A New Class of Geometric Analog Error Correction Codes for Crossbar Based In-Memory Computing

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***Abstract*—Analog error correction codes over the real field have recently been proposed for analog in-memory computing on resistive crossbars, a promising circuit for accelerating vector–matrix multiplication in machine learning. Unlike traditional communication or storage channels, this scenario involves a mixed noise model consisting of small perturbations and outlier errors. Existing analog codes either focus on single outlier correction or can correct multiple outliers but lack a systematic construction. In this paper, we introduce a new class of geometric codes that can correct multiple outlier errors with a systematic construction. Then we provide analysis and comparisons with the best existing codes, showing that the proposed geometric codes have superior error correction capability across various settings.**

I. Introduction

Analog in-memory computing is a cutting-edge technology that integrates data storage and computation directly within memory cells, enabling significant acceleration of deep neural network (DNN) computations [1], [2]. The primary motivation for analog in-memory computing is to overcome the "von Neumann bottleneck" by avoiding the need for massive data transfers between processors and memory [3]–[5]. This approach promises substantial improvements in speed and energy efficiency by exploiting the vector-matrix multiplications within crossbar architectures [6], a fundamental operation in DNNs. In recent years, significant progress has been made in the development of analog computing chips that implement deep neural networks (DNNs) for training [7] and inference [8] directly in analog circuits. They achieve performance comparable to software-based implementations while having substantially higher speed and energy efficiency than digital circuits.

However, a challenge for analog in-memory computing is the reliability of computing against errors. Non-volatile memories (NVMs) are known to suffer from a range of noise mechanisms, including cell-programming noise, cell-level drift, random noise, read/write disturbances, stuck cells, and short cells, etc [9]. In general, these errors can be broadly categorized into two types: small but ubiquitous errors, such as programming noise, and more isolated but potentially severe errors, such as stuck cells.

Analog ECCs have been proposed to address the above challenge [10], [11]. These codes are specifically designed to correct errors in codewords transmitted through channels that introduce two primary types of additive noise: limited-magnitude errors (LMEs) and unlimited-magnitude errors (UMEs). LMEs are small but widespread, arising from effects like cell-programming noise and random read/write disturbances in NVM arrays. In contrast, UMEs, such as stuck cells or short cells, occur less frequently but can be significantly more disruptive. While DNNs can often tolerate minor, distributed noise, they are particularly vulnerable to large, isolated errors, making robust error correction essential for reliable in-memory computation [1].

This has motivated the development of several analog error correction code constructions [10]–[12]. However, most existing codes focus primarily on detecting or correcting only a single UME [10], [11]. Some promising short codes for multiple UMEs have been generated using genetic algorithms [12]. Despite this progress, the design of Analog ECCs remains a significant challenge. A critical metric for the analysis of analog codes is the m-height [10], which quantifies a code's ability to correct UMEs, analogous to the minimum distance in conventional ECCs over finite fields. Calculating the exact m-height has a computational complexity of approximately $O(n^{m+1})$ [12].

To explore codes with systematic construction and the ability to correct multiple UMEs, this paper introduces a class of geometry-based analog codes: polygonal codes, polyhedral codes and dual polyhedral codes. The polygonal codes have dimension $k = 2$ with configurable length n. There are two polyhedra we investigated: the icosahedron and dodecahedron. The icosahedral code and its dual have parameters $(n, k) = (6, 3)$, while the dodecahedral code and its dual have parameters $(n, k) = (10, 7)$ and $(n, k) = (10, 3)$, respectively. We present explicit constructions of these codes and analyze their m-heights. A comparison with the best known codes demonstrates their superior performance.

The paper is organized as follows. Section II introduces the necessary background and notation. Section III presents the construction and analysis of the polygonal codes. Section **??**

introduces the icosahedral and dodecahedral codes, together with their corresponding analyses. Section IV discusses the dual icosahedral and dual dodecahedral codes and analyzes their error correction capability. Finally, Section V concludes the paper.

II. Preliminaries

Let $\mathcal{C} \subset \mathbb{R}^n$ denote a real-valued linear code of dimension k, generated by a real-valued $k \times n$ matrix $\mathbf{G}$. Each codeword takes the form

$$\mathbf{c} = \mathbf{u}\,\mathbf{G}, \qquad \mathbf{u} \in \mathbb{R}^k,\ \mathbf{G} \in \mathbb{R}^{k\times n},\ \mathbf{c} \in \mathbb{R}^n,$$

where $\mathbf{u}$ is an arbitrary real input vector. For a codeword $\mathbf{c} = (c_0, c_1, \ldots, c_{n-1}) \in \mathcal{C}$, consider the absolute values

$$|c_0|,\ |c_1|,\ \ldots,\ |c_{n-1}|.$$

Rearrange these n values in nonincreasing order and denote the order statistics by

$$|c_{(0)}| \ \geq\ |c_{(1)}| \ \geq\ \cdots\ \geq\ |c_{(n-1)}|,$$

where $c_{(j)}$ denotes the entry of $\mathbf{c}$ whose absolute value is the j-th largest among $\{|c_0|, \ldots, |c_{n-1}|\}$.

The m-height of the codeword $\mathbf{c}$ is then defined as the ratio of the largest-magnitude entry to the $(m+1)$-th largest-magnitude entry:

$$h_m(\mathbf{c}) \ =\ \left|\frac{c_{(0)}}{c_{(m)}}\right|.$$

The m-height of the code $\mathcal{C}$ is defined as

$$h_m(\mathcal{C}) \ =\ \max_{\mathbf{c}\in\mathcal{C}}\ h_m(\mathbf{c}),$$

that is, the largest possible m-height achieved over all codewords.

Theorem 2 of [12] gives a general method to compute $h_m(\mathcal{C})$ by solving a family of linear programs that jointly determine the m-height and the corresponding optimal input vector $\mathbf{u}_{\text{opt}}$. This procedure requires solving

$$n(n-1)\binom{n-2}{m-1}2^m$$

linear programs (each with k variables).

Let δ and Δ be positive real thresholds, with $\Delta > \delta > 0$. An error vector $\boldsymbol{\varepsilon} = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{n-1}) \in \mathbb{R}^n$ is called a *limited-magnitude error* (LME) vector if

$$\varepsilon_i \in [-\delta, \delta], \qquad \forall\, i \in \{0, 1, \ldots, n-1\}.$$

For a vector $\mathbf{e} = (e_0, e_1, \ldots, e_{n-1}) \in \mathbb{R}^n$, define its Δ-support as

$$\mathrm{Supp}_\Delta(\mathbf{e}) \ :=\ \left\{\, i \in \{0, 1, \ldots, n-1\} : |e_i| > \Delta \,\right\}.$$

This definition naturally extends to $\Delta = 0$, in which case

$$\mathrm{Supp}_0(\mathbf{e}) = \mathrm{Supp}(\mathbf{e})$$

is the ordinary support. The Hamming weight of $\mathbf{e}$ is then defined as

$$W_H(\mathbf{e}) = \big|\mathrm{Supp}_0(\mathbf{e})\big|.$$

An error vector $\mathbf{e} = (e_0, \ldots, e_{n-1}) \in \mathbb{R}^n$ is called an *unlimited-magnitude error* (UME) vector of Hamming weight w if $W_H(\mathbf{e}) = w$.

A noisy received word $\mathbf{y} = (y_0, \ldots, y_{n-1}) \in \mathbb{R}^n$ is given by

$$\mathbf{y} = \mathbf{c} + \boldsymbol{\varepsilon} + \mathbf{e},$$

where $\mathbf{c} \in \mathcal{C}$ is the transmitted codeword, $\boldsymbol{\varepsilon}$ is the LME vector, and $\mathbf{e}$ is the UME vector. The analog error correction code is designed to correct UMEs. A fundamental condition characterizing this error correction capability was derived in [10], as stated in the following theorem.

Theorem II.1. *Let $\mathcal{C}$ be a linear (n,k) code over $\mathbb{R}$. Given $\delta, \Delta \in \mathbb{R}^+$ with $\delta < \Delta$ and a positive integer t, there exists a decoder for $\mathcal{C}$ that corrects t UMEs if and only if*

$$\Delta \ \geq\ 2\big(h_{2t}(\mathcal{C}) + 1\big)\delta.$$

Thus, achieving smaller m-height directly translates to stronger UME correction capability.

III. Dual Polygonal Codes

In this section, we construct a class of $k = 2$ analog codes whose generator matrices consist of evenly spaced unit vectors over a half circle. It is the dual of the code introduced in Example 1 of [13].

A. Code Construction

Let $k = 2$ and let $G = [g_0, \ldots, g_{n-1}] \in \mathbb{R}^{2\times n}$ consist of unit columns

$$g_j = \begin{bmatrix}\cos\theta_j \\ \sin\theta_j\end{bmatrix}, \qquad \theta_j \in [0, \pi).$$

Choose

$$\theta_j = \frac{\pi}{n}\cdot j, \qquad j = 0, 1, \ldots, n-1.$$

Then the set $\{g_0, \ldots, g_{n-1}\}$ forms an evenly spaced set of vectors over a half circle. An example with $n = 3$ is illustrated in Fig. 1. The dashed lines mark the antipodal directions of the generator vectors. Because only the absolute values of the codeword $|c_j|$ affect the m-height, using g_j or $-g_j$ yields codes with the same error correction capability. For simplicity, we select vectors within the angular range $[0, \pi)$.

