 in [1,2] introduced a particular decoder for locating τ significant UMEs and detecting $\tau + \sigma$ UMEs. Specifically, if $\boldsymbol{e}$ is a UME vector with $|\,\mathrm{Supp}_0(\boldsymbol{e})| \leq \tau + \sigma$, then the decoder $\mathcal{D}$ operates as follows:

(i) The output $\mathcal{D}(\boldsymbol{y})$ is “e” if there is no compatible UME vector $\boldsymbol{e}'$ such that $|\,\mathrm{Supp}_0(\boldsymbol{e}')| \leq \tau$.

(ii) Otherwise, the output $\mathcal{D}(\boldsymbol{y})$ is a set of size at most τ, defined as the intersection of the supports of all compatible UME vectors $\boldsymbol{e}'$ with $|\,\mathrm{Supp}_0(\boldsymbol{e}')| \leq \tau$. By Lemma 2 in [2], this set is guaranteed to contain $\mathrm{Supp}_\Delta(\boldsymbol{e})$.

The following example further illustrates the behavior of the decoder for the $[n, 1]$ repetition code over $\mathbb{R}$ discussed in Example 1 of [2].

Example 1. *Consider the repetition code*

$$\mathcal{C} = \{(a, a, \ldots, a) \in \mathbb{R}^{12} : a \in \mathbb{R}\},$$

with parameters

$$n = 12, \qquad k = 1, \qquad \delta = 1, \qquad \Delta = 4, \qquad \tau = 5, \qquad \sigma = 1.$$

Assume that the transmitted input is $u = 0$, so the transmitted codeword is

$$\boldsymbol{c} = (0, 0, \ldots, 0).$$

The received vector is

$$\boldsymbol{y} = \boldsymbol{c} + \boldsymbol{e} + \boldsymbol{\varepsilon},$$

where $|\varepsilon_i| \leq 1$ for all $i \in [12\rangle$.

We consider the following three cases.

(i) ***A case with*** $|\operatorname{Supp}_0(\boldsymbol{e})| \le \tau$***.*** *Let*

$$\boldsymbol{e} = (5,\ 4.5,\ 2.6,\ 2,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0), \qquad \boldsymbol{\varepsilon} = \mathbf{0}.$$

Then

$$\boldsymbol{y} = (5,\ 4.5,\ 2.6,\ 2,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0),$$

and

$$\operatorname{Supp}_0(\boldsymbol{e}) = \{0,1,2,3\}, \qquad \operatorname{Supp}_\Delta(\boldsymbol{e}) = \{0,1\}.$$

The possible compatible codewords $\mathbf{c}' = (a,\dots,a)$ *with a compatible UME vector* $\mathbf{e}'$ *satisfying*

$$|\operatorname{Supp}_0(\boldsymbol{e}')| \le 5$$

are those with $a \in [-1,1]$.

If $a \in [-1,1)$, *then*

$$\operatorname{Supp}_0(\boldsymbol{e}') = \{0,1,2,3\}.$$

If $a = 1$, *then*

$$\operatorname{Supp}_0(\boldsymbol{e}') = \{0,1,2\}.$$

Hence, the possible supports are

$$\{0,1,2,3\} \qquad \textit{and} \qquad \{0,1,2\},$$

and therefore

$$\mathcal{D}(\boldsymbol{y}) = \{0,1,2\}.$$

(ii) ***A case with*** $\tau < |\operatorname{Supp}_0(\boldsymbol{e})| \le \tau + \sigma$***.*** *Let*

$$\boldsymbol{e} = (5,\ 5,\ 4.6,\ 4.3,\ 2.7,\ 2.4,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0), \qquad \boldsymbol{\varepsilon} = \mathbf{0}.$$

Then

$$\boldsymbol{y} = (5,\ 5,\ 4.6,\ 4.3,\ 2.7,\ 2.4,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0),$$

and

$$\operatorname{Supp}_0(\boldsymbol{e}) = \{0,1,2,3,4,5\}, \qquad \operatorname{Supp}_\Delta(\boldsymbol{e}) = \{0,1,2,3\}.$$

There is no compatible codeword $\mathbf{c}' = (a,\dots,a)$ *for which the corresponding compatible UME vector* $\mathbf{e}'$ *satisfies*

$$|\operatorname{Supp}_0(\boldsymbol{e}')| \le 5.$$

Therefore,

$$\mathcal{D}(\boldsymbol{y}) = \text{“}e\text{”}.$$

(iii) ***A case with*** $|\operatorname{Supp}_0(\boldsymbol{e})| > \tau + \sigma$***.*** *Let*

$$\boldsymbol{e} = (5,\ 5,\ 5,\ 5,\ 5,\ 5,\ 5,\ 3,\ 3,\ 3,\ 3,\ 3), \qquad \boldsymbol{\varepsilon} = \mathbf{0}.$$

Then

$$\boldsymbol{y} = (5,\ 5,\ 5,\ 5,\ 5,\ 5,\ 5,\ 3,\ 3,\ 3,\ 3,\ 3),$$

and

$$|\operatorname{Supp}_0(\boldsymbol{e})| = 12 > \tau + \sigma = 6.$$

Taking

$$\boldsymbol{c}' = (4, 4, \ldots, 4),$$

we have

$$|y_i - 4| \leq 1, \qquad \forall i \in [12\rangle,$$

so there exists a compatible decomposition with

$$\boldsymbol{e}' = \boldsymbol{0}.$$

Hence,

$$\mathcal{D}(\boldsymbol{y}) = \varnothing.$$

References

[1] R. M. Roth, "Analog error-correcting codes," *IEEE Trans. Inf. Theory*, vol. 66, no. 7, pp. 4075-4088, Jul. 2020.

[2] R. M. Roth, "Corrections to Analog error-correcting codes," *IEEE Trans. Inf. Theory*, vol. 69, no. 6, pp. 3793-3794, Jun. 2023.