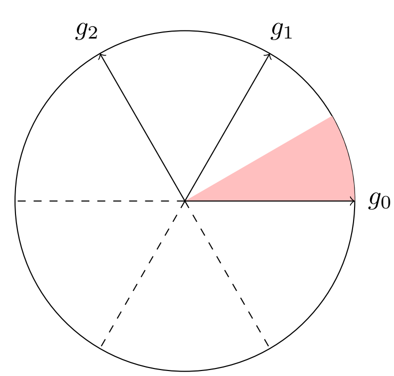

Fig. 1. An example of polygonal codes for $n = 3$.

B. m-height profile

According to the definition of m-height, the magnitude of the information vector does not matter. For a unit information vector $u(\varphi) = (\cos\varphi, \sin\varphi)^\top$, the resulting codeword is

$$c(\varphi) = u(\varphi)^\top G \in \mathbb{R}^n,$$

whose entries are

$$c_j(\varphi) = \cos(\theta_j - \varphi).$$

Let $c_{(0)}(\varphi) \geq c_{(1)}(\varphi) \geq \cdots \geq c_{(n-1)}(\varphi)$ denote the elements of $\{|c_j(\varphi)|\}_{j=0}^{n-1}$ sorted in nonincreasing order. Then the m-height of the code is

$$h_m(\mathcal{C}) = \sup_{\varphi\in[0,2\pi)} \frac{c_{(0)}(\varphi)}{c_{(m)}(\varphi)}.$$

Information directions $\varphi \in \left[0, \frac{\pi}{n}\right]$ generate a subset of codewords. Due to the symmetry of the construction, all other codewords can be obtained from this subset by suitable permutations (and sign changes) of the coordinates. Since permutations do not affect the m-height, the interval $\varphi \in \left[0, \frac{\pi}{n}\right]$ constitutes a complete and sufficient domain for analyzing the m-height of the code, as illustrated in Fig. 1. Hence, the m-height simplifies to

$$h_m(\mathcal{C}) = \sup_{\varphi\in[0,2\pi)} \frac{c_{(0)}(\varphi)}{c_{(m)}(\varphi)} = \sup_{\varphi\in[0,\frac{\pi}{n}]} \frac{c_{(0)}(\varphi)}{c_{(m)}(\varphi)}.$$

Lemma III.1 (Order statistics of $|c_j(\varphi)|$)**.** *Let $n \geq 2$ and fix $\varphi \in \left[0, \frac{\pi}{n}\right]$. Then the order statistics $c_{(k)}(\varphi)$ of $|c_j(\varphi)|$, are attained at the indices*

$$c_{(k)}(\varphi) = |c_{j_k}(\varphi)|, \qquad k = 0, 1, \ldots, n-1,$$

where

$$j_k = \begin{cases} 0, & k = 0, \\ \dfrac{k+1}{2}, & k \text{ odd}, \\ n - \dfrac{k}{2}, & k \text{ even and } k \geq 2. \end{cases}$$

Thus, by Lemma III.1, the m-height computation can be simplified as follows.

$$h_m(C) = \sup_{\varphi\in[0,\frac{\pi}{n}]} \frac{c_{(0)}(\varphi)}{c_{(m)}(\varphi)} = \sup_{\varphi\in[0,\frac{\pi}{n}]} \frac{\cos\varphi}{\cos\theta_{j_m}}.$$

For the polygonal code, the direction $u(\varphi)$ can be orthogonal to at most one generator vector g_j, that is, there is at most one j such that $\cos(\theta_j - \varphi) = 0$. Hence at most the smallest order statistic $c_{(n-1)}(\varphi)$ can vanish. When $m = n-1$, the denominator $c_{(n-1)}(\varphi)$ vanishes for some φ, so $h_{n-1}(C) = \infty$, as expected for an MDS code. In particular, for every $m \leq n-2$, the denominator $c_{(m)}(\varphi)$ is strictly positive for all $\varphi \in [0, \frac{\pi}{n}]$, and the corresponding m-height is finite. We therefore restrict attention to $m \leq n-2$.

Theorem III.2 (Critical points and maximal m-height)**.** *For the polygonal codes, and $0 < m \leq n-2$, the m-height achieves its maximum at*

$$\arg\max_{\varphi\in[0,\frac{\pi}{n}]} h_m(\varphi) = \begin{cases} \varphi = \frac{\pi}{n}, & m \text{ even}, \\ \varphi = 0, & m \text{ odd}. \end{cases}$$

Moreover, the m-height is

$$h_m(C) = \begin{cases} \dfrac{\cos\frac{\pi}{2n}}{\cos\left((m+1)\frac{\pi}{2n}\right)}, & m \text{ even}, \\ \dfrac{1}{\cos\left((m+1)\frac{\pi}{2n}\right)}, & m \text{ odd}. \end{cases}$$

C. Comparison with best-known codes

We compared our polygonal codes with the best known codes with $k = 2$ and $n \leq 10$ in the literature, namely the codes found by genetic programming in [12]. The proposed polygonal codes outperform the existing codes for most (n, m) parameter pairs, and they achieve strictly smaller m-height in the cases listed in Table I.

TABLE I
COMPARISON BETWEEN PREVIOUS BEST CODES AND POLYGONAL CODES.

(n,m)	Genetic	Polygonal codes
$(5,2)$	1.83	1.6180
$(5,3)$	3.25	3.2361
$(6,3)$	2.28	2.0000
$(6,4)$	4.10	3.7321
$(8,4)$	2.88	1.7654
$(10,5)$	1.92	1.7013
$(10,6)$	3.88	2.1756
$(10,7)$	3.88	3.2361
$(10,8)$	28.74	6.3138

IV. DUAL POLYHEDRAL CODES

In this section, we introduce two classes of graph-based codes derived from three-dimensional geometric structures: the icosahedron and the dodecahedron. These codes are the duals of the codes introduced in Examples 2 and 3 of [13].

A. Dual Icosahedral Codes

We first consider a class of graph codes based on three-dimensional geometric structures, using the regular icosahedron as the prototype.

1) Construction: The icosahedron has 12 vertices, 20 faces, and 30 edges. By placing its vertices on the sphere, we obtain 6 symmetric axes $\{g_1, \ldots, g_6\}$, where each axis corresponds to a pair of antipodal vertices. These six axes can be represented by the following matrix:

$$G = \begin{bmatrix} 0 & 0 & 1 & 1 & \phi & \phi \\ 1 & 1 & \phi & -\phi & 0 & 0 \\ \phi & -\phi & 0 & 0 & 1 & -1 \end{bmatrix}, \qquad \phi = \frac{1+\sqrt{5}}{2}.$$

2) m-height profile: When searching for the optimal information vector u, only its direction matters, and its magnitude does not affect the m-height. Therefore, it is sufficient to restrict the analysis to the faces of the solid. Among the 20 faces, all are symmetric, and furthermore, each face contains subregions that are themselves symmetric. Thus, analyzing a single representative subregion is equivalent to analyzing the entire space.

Motivated by this symmetry, we restrict our attention to the subregion whose direction is closest to g_1, as illustrated in Fig. 2. We define the smaller triangle as

$$T := \operatorname{conv}\{v_1, v_2, v_3\}$$

with

$$v_1 := g_1, \qquad v_2 := \frac{g_1 + g_3}{2}, \qquad v_3 := \frac{g_1 + g_3 + g_5}{3}.$$

Fig. 2. Icosahedron with a shaded triangular region indicating the fundamental search space used to locate the optimal directions within one face.

Any $x \in T$ can be written in barycentric form

$$x(u,v) := uv_1 + vv_2 + (1-u-v)v_3.$$

The parameter domain is the standard triangle

$$D := \{(u,v) \in \mathbb{R}^2 : u \geq 0,\ v \geq 0,\ u+v \leq 1\}.$$

Lemma IV.1 (Global ordering of $|x \cdot g_i|$)**.** *For every $x \in T$, the six absolute values $|x \cdot g_i|$, $i = 1, \ldots, 6$, satisfy the global order*

$$|x \cdot g_1| \;\geq\; |x \cdot g_3| \;\geq\; |x \cdot g_5| \;\geq\; |x \cdot g_4| \;\geq\; |x \cdot g_2| \;\geq\; |x \cdot g_6|.$$

For $m = 0$, the m-height is trivial. Moreover, since the code is MDS, we have $h_m(\mathcal{C}) = \infty$ for $m = 4, 5$. Hence, it suffices to consider $m = 1, 2, 3$. From the proof of Lemma IV.1, we have shown that for all $x \in T$,

$$x \cdot g_1 \;\geq\; x \cdot g_3 \;\geq\; x \cdot g_5 \;\geq\; 0 \;\geq\; x \cdot g_4.$$

Therefore, for $m = 1, 2, 3$, the m-height optimization reduces to maximizing

$$f_m(x) := \frac{x \cdot g_1}{d_m(x)}, \qquad x \in T,$$

where the denominator is given by

$$d_1(x) = x \cdot g_3, \qquad d_2(x) = x \cdot g_5, \qquad d_3(x) = -\,x \cdot g_4.$$

Theorem IV.2. *Let $x \in T$ and let f_1, f_2, f_3 be defined as above. Then the maximizers over T and the corresponding maximal values are:*

1) $\arg\max_{x \in T} f_1(x) = v_1$, *and* $f_1(g_1) = \sqrt{5}$.
2) $\arg\max_{x \in T} f_2(x) = v_1$, *and* $f_2(g_1) = \sqrt{5}$.
3) $\arg\max_{x \in T} f_3(x) = v_3$, *and* $f_3(v_3) = 2 + \sqrt{5}$.

B. Dual dodecahedral codes

We now consider the graph code derived from the regular dodecahedron.

1) Construction: The dodecahedron has 20 vertices, 12 faces, and 30 edges. Placing its vertices on the sphere yields 10 symmetric axes $\{g_1, \ldots, g_{10}\}$ (each axis connects a pair of antipodal vertices), represented by

$$G = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & \varphi^{-1} & \varphi^{-1} & \varphi & \varphi \\ 1 & 1 & -1 & -1 & \varphi & \varphi & 0 & 0 & \varphi^{-1} & -\varphi^{-1} \\ 1 & -1 & 1 & -1 & \varphi^{-1} & -\varphi^{-1} & \varphi & -\varphi & 0 & 0 \end{bmatrix}.$$

2) m-height profile: Similar to the icosahedral case, symmetry allows us to restrict attention to a fundamental triangular region on a single dodecahedral face, as illustrated in Fig. 3, with vertices

$$x_A := g_1, \quad x_B := \frac{g_1 + g_5}{2}, \quad x_C := \frac{g_1 + g_2 + g_5 + g_6 + g_9}{5}.$$

Let $T' = \operatorname{conv}\{x_A, x_B, x_C\}$, $x(u,v) = u\,x_A + v\,x_B + (1-u-v)\,x_C$, where $u, v \geq 0$ and $u + v \leq 1$, and define $\beta_j(x) := |x \cdot g_j|$. Let $\beta_{[j]}(x)$ be the order statistics of $\beta_j(x)$.

Lemma IV.3. *For any $x \in T'$ we have the following inequalities:*

$$\beta_1 \geq \beta_5 \geq \beta_9 \geq \max\{\beta_6, \beta_7\}, \quad (1)$$
$$\beta_6 \geq \beta_2, \qquad \beta_6 \geq \beta_4, \qquad \beta_7 \geq \beta_4, \quad (2)$$
$$\beta_4 \geq \max\{\beta_3, \beta_8, \beta_{10}\}, \qquad \beta_2 \geq \max\{\beta_3, \beta_8, \beta_{10}\}, \quad (3)$$
$$\beta_3 \leq \beta_{10}. \quad (4)$$

Corollary IV.4. *Let $x \in T'$ and let $\beta_{[k]}(x)$ be the order statistics of $\beta_j(x)$. Under the inequalities* (1)–(4)*, the index attaining rank k can only belong to:*

$$k = 1 : \{1\}, \quad k = 2 : \{5\}, \quad k = 3 : \{9\}, \quad k = 4 : \{6, 7\},$$
$$k = 5 : \{2, 6, 7\}, \quad k = 6 : \{2, 4, 7\}, \quad k = 7 : \{2, 4\},$$
$$k = 8 : \{8, 10\}, \quad k = 9 : \{3, 8, 10\}, \quad k = 10 : \{3, 8\}.$$

Lemma IV.5. *For each $j \in \{2, 4, 5, 6, 7, 9, 10\}$, define*

$$f_j(u,v) := \frac{x(u,v) \cdot g_1}{|x(u,v) \cdot g_j|}, \qquad (u,v) \in T'.$$

Then f_j has no stationary point in T'. In fact, on T' we have

$$\partial_u f_2 > 0,\ \partial_v f_2 > 0; \qquad \partial_u f_5 > 0,\ \partial_v f_5 \leq 0;$$
$$\partial_u f_6 > 0,\ \partial_v f_6 > 0; \qquad \partial_u f_7 < 0,\ \partial_v f_7 < 0;$$
$$\partial_u f_9 > 0,\ \partial_v f_9 > 0; \qquad \partial_u f_{10} < 0,\ \partial_v f_{10} > 0;$$

and for $j = 4$, $\partial_v f_4 < 0$.

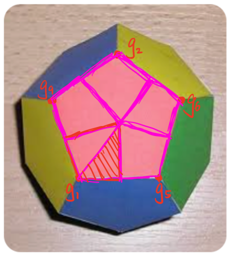

Fig. 3. Dodecahedron with a shaded triangular region indicating the fundamental search space used to locate the optimal directions within one face.

TABLE II
COMPARISON BETWEEN PREVIOUS BEST CODES AND POLYHEDRAL CODES.

n	k	m	Genetic	Polyhedral codes
6	3	1	1	2.24
6	3	2	2.87	2.24
6	3	3	7	4.24
10	3	1	1	1.34
10	3	2	1	1.62
10	3	3	2.12	1.76
10	3	4	4.36	3
10	3	5	5.97	4.24
10	3	6	17.18	4.24
10	3	7	69.44	9.47

Corollary IV.6. *Let $x \in T'$. For $m = 1, 2$, the corresponding m-height objective functions are*

$$f_5(x) := \frac{x \cdot g_1}{x \cdot g_5}, \qquad f_9(x) := \frac{x \cdot g_1}{x \cdot g_9}.$$

And:

1) *The maximizer of f_5 over T' is $x = x_A = g_1$, and*

$$f_5(g_1) = \frac{3}{\sqrt{5}}.$$

2) *The maximizer of f_9 over T' is $x = x_B = \frac{g_1 + g_5}{2}$, and*

$$f_9(x_B) = \varphi.$$

Theorem IV.7. *For each $m \in \{3, 4, 5, 6, 7\}$, any maximizer of the corresponding m-height over T' must lie in the candidate set*

$$\mathcal{S} = \Big\{ (u, v) \in T' : \ (1, 0), \ (0, 0), \ (0, 1),$$

$$\big(0, \tfrac{1+3\sqrt{5}}{11}\big), \ \big(\tfrac{\varphi}{3}, 0\big), \ \big(0, 2\sqrt{5} - 4\big) \Big\}.$$

Evaluating all candidates in $\mathcal{S}$ and taking the maximum yields

$$h_3 = 4 - \sqrt{5}, \qquad h_4 = h_5 = 3,$$

$$h_6 = 2 + \frac{3\sqrt{5}}{5}, \qquad h_7 = 5 + 2\sqrt{5}.$$

Moreover, the maxima are attained at $x = x_C$ for $m \in \{3, 6\}$, at $x = x_A = g_1$ for $m \in \{4, 5\}$, and at $(u, v) = \big(0, 2\sqrt{5} - 4\big)$ for $m = 7$.

C. Comparison with best-known codes

We compare our dual polyhedral codes with the best known codes in the literature. Table II summarizes the results: the column "Genetic" lists the smallest m-height values obtained via the genetic search, as can be found in [12]. The comparison shows that the proposed dual polyhedral codes outperform the existing codes for most (n, m) parameter pairs, especially for larger m values.

V. CONCLUSION

In this paper, we proposed several geometric analog error-correction codes, namely polygonal codes, polyhedral codes and their duals. We analyzed their m-heights, a key metric that characterizes the error correction capability of analog codes. Through comparison with the best previously known codes, we showed that the proposed codes outperform existing designs for most parameter settings. A limitation of the current constructions is that their dimensions are restricted to $k = 2$ and $k = 3$, and for $k = 3$ the code lengths are limited to $n = 6$ and $n = 10$. Extending these geometric constructions to more general values of k and n remains an interesting direction for future work.

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Appendix

a) Proof of Lemma III.1: : Recall that

$$c_j(\varphi) = \cos\Big(\tfrac{\pi j}{n} - \varphi\Big), \qquad j = 0, 1, \dots, n-1,$$

and by definition $c_{(k)}(\varphi)$ denotes the $(k+1)$-th largest value among $\{|c_j(\varphi)|\}_{j=0}^{n-1}$.

Since $|\cos x|$ is π-periodic and even, it is convenient to reduce to the principal strip $(-\frac{\pi}{2}, \frac{\pi}{2}]$. Define

$$P(x) \;:=\; x - \pi \Big\lfloor \frac{x + \frac{\pi}{2}}{\pi} \Big\rfloor \in \big(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\big],$$

so that

$$\big|\cos P(x)\big| = \big|\cos x\big| \quad \text{for all } x \in \mathbb{R}.$$

Let

$$\mathcal{M} := \big\{-\varphi + j\tfrac{\pi}{n} : j = 0, 1, \dots, n-1\big\},$$
$$\mathcal{L} := P(\mathcal{M}) = \big\{\, P(-\varphi + j\tfrac{\pi}{n}) : j = 0, 1, \dots, n-1 \,\big\}.$$

Then

$$\big\{|c_j(\varphi)|\big\}_{j=0}^{n-1} = \big\{|\cos x| : x \in \mathcal{M}\big\}$$
$$= \big\{|\cos x| : x \in \mathcal{L}\big\} = \big\{\cos x : x \in \mathcal{L}\big\}.$$

because $\mathcal{L} \subset (-\frac{\pi}{2}, \frac{\pi}{2}]$ and $\cos x > 0$ on this interval.

On $[0, \frac{\pi}{2}]$ the function $x \mapsto \cos x$ is strictly decreasing, and $\cos x$ is even. Hence for $|\alpha|, |\beta| \le \frac{\pi}{2}$ we have

$$\cos\alpha \ge \cos\beta \iff |\alpha| \le |\beta|.$$

Therefore, within $\mathcal{L}$, ordering the values $\{\cos x : x \in \mathcal{L}\}$ in nonincreasing order is equivalent to ordering the absolute values $\{|x| : x \in \mathcal{L}\}$ in nondecreasing order.

We now describe $\mathcal{L}$ explicitly. First, for $j = 0, 1, \dots, \lfloor \frac{n}{2} \rfloor$ we have

$$-\varphi + j\tfrac{\pi}{n} \in \big(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\big],$$
$$P\Big(-\varphi + j\tfrac{\pi}{n}\Big) = -\varphi + j\tfrac{\pi}{n}.$$

These contribute the points

$$-\varphi,\ \tfrac{\pi}{n} - \varphi,\ \tfrac{2\pi}{n} - \varphi,\ \tfrac{3\pi}{n} - \varphi,\ \dots$$

up to the last j for which the expression stays in $(-\frac{\pi}{2}, \frac{\pi}{2}]$.

Next, consider $j = \frac{n}{2} + r$ (if n is even) or $j = \frac{n-1}{2} + 1 + r$ (if n is odd); in either case we can write

$$j = n - r, \qquad r = 1, 2, \dots, \Big\lfloor \tfrac{n-1}{2} \Big\rfloor.$$

Then

$$-\varphi + j\tfrac{\pi}{n} = -\varphi + (n-r)\tfrac{\pi}{n} = \pi - \Big(\tfrac{\pi r}{n} + \varphi\Big),$$

which lies in $(\frac{\pi}{2}, \pi)$ for the relevant r. Applying P gives

$$P\Big(-\varphi + (n-r)\tfrac{\pi}{n}\Big) = -\Big(\tfrac{\pi r}{n} + \varphi\Big),$$

so these contribute the points

$$-\Big(\tfrac{\pi}{n} + \varphi\Big),\ -\Big(\tfrac{2\pi}{n} + \varphi\Big),\ -\Big(\tfrac{3\pi}{n} + \varphi\Big),\ \dots.$$

Collecting both parts, the elements of $\mathcal{L}$ in $(-\frac{\pi}{2}, \frac{\pi}{2}]$ can be written as

$$-\varphi,\ \tfrac{\pi}{n} - \varphi,\ \tfrac{2\pi}{n} - \varphi,\ \dots \quad \text{and} \quad -\Big(\tfrac{\pi}{n} + \varphi\Big),\ -\Big(\tfrac{2\pi}{n} + \varphi\Big),\ \dots$$

hence their absolute values form the interleaving sequence

$$\varphi,\ \tfrac{\pi}{n} - \varphi,\ \tfrac{\pi}{n} + \varphi,\ \tfrac{2\pi}{n} - \varphi,\ \tfrac{2\pi}{n} + \varphi,\ \tfrac{3\pi}{n} - \varphi,\ \tfrac{3\pi}{n} + \varphi,\ \dots,$$

because for every integer $r \ge 1$ and every $\varphi \in [0, \frac{\pi}{n}]$,

$$r\tfrac{\pi}{n} - \varphi \;\le\; r\tfrac{\pi}{n} + \varphi \;\le\; (r+1)\tfrac{\pi}{n} - \varphi.$$

We now map these distances back to indices j. By construction,

- The smallest distance φ corresponds to $j = 0$.
- The next distance $\frac{\pi}{n} - \varphi$ corresponds to $j = 1$.
- The distance $\frac{\pi}{n} + \varphi$ corresponds to $j = n-1$.
- The distance $\frac{2\pi}{n} - \varphi$ corresponds to $j = 2$.
- The distance $\frac{2\pi}{n} + \varphi$ corresponds to $j = n-2$.

and so on. In general, the interleaving pattern of distances yields the index sequence

$$0,\ 1,\ n-1,\ 2,\ n-2,\ 3,\ n-3,\ \dots,$$

i.e.,

$$j_k = \begin{cases} 0, & k = 0, \\ \dfrac{k+1}{2}, & k \text{ odd}, \\ n - \dfrac{k}{2}, & k \text{ even and } k \ge 2. \end{cases}$$

Since $c_{(k)}(\varphi)$ is, by definition, the $(k+1)$-th largest element of $\{|c_j(\varphi)|\}_{j=0}^{n-1}$, and we have just shown that the corresponding indices appear in the order $j_0, j_1, j_2, \dots$ given above, which proves the lemma. □

b) Proof of Theorem III.2: :

From Lemma III.1, the ordered term for $0 < m \le n-2$ satisfies

$$c_{(m)}(\varphi) = \cos\big(A_{(m)} + s_{(m)}\varphi\big),$$

where

$$A_{(m)} = \frac{\pi}{n}\Big\lceil \frac{m}{2} \Big\rceil, \qquad s_{(m)} = (-1)^m,$$

and therefore

$$h_m(\varphi) = \frac{\cos\varphi}{\cos\big(A_{(m)} + s_{(m)}\varphi\big)}.$$

Taking a derivative and simplifying,

$$h'_m(\varphi) = \frac{s_{(m)} \sin A_{(m)}}{\cos^2\big(A_{(m)} + s_{(m)}\varphi\big)}.$$

Since $A_{(m)} \in (0, \frac{\pi}{2}]$, the numerator has the same sign as $s_{(m)}$, and the denominator is strictly positive. Thus,

$$\operatorname{sgn} h'_m(\varphi) = \operatorname{sgn}(s_{(m)}) = \begin{cases} +1, & m \text{ even}, \\ -1, & m \text{ odd}. \end{cases}$$

Hence $h_m(\varphi)$ is strictly increasing when m is even and strictly decreasing when m is odd, proving

$$\arg\max h_m = \begin{cases} \varphi = \frac{\pi}{n}, & m \text{ even}, \\ \varphi = 0, & m \text{ odd}. \end{cases}$$

Finally, evaluating at the maximizing point:

$$h_m(C) = \begin{cases} \dfrac{\cos\frac{\pi}{2n}}{\cos\big((m+1)\frac{\pi}{2n}\big)}, & m \text{ even}, \\ \dfrac{1}{\cos\big((m+1)\frac{\pi}{2n}\big)}, & m \text{ odd}. \end{cases}$$

This completes the proof. □

c) Proof of Lemma IV.1: : A direct computation with the parametrization $x(u,v)$ yields

$$x \cdot g_1 = \frac{4u+v}{3} + \frac{\sqrt{5}}{2} + \frac{7}{6},$$
$$x \cdot g_3 = \frac{-2u+v}{3} + \frac{\sqrt{5}}{2} + \frac{7}{6},$$
$$x \cdot g_5 = \frac{-2u-2v}{3} + \frac{\sqrt{5}}{2} + \frac{7}{6},$$

and

$$x \cdot g_2 = -\frac{\phi}{3}\,(2u - v + 1),$$
$$x \cdot g_4 = -\frac{\phi}{3}\,(2u + 2v + 1),$$
$$x \cdot g_6 = -\frac{\phi}{3}\,(4u + v - 1).$$

From these expressions we first obtain, for all $(u,v) \in T$,

$$x\cdot g_1 > 0, \quad x\cdot g_3 > 0, \quad x\cdot g_5 > 0, \quad x\cdot g_4 < 0, \quad x\cdot g_2 \le 0,$$

and $x \cdot g_6$ changes sign across the line $4u + v = 1$.

For g_1, g_3, g_5, we have

$$x \cdot g_1 - x \cdot g_3 = 2u \ge 0, \qquad x \cdot g_3 - x \cdot g_5 = v \ge 0,$$

with equality only on the edges $u = 0$ and $v = 0$, respectively. Since these three quantities are always positive, this implies

$$|x \cdot g_1| \ge |x \cdot g_3| \ge |x \cdot g_5|$$

for all $x \in T$.

To compare $|x \cdot g_5|$ with $|x \cdot g_4|$, note that $x \cdot g_5 > 0$ and $x \cdot g_4 < 0$, so

$$|x \cdot g_5| \ge |x \cdot g_4| \quad \iff \quad x \cdot g_5 + x \cdot g_4 \ge 0.$$

Using the formulas above, one finds

$$x \cdot g_5 + x \cdot g_4 = \frac{\sqrt{5}+3}{3}\,(1 - u - v),$$

which is nonnegative on T and vanishes exactly when $u+v = 1$. Similarly, since $x \cdot g_2 \le 0$ and $x \cdot g_4 < 0$, we compare

$$|x \cdot g_4| \ge |x \cdot g_2| \quad \iff \quad x \cdot g_4 - x \cdot g_2 \le 0,$$

and a direct computation shows

$$x \cdot g_4 - x \cdot g_2 = -\phi\, v,$$

which is nonpositive on T, with equality only when $v = 0$. For $|x \cdot g_2|$ and $|x \cdot g_6|$,

$$|x \cdot g_2| \ge |x \cdot g_6| \iff 2u - v + 1 \;\ge\; |\,4u + v - 1\,|.$$

Split into two cases:

(i) If $4u + v \ge 1$, then $|4u + v - 1| = 4u + v - 1$ and

$$2u - v + 1 \ge 4u + v - 1 \iff 2 \ge 2u + 2v \iff 1 \ge u + v,$$

which holds on T, with equality exactly on the edge $u+v = 1$.

(ii) If $4u + v \le 1$, then $|4u + v - 1| = 1 - 4u - v$ and

$$2u - v + 1 \ge 1 - 4u - v \iff 6u \ge 0,$$

always true, with equality exactly on the edge $u = 0$.

Combining (i)–(ii), we obtain $|x\cdot g_2| \ge |x\cdot g_6|$ for all $(u,v) \in T$, and equality holds only on $u = 0$ or $u + v = 1$.

Combining these comparisons yields the global chain

$$|x\cdot g_1| \;\ge\; |x\cdot g_3| \;\ge\; |x\cdot g_5| \;\ge\; |x\cdot g_4| \;\ge\; |x\cdot g_2| \;\ge\; |x\cdot g_6|$$

for all $x \in T$, with equalities only in the boundary cases listed above. □

d) Proof of Theorem IV.2: : We prove each claim by analyzing the monotonicity of the corresponding objective on the domain D.

(1) For f_1**.** For $(u,v) \in D$,

$$f_1(u,v) = \frac{4u + v + \frac{3\sqrt{5}}{2} + \frac{7}{2}}{-2u - 2v + \frac{3\sqrt{5}}{2} + \frac{7}{2}}.$$

A direct differentiation simplifies to

$$\partial_u f_1(u,v) = \frac{-6v + 9\sqrt{5} + 21}{\big(-2u - 2v + \frac{3\sqrt{5}}{2} + \frac{7}{2}\big)^2} > 0,$$
$$\partial_v f_1(u,v) = \frac{6u + \frac{9\sqrt{5}}{2} + \frac{21}{2}}{\big(-2u - 2v + \frac{3\sqrt{5}}{2} + \frac{7}{2}\big)^2} > 0.$$

Hence f_1 is increasing in both u and v on D, so the maximum must lie on the edge $u + v = 1$. Restricting to this edge,

$$f_1(u, 1-u) = \frac{3u + 1 + \frac{3\sqrt{5}}{2} + \frac{7}{2}}{-2 + \frac{3\sqrt{5}}{2} + \frac{7}{2}},$$

which is strictly increasing in u. Therefore the unique maximizer is $u = 1$, i.e., $(u,v) = (1,0)$, corresponding to $x = v_1 = g_1$. Evaluating yields $f_1(g_1) = \sqrt{5}$.

(2) For f_2**.** For $(u,v) \in D$,

$$f_2(u,v) = \frac{8u + 2v + 3\sqrt{5} + 7}{-4u + 2v + 3\sqrt{5} + 7}.$$

Differentiation gives $\partial_u f_2(u,v) > 0$ and $\partial_v f_2(u,v) \le 0$ on D. Thus, for any $(u,v) \in D$ we have $f_2(u,0) \ge f_2(u,v)$, and then $f_2(1,0) \ge f_2(u,0)$. Hence the maximum is attained at $(u,v) = (1,0)$, i.e., $x = v_1 = g_1$, and $f_2(g_1) = \sqrt{5}$.

(3) For f_3**.** For $(u,v) \in D$,

$$f_3(u,v) = \frac{8u + 2v + 3\sqrt{5} + 7}{(1+\sqrt{5})(2u + 2v + 1)}.$$

Since $(1+\sqrt{5})^{-1} > 0$, it suffices to maximize

$$g(u,v) = \frac{8u + 2v + 3\sqrt{5} + 7}{2u + 2v + 1}.$$

One checks that $\partial_u g(u,v) < 0$ and $\partial_v g(u,v) < 0$ on D. Hence g (and thus f_3) is strictly decreasing as either u or

v increases, so the maximum over D is attained at $(u, v) = (0, 0)$, corresponding to $x = v_3$. Evaluating yields $f_3(v_3) = 2 + \sqrt{5}$.

This completes the proof. $\square$

e) Proof of Lemma IV.3: : We first compute

$$x \cdot g_1 = \Big(2 - \frac{2\sqrt{5}}{5}\Big)u + \Big(\frac{1}{2} + \frac{\sqrt{5}}{10}\Big)v + \Big(1 + \frac{2\sqrt{5}}{5}\Big),$$
$$x \cdot g_5 = \Big(-1 + \frac{3\sqrt{5}}{5}\Big)u + \Big(\frac{1}{2} + \frac{\sqrt{5}}{10}\Big)v + \Big(1 + \frac{2\sqrt{5}}{5}\Big),$$
$$x \cdot g_9 = \Big(-1 + \frac{3\sqrt{5}}{5}\Big)u + \Big(-\frac{1}{2} + \frac{\sqrt{5}}{10}\Big)v + \Big(1 + \frac{2\sqrt{5}}{5}\Big).$$
$$x \cdot g_2 = \Big(-\frac{2\sqrt{5}}{5}\Big)u + \Big(-\frac{2\sqrt{5}}{5}\Big)v + \Big(1 + \frac{2\sqrt{5}}{5}\Big),$$
$$x \cdot g_6 = \Big(-\frac{2\sqrt{5}}{5}\Big)u + \Big(-\frac{2\sqrt{5}}{5} + \frac{\sqrt{5}-1}{2}\Big)v + \Big(1 + \frac{2\sqrt{5}}{5}\Big),$$
$$x \cdot g_7 = \Big(\frac{4\sqrt{5}}{5}\Big)u + \Big(\frac{5+3\sqrt{5}}{10}\Big)v + \frac{\sqrt{5}}{5}.$$
$$x \cdot g_4 = \Big(-1 + \frac{\sqrt{5}}{5}\Big)u - \Big(\frac{1}{2} + \frac{3\sqrt{5}}{10}\Big)v - \frac{\sqrt{5}}{5},$$
$$x \cdot g_{10} = \Big(1 - \frac{\sqrt{5}}{5}\Big)u - \frac{\sqrt{5}}{5}\,v + \frac{\sqrt{5}}{5}.$$
$$x \cdot g_3 = \Big(1 + \frac{\sqrt{5}}{5}\Big)u + \frac{\sqrt{5}}{5}\,v - \frac{\sqrt{5}}{5}.$$
$$x \cdot g_8 = \frac{\sqrt{5}}{5} - \Big(1 + \frac{\sqrt{5}}{5}\Big)(u+v).$$

On T', one checks that $x \cdot g_4 \le 0$ everywhere, while

$$x \cdot g_j \ge 0 \quad \text{for } j \in \{1, 2, 5, 6, 7, 9, 10\}.$$

The only inner products that may change sign on T' are $x \cdot g_3$ and $x \cdot g_8$. Therefore, we may drop absolute values for $x \cdot g_j$ with $j \in \{1, 2, 5, 6, 7, 9, 10\}$, keep a minus sign for $|x \cdot g_4| = -(x \cdot g_4)$, and retain absolute values for $|x \cdot g_3|$ and $|x \cdot g_8|$.

Proof of (1). Compute the differences:

$$(x\cdot g_1) - (x\cdot g_5) = (3-\sqrt{5})\,u \;\ge\; 0, \quad (x\cdot g_5) - (x\cdot g_9) = v \;\ge\; 0,$$
$$(x \cdot g_9) - (x \cdot g_6) = \Big(\frac{5-\sqrt{5}}{5}\Big)u + \frac{\sqrt{5}-1}{2}\,v \;\ge\; 0,$$
$$(x \cdot g_9) - (x \cdot g_7) = \Big(1 + \frac{\sqrt{5}}{5}\Big)\big(1 - u - v\big) \;\ge\; 0.$$

which gives $\beta_1 \ge \beta_5 \ge \beta_9 \ge \max\{\beta_6, \beta_7\}$.

Proof of (2). First,

$$(x \cdot g_6) - (x \cdot g_2) = \frac{\sqrt{5}-1}{2}\,v \;\ge\; 0 \quad \Longrightarrow \quad \beta_6 \ge \beta_2.$$

Next,

$$\beta_6 - \beta_4 = (x\cdot g_6) + (x\cdot g_4) = 1 + \frac{\sqrt{5}}{5} - \Big(1 + \frac{\sqrt{5}}{5}\Big)(u+v) \;\ge\; 0,$$

so $\beta_6 \ge \beta_4$. Also,

$$\beta_7 - \beta_4 = (x \cdot g_7) + (x \cdot g_4) = (\sqrt{5} - 1)\,u \;\ge\; 0.$$

so $\beta_7 \ge \beta_4$.

Proof of (3). We show $|x \cdot g_j| \le \beta_4$ and $|x \cdot g_j| \le \beta_2$ for $j \in \{3, 8, 10\}$. For $j = 10$ (no absolute needed):

$$\beta_4 - \beta_{10} = -(x \cdot g_4) - (x \cdot g_{10}) = \frac{1+\sqrt{5}}{2}\,v \;\ge\; 0,$$
$$\beta_2 - \beta_{10} = 1 + \frac{\sqrt{5}}{5} - \Big(1 + \frac{\sqrt{5}}{5}\Big)u - \frac{\sqrt{5}}{5}\,v \;\ge\; 0.$$

For $j = 3$, it suffices to check $\beta_4 \pm (x \cdot g_3) \ge 0$ and $\beta_2 \pm (x \cdot g_3) \ge 0$:

$$\beta_4 - (x \cdot g_3) = \frac{2\sqrt{5}}{5}(1-u) + \frac{5+\sqrt{5}}{10}v \;\ge\; 0,$$
$$\beta_4 + (x \cdot g_3) = 2u + \Big(\frac{1}{2} + \frac{\sqrt{5}}{2}\Big)v \;\ge\; 0,$$
$$\beta_2 - (x \cdot g_3) = 1 + \frac{3\sqrt{5}}{5} - \Big(1 + \frac{3\sqrt{5}}{5}\Big)u - \frac{3\sqrt{5}}{5}v \;\ge\; 0,$$
$$\beta_2 + (x \cdot g_3) = 1 + \frac{\sqrt{5}}{5} + \Big(1 - \frac{\sqrt{5}}{5}\Big)u - \frac{\sqrt{5}}{5}v \;\ge\; 0.$$

For $j = 8$, similarly check $\beta_4 \pm (x \cdot g_8) \ge 0$ and $\beta_2 \pm (x \cdot g_8) \ge 0$:

$$\beta_4 - (x \cdot g_8) = 2u + \Big(\frac{1}{2} + \frac{3\sqrt{5}}{10}\Big)v + \frac{\sqrt{5}}{5} \;\ge\; 0,$$
$$\beta_4 + (x \cdot g_8) = \frac{2\sqrt{5}}{5} - \frac{2\sqrt{5}}{5}u - \frac{5-\sqrt{5}}{10}\,v \;\ge\; 0,$$
$$\beta_2 - (x \cdot g_8) = 1 + \frac{3\sqrt{5}}{5} - \Big(1 - \frac{\sqrt{5}}{5}\Big)u - \Big(1 - \frac{3\sqrt{5}}{5}\Big)v \;\ge\; 0,$$
$$\beta_2 + (x \cdot g_8) = 1 + \frac{\sqrt{5}}{5} + \Big(1 + \frac{\sqrt{5}}{5}\Big)u + \frac{3\sqrt{5}}{5}v \;\ge\; 0.$$

Thus $\beta_4 \ge \max\{\beta_3, \beta_8, \beta_{10}\}$ and $\beta_2 \ge \max\{\beta_3, \beta_8, \beta_{10}\}$.

Proof of (4). Since $\beta_{10} = x \cdot g_{10} \ge 0$, it suffices to show $\beta_{10} \pm (x \cdot g_3) \ge 0$:

$$\beta_{10} - (x \cdot g_3) = \frac{2\sqrt{5}}{5}(1 - u - v) \;\ge\; 0,$$
$$\beta_{10} + (x \cdot g_3) = 2u \;\ge\; 0.$$

hence $|x \cdot g_3| \le x \cdot g_{10}$, i.e., $\beta_3 \le \beta_{10}$. $\square$

f) Proof of Corollary IV.4: : From (1)–(4), we know that ranks $1, 2, 3$ are attained uniquely by $\{1\}, \{5\}, \{9\}$, and the remaining ranks are determined by $\{\beta_2, \beta_3, \beta_4, \beta_6, \beta_7, \beta_8, \beta_{10}\}$.

From (3), we have

$$\max\{\beta_3, \beta_8, \beta_{10}\} \le \min\{\beta_2, \beta_4\},$$

so $\{3, 8, 10\}$ must occupy the bottom three ranks $k = 8, 9, 10$, and $\{2, 4, 6, 7\}$ must occupy ranks $k = 4, 5, 6, 7$.

Within $\{2, 4, 6, 7\}$, inequality (2) gives

$$\beta_6 \ge \beta_2, \qquad \beta_6 \ge \beta_4, \qquad \beta_7 \ge \beta_4.$$

Hence β_2 cannot be the largest among $\{\beta_2, \beta_4, \beta_6, \beta_7\}$ since $\beta_6 \ge \beta_2$, and β_4 cannot be the largest since $\beta_6 \ge \beta_4$; therefore the largest must be attained at index 6 or 7, i.e., $k = 4 : \{6, 7\}$. Moreover, since both β_6 and β_7 dominate β_4, index 4 cannot be the second-largest, so the second-largest must lie in $\{2, 6, 7\}$, i.e., $k = 5 : \{2, 6, 7\}$. Next, because $\beta_6 \ge \beta_2$

and $\beta_6 \geq \beta_4$, index 6 cannot be the third- or fourth-largest within this subset; thus the third-largest must lie in $\{2,4,7\}$, i.e., $k = 6 : \{2,4,7\}$. Finally, since $\beta_7 \geq \beta_4$, index 7 cannot be the smallest among the four, so the smallest must lie in $\{2,4\}$, i.e., $k = 7 : \{2,4\}$.

Within the bottom group $\{3,8,10\}$, inequality (4) gives $\beta_3 \leq \beta_{10}$, so index 3 cannot be the largest among $\{3,8,10\}$ and index 10 cannot be the smallest among $\{3,8,10\}$. Hence

$$k = 8 : \{8,10\}, \qquad k = 9 : \{3,8,10\}, \qquad k = 10 : \{3,8\}.$$

□

g) Proof of Lemma IV.5: : On T', we have $x \cdot g_4 < 0$ and $x \cdot g_j > 0$ for $j \in \{2,5,6,7,9,10\}$. Hence $|x \cdot g_4| = -(x \cdot g_4)$ and $|x \cdot g_j| = x \cdot g_j$ for $j \in \{2,5,6,7,9,10\}$.

Write $N := x \cdot g_1 = a_1 u + b_1 v + c_1$ and $D_j := |x \cdot g_j| = a_j u + b_j v + c_j$. We have

$$\partial_u f_j = \frac{a_1 D_j - a_j N}{D_j^2}, \qquad \partial_v f_j = \frac{b_1 D_j - b_j N}{D_j^2}.$$

Substituting the explicit formulas of $x \cdot g_1$ and $x \cdot g_j$ yields the following:

$$\partial_u f_2 = \frac{(10+4\sqrt{5}) + (5-3\sqrt{5})v}{5\,D_2^2} > 0,$$
$$\partial_v f_2 = \frac{(15+7\sqrt{5}) + (-10+6\sqrt{5})u}{10\,D_2^2} > 0;$$
$$\partial_u f_5 = \frac{(5+\sqrt{5}) + (5-\sqrt{5})v}{5\,D_5^2} > 0,$$
$$\partial_v f_5 = \frac{(\sqrt{5}-5)u}{5\,D_5^2} < 0;$$
$$\partial_u f_6 = \frac{(10+4\sqrt{5}) + (-5+3\sqrt{5})v}{5\,D_6^2} > 0,$$
$$\partial_v f_6 = \frac{(5+2\sqrt{5}) + (5-3\sqrt{5})u}{5\,D_6^2} > 0;$$
$$\partial_u f_7 = -\frac{2(5+\sqrt{5})}{5\,D_7^2} < 0,$$
$$\partial_v f_7 = -\frac{5+2\sqrt{5}}{5\,D_7^2} < 0;$$
$$\partial_u f_9 = \frac{(5+\sqrt{5}) + (-5+\sqrt{5})v}{5\,D_9^2} > 0,$$
$$\partial_v f_9 = \frac{(5+2\sqrt{5}) + (5-\sqrt{5})u}{5\,D_9^2} > 0;$$
$$\partial_u f_{10} = -\frac{(5-\sqrt{5}) + 2\sqrt{5}\,v}{5\,D_{10}^2} < 0,$$
$$\partial_v f_{10} = \frac{(5+3\sqrt{5}) + 4\sqrt{5}\,u}{10\,D_{10}^2} > 0.$$

For $j = 4$ we use $D_4 = |x \cdot g_4| = -(x \cdot g_4) = \big(1 - \frac{\sqrt{5}}{5}\big)u + \big(\frac{1}{2} + \frac{3\sqrt{5}}{10}\big)v + \frac{\sqrt{5}}{5}$, and obtain

$$\partial_v f_4 = -\frac{2\sqrt{5}\,u + (5+2\sqrt{5})}{5\,D_4^2} < 0.$$

In each case, ∇f_j cannot vanish on T' (indeed, at least one partial derivative has a fixed nonzero sign), so f_j has no stationary point in T'. □

h) Proof of Corollary IV.6: : **(1) Case** $m = 1$**.** From the proof of Lemma IV.5, we have

$$\partial_u f_5(u,v) > 0, \qquad \partial_v f_5(u,v) < 0.$$

Hence f_5 is increasing in u and decreasing in v on T, so the maximizer is attained at $(u,v) = (1,0)$, i.e., $x = x_A = g_1$. Evaluating gives

$$f_5(g_1) = \frac{g_1 \cdot g_1}{g_1 \cdot g_5} = \frac{3}{\sqrt{5}}.$$

(2) Case $m = 2$. We have

$$\partial_u f_9(u,v) > 0, \qquad \partial_v f_9(u,v) > 0.$$

Thus f_9 is increasing in both u and v, and by monotonicity the maximizer lies on the edge $u + v = 1$. Along this edge,

$$f_9(u, 1-u) = \frac{(3-\sqrt{5})u + (3+\sqrt{5})}{(\sqrt{5}-1)u + (1+\sqrt{5})}$$

is strictly decreasing in u, so the unique maximizer is $(u,v) = (0,1)$, i.e., $x = x_B = \frac{g_1+g_5}{2}$. Evaluating gives

$$f_9(x_B) = \frac{x_B \cdot g_1}{x_B \cdot g_9} = \varphi.$$

This completes the proof. □

i) Proof of Theorem IV.7: : We treat the three cases $m \in \{3,4\}$, $m \in \{5,6\}$, and $m = 7$ separately.

Case I: $m = 3,4$ **(the denominator is in** $\{g_2, g_6, g_7\}$**).** By Corollary IV.4, for $m = 3,4$, the $(m+1)$-st largest projection must be attained at index 2, 6, or 7.

Define

$$L_{i,j} := \{x \in T' : \ |x \cdot g_i| = |x \cdot g_j|\}.$$

Since $x \cdot g_2,\ x \cdot g_6,\ x \cdot g_7 \geq 0$ on T', we drop the absolute values and write $L_{i,j} = \{x \in T' : x \cdot g_i = x \cdot g_j\}$ for $i,j \in \{2,6,7\}$. The switching boundaries among these three denominators are given by (we do not need to consider the switching boundary $L_{2,6}$ since by Lemma IV.3, $x \cdot g_6 \geq x \cdot g_2$)

$$L_{2,7} : \ x \cdot g_2 = x \cdot g_7 \iff 12\sqrt{5}\,u + (7\sqrt{5}+5)\,v = 10 + 2\sqrt{5},$$
$$L_{6,7} : \ x \cdot g_6 = x \cdot g_7 \iff 6\sqrt{5}\,u + (5+\sqrt{5})\,v = 5 + \sqrt{5}.$$

By Lemma IV.5, none of f_2, f_6, or f_7 has a stationary point in T'. The switching boundaries partition T' into three subregions, as illustrated in Fig. 4. Within each subregion, the ordering among the denominators is fixed, and the relevant denominator is g_j for some $j \in \{2,6,7\}$. Consequently, the m-height optimization reduces to maximizing a single ratio $f_j(u,v)$ over that subregion. We thus analyze the three subregions separately, maximizing $f_2(u,v)$, $f_6(u,v)$, or $f_7(u,v)$ on the corresponding region.

(A) Maximizers in $\mathbf{T_1}$. From Lemma IV.5, both $\partial_u f_2$ and $\partial_v f_2$ are strictly positive. Hence f_2 is strictly increasing in both u and v, and its maximum over $\mathbf{T_1}$ must be attained on the edge $L_{2,7}$. Evaluating f_2 along this edge shows that

$$f_2\Big(u, \frac{10+2\sqrt{5}-12\sqrt{5}\,u}{7\sqrt{5}+5}\Big) = \frac{(13-5\sqrt{5})u + (13+6\sqrt{5})}{(4\sqrt{5}-6)u + (5+4\sqrt{5})},$$

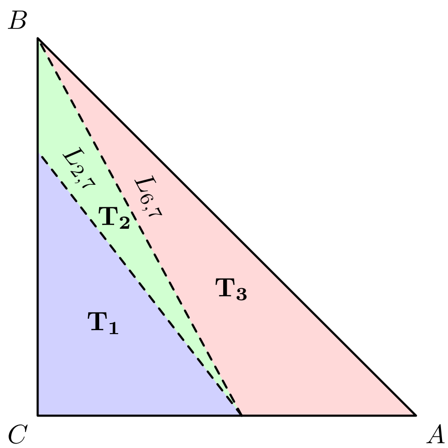

Fig. 4. Triangle T' in (u,v)-coordinates and switching lines $L_{2,7}$ and $L_{6,7}$, partitioning T' into three subregions.

with derivative

$$\frac{d}{du}(\cdot) = \frac{11(\sqrt{5}-7)}{\big((4\sqrt{5}-6)u+(5+4\sqrt{5})\big)^2} < 0.$$

Hence the maximum is attained at $u=0$, i.e.,

$$(u,v) = \Big(0,\ \frac{1+3\sqrt{5}}{11}\Big).$$

For f_6, both $\partial_u f_6$ and $\partial_v f_6$ are strictly positive. Evaluating f_6 along $L_{2,7}$ shows that

$$f_6\big(u,v(u)\big) = \frac{(6\sqrt{5}-10)u+(25+11\sqrt{5})}{(4\sqrt{5}-20)u+(15+9\sqrt{5})}.$$

Differentiating yields

$$\frac{d}{du}f_6\big(u,v(u)\big) = \frac{40(10+3\sqrt{5})}{\big((4\sqrt{5}-20)u+(15+9\sqrt{5})\big)^2} > 0.$$

Hence f_6 is strictly increasing along $L_{2,7}$ inside T', so its maximum on this segment is attained at the endpoint with largest u, namely where $v=0$:

$$(u,v) = \Big(\frac{10+2\sqrt{5}}{12\sqrt{5}}, 0\Big) = \Big(\frac{1+\sqrt{5}}{6}, 0\Big).$$

For f_7, both partial derivatives satisfy $\partial_u f_7 < 0$ and $\partial_v f_7 < 0$ on T', so f_7 is strictly decreasing in both variables. Hence its maximum over $\mathbf{T_1}$ is attained at the vertex $(u,v)=(0,0)$.

(B) Maximizers in $\mathbf{T_2}$.

For f_2, both $\partial_u f_2$ and $\partial_v f_2$ are strictly positive. Evaluating f_2 along $L_{6,7}$ shows that

$$f_2\Big(u,\ 1+\tfrac{3}{2}(1-\sqrt{5})u\Big) = \frac{(4-2\sqrt{5})u+(3+\sqrt{5})}{2\big((3-\sqrt{5})u+1\big)}.$$

Differentiating gives

$$\frac{d}{du}(\cdot) = \frac{-4\sqrt{5}}{\big(2((3-\sqrt{5})u+1)\big)^2} < 0,$$

hence the maximum is attained at $u=0$, i.e., at $(u,v)=(0,1)$.

For f_6, both $\partial_u f_6$ and $\partial_v f_6$ are strictly positive. Evaluating f_6 along $L_{6,7}$ shows that

$$f_6\Big(u,\ 1+\tfrac{3}{2}(1-\sqrt{5})u\Big) = \frac{(4-2\sqrt{5})u+(3+\sqrt{5})}{(\sqrt{5}-3)u+(1+\sqrt{5})}.$$

with derivative

$$\frac{d}{du}(\cdot) = \frac{10(1+\sqrt{5})}{\big((\sqrt{5}-3)u+(1+\sqrt{5})\big)^2} > 0.$$

Thus the maximum on $L_{6,7}$ is attained at the largest feasible u, which occurs at $v=0$, i.e.,

$$(u,v) = \Big(\frac{1+\sqrt{5}}{6},\ 0\Big).$$

For f_7, both $\partial_u f_7$ and $\partial_v f_7$ are strictly negative. Evaluating f_7 along $L_{2,7}$ shows that

$$f_7\Big(u, \frac{10+2\sqrt{5}-12\sqrt{5}\,u}{7\sqrt{5}+5}\Big) = \frac{(6\sqrt{5}-10)u+(25+11\sqrt{5})}{(10-2\sqrt{5})u+(15+5\sqrt{5})}.$$

Its derivative is

$$\frac{d}{du}(\cdot) = -\frac{20(7+\sqrt{5})}{\big((10-2\sqrt{5})u+(15+5\sqrt{5})\big)^2} < 0.$$

Hence the maximum on this segment is attained at the endpoint with smallest u, i.e.,

$$(u,v) = \Big(0,\ \frac{1+3\sqrt{5}}{11}\Big).$$

(C) Maximizers in $\mathbf{T_3}$.

For f_2, both $\partial_u f_2$ and $\partial_v f_2$ are strictly positive. Evaluating f_2 along edge AB shows that

$$f_2(u,1-u) = \big(\frac{3}{2}-\frac{\sqrt{5}}{2}\big)u + \big(\frac{3}{2}+\frac{\sqrt{5}}{2}\big).$$

Differentiating yields

$$\frac{d}{du}(\cdot) = \frac{3}{2}-\frac{\sqrt{5}}{2} > 0,$$

so f_2 is strictly increasing along AB. Hence the maximum on AB is attained at $u=1$, i.e., at the vertex $A=(1,0)$.

For f_6, both $\partial_u f_6$ and $\partial_v f_6$ are strictly positive. Evaluating f_6 along edge AB shows that

$$f_6(u,1-u) = \frac{(3-\sqrt{5})u+(3+\sqrt{5})}{(1-\sqrt{5})u+(1+\sqrt{5})}.$$

Its derivative is

$$\frac{d}{du}f_6(u,1-u) = \frac{4\sqrt{5}}{\big((1-\sqrt{5})u+(1+\sqrt{5})\big)^2} > 0,$$

so $f_6(u,1-u)$ is strictly increasing on u. Therefore the maximum on edge AB is attained at $u=1$ (i.e., $(u,v)=(1,0)$, the vertex A).

For f_7, both $\partial_u f_7$ and $\partial_v f_7$ are strictly negative. Evaluating f_7 along edge $L_{6,7}$ shows that

$$f_7\Big(u,\, 1+\tfrac{3}{2}(1-\sqrt{5})u\Big) = \frac{(2-\sqrt{5})u + \frac{3+\sqrt{5}}{2}}{\frac{\sqrt{5}-3}{2}\,u + \frac{1+\sqrt{5}}{2}}.$$

Its derivative is

$$\frac{d}{du} f_7\Big(u,\, 1+\tfrac{3}{2}(1-\sqrt{5})u\Big) = \frac{\frac{\sqrt{5}-1}{2}}{\big(\frac{\sqrt{5}-3}{2}\,u + \frac{1+\sqrt{5}}{2}\big)^2} > 0.$$

so f_7 is strictly increasing along $L_{6,7}$.

Therefore, the maximum on this segment is attained at the endpoint with largest u, namely

$$(u,v) = \Big(\frac{1+\sqrt{5}}{6},\, 0\Big).$$

Case II: $m = 5, 6$ **(the denominator is in** $\{g_2, g_4, g_7\}$**).** For $m = 5$ or $m = 6$, the $(m+1)$-st largest magnitude must be attained at index 2, 4, or 7. On T, $x \cdot g_2 \geq 0$ and $x \cdot g_7 \geq 0$, while $x \cdot g_4 < 0$, so $|x \cdot g_2| = x \cdot g_2$, $|x \cdot g_7| = x \cdot g_7$, and $|x \cdot g_4| = -(x \cdot g_4)$. The switching interfaces are

$$L_{2,4} : 2(5+\sqrt{5})\,u + (5+7\sqrt{5})\,v = 10 + 2\sqrt{5},$$
$$L_{2,7} : 12\sqrt{5}\,u + (7\sqrt{5}+5)\,v = 10 + 2\sqrt{5}.$$

Similar to Case I, the corresponding subregion partition is illustrated in Fig. 5.

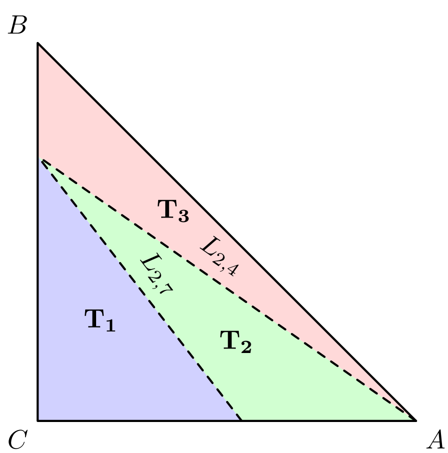

Fig. 5. Triangle T' in (u, v)-coordinates and switching lines $L_{2,4}$ and $L_{2,7}$, partitioning T' into three subregions.

(A) Maximizers in T_1**.** Maximizing f_2 and f_7 in T_1 is already done in case I, thus we skip them.

For f_4, From Lemma IV.5, $\partial_v f_4 < 0$. Thus the maximizer is on $v = 0$, which is edge CA

$$f_4(u, 0) = \frac{(10-2\sqrt{5})u + (5+2\sqrt{5})}{(5-\sqrt{5})u + \sqrt{5}}.$$

Its derivative is

$$\frac{d}{du} f_4(u, 0) = \frac{5(\sqrt{5}-5)}{\big((5-\sqrt{5})u + \sqrt{5}\big)^2} < 0.$$

Therefore $f_4(u, 0)$ is strictly decreasing on $u \in [0, 1]$, and the maximum on the edge $v = 0$ is attained at $u = 0$.

(B) Maximizers in T_2**.**

For f_2, both $\partial_u f_2$ and $\partial_v f_2$ are strictly positive. Evaluating f_2 along edge $L_{2,4}$ shows that

$$f_2\Big(u,\, \frac{10+2\sqrt{5}-2(5+\sqrt{5})\,u}{5+7\sqrt{5}}\Big) = \frac{(-10+10\sqrt{5})\,u + (25+11\sqrt{5})}{(-10+2\sqrt{5})\,u + (15+5\sqrt{5})}.$$

Its derivative is

$$\frac{d}{du}\, f_2(u, v(u)) = \frac{240+160\sqrt{5}}{\big((-10+2\sqrt{5})\,u + (15+5\sqrt{5})\big)^2} > 0.$$

Therefore, the maximum is attained at the endpoint with the largest u, i.e., at the vertex A.

For f_4, we have $\partial_v f_4 < 0$. Hence any maximizer must lie on $CA \cap \mathbf{T}_2$ or on $L_{2,7}$. From the previous calculation, along CA we also have $\partial_u f_4 < 0$. Therefore, if the maximizer lies on $CA \cap \mathbf{T}_2$, it must occur at the endpoint where $CA \cap \mathbf{T}_2$ meets $L_{2,7}$. Consequently, it suffices to restrict attention to $L_{2,7}$. Evaluating f_4 along edge $L_{2,7}$ shows that

$$f_4\Big(u,\, \frac{10+2\sqrt{5}-12\sqrt{5}\,u}{7\sqrt{5}+5}\Big) = \frac{(13-5\sqrt{5})u + (13+6\sqrt{5})}{(5-7\sqrt{5})u + (5+4\sqrt{5})}.$$

Its derivative is

$$\frac{d}{du}\, f_4 = \frac{110+88\sqrt{5}}{\big((5-7\sqrt{5})u + (5+4\sqrt{5})\big)^2} > 0.$$

Therefore the maximum of f_4 over $L_{2,7}$ is attained at the endpoint where $v = 0$, i.e.,

$$(u,v) = \Big(\frac{1+\sqrt{5}}{6},\, 0\Big).$$

For f_7, both $\partial_u f_7$ and $\partial_v f_7$ are strictly negative. Thus the maximum is on edge $L_{2,7}$, which has already been calculated in Case I.

(C) Maximizers in T_3**.**

The maximuzer for f_2 in $\mathbf{T_3}$ is on edge AB, maximizing f_2 along AB is already done in previous case.

For f_4, since $\partial_v f_4 < 0$, the mazimizer is on edge $L_{2,4}$. Since on that edge, $f_2 = f_4$, it is equivalent as evaluating f_2 along $L_{2,4}$, which is already done in previous case.

For f_7, since $\partial_v f_7 < 0$, the mazimizer is on edge $L_{2,4}$.

$$f_7\Big(u,\, \frac{10+2\sqrt{5}-2(5+\sqrt{5})\,u}{5+7\sqrt{5}}\Big) = \frac{(10\sqrt{5}-10)\,u + (25+11\sqrt{5})}{20u + (15+5\sqrt{5})}.$$

Differentiating gives

$$\frac{d}{du}\, f_7 = -\frac{40(10+3\sqrt{5})}{\big(20u + (15+5\sqrt{5})\big)^2} < 0.$$

Therefore, the maximum of f_7 over $L_{2,4}$ is attained at $u = 0$, i.e.,

$$(u,v) = \Big(0,\, \frac{1+3\sqrt{5}}{11}\Big).$$

Case III: $m = 7$ **(the denominator is in** $\{g_8, g_{10}\}$**).**

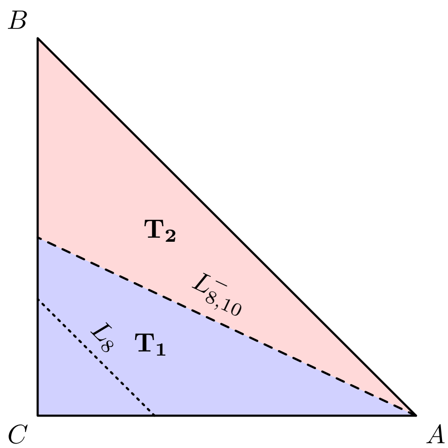

Fig. 6. Triangle T' in (u, v)-coordinates, the switching line $L^-_{8,10}$, and the zero line $L_8 :\ x \cdot g_8 = 0$.

The switching condition $|x \cdot g_8| = |x \cdot g_{10}|$ splits into

$$L^+_{8,10} :\ x \cdot g_8 = x \cdot g_{10}, \qquad L^-_{8,10} :\ x \cdot g_8 = -x \cdot g_{10},$$

which reduce to

$$L^+_{8,10} :\ v = -2u, \qquad L^-_{8,10} :\ v = (2\sqrt{5} - 4)(1 - u).$$

The switching line $L^+_{8,10} :\ v = -2u$ intersects T' only at the vertex $C = (0, 0)$. Hence, it suffices to consider the two subregions partitioned by

$$L^-_{8,10} :\ v = (2\sqrt{5} - 4)(1 - u),$$

as illustrated in Fig. 6. We further denote by L_8 the zero line of $x \cdot g_8$, i.e., $x \cdot g_8 = 0$.

In subregion $\mathbf{T_1}$, the ordering satisfies

$$|x \cdot g_{10}| \geq |x \cdot g_8|.$$

Moreover, since $x \cdot g_{10} \geq 0$ throughout $\mathbf{T}'$, the m-height function for $m = 7$ reduces to

$$f_{10}(u, v) = \frac{x \cdot g_1}{x \cdot g_{10}}.$$

By Lemma IV.5, we have

$$\partial_u f_{10} < 0, \qquad \partial_v f_{10} > 0 \quad \text{on } T'.$$

Therefore, the maximum of f_{10} over $\mathbf{T_1}$ is attained at the vertex with minimal u and maximal v, namely at the intersection of $L^-_{8,10}$ and the edge BC, which is

$$(u, v) = (0,\ 2\sqrt{5} - 4).$$

In subregion $\mathbf{T_2}$, the ordering is reversed and the m-height function is

$$f_8(u, v) = \frac{x \cdot g_1}{-x \cdot g_8}.$$

Then direct differentiation yields

$$\partial_u f_8 = \frac{\left(1 - \frac{\sqrt{5}}{5}\right)v - (1 + \sqrt{5})}{D(u, v)^2} < 0,$$

$$\partial_v f_8 = \frac{\left(\frac{\sqrt{5}}{5} - 1\right)u - \left(\frac{3}{2} + \frac{7\sqrt{5}}{10}\right)}{D(u, v)^2} < 0.$$

Hence f_8 is strictly decreasing in both variables, and its maximum over $\mathbf{T_2}$ must be attained on the boundary $L^-_{8,10}$.

Evaluating f_8 along $L^-_{8,10} :\ v = (2\sqrt{5} - 4)(1 - u)$ gives

$$f_8\Big(u,\ (2\sqrt{5} - 4)(1 - u)\Big) = \frac{(3 - \sqrt{5})u + \sqrt{5}}{(3 - \sqrt{5})u + (\sqrt{5} - 2)}.$$

Differentiating,

$$\frac{d}{du} f_8\Big(u,\ (2\sqrt{5}-4)(1-u)\Big) = \frac{2(\sqrt{5} - 3)}{\big((3 - \sqrt{5})u + (\sqrt{5} - 2)\big)^2} < 0,$$

so this restriction is strictly decreasing in u. Consequently, the maximum of f_8 on $\mathbf{T_2}$ is attained at the endpoint $u = 0$, i.e.,

$$(u, v) = (0,\ 2\sqrt{5} - 4).$$

□