# Improved Hybrid RM-Polar Codes and Decoding on Stable Permuted Factor Graphs

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***Abstract*—A new family of modified polar codes and a new permutation selection scheme for belief propagation list (BPL) decoding are presented. We first propose a new code construction methodology to interpolate between Reed-Muller (RM) codes and polar codes. By taking advantage of an existing partial order on bit-channels whose corresponding indices share the same Hamming weight, we analyze the complexity of the new construction method. It is shown that we need to compute the reliability of roughly a fraction $1/\log^{3/2} N$ of all the bit-channels contained in the subset. Then, we explore a special family of factor-graph layer permutations called stable permutations (SPs) that preserve a specified information set when the corresponding bit permutations are applied to message bit indices. Simulation results show that the error-rate performance of the new family of codes is better than that of 5G polar codes under successive cancellation list (SCL) decoding. In addition, the SP selection scheme stands out as a preferred one for BPL decoding in terms of error-rate performance, while the average number of iterations per belief propagation (BP) decoder on the permuted factor graph is close to that of the original BP decoder when an early stopping condition is applied.**

## I. Introduction

Polar codes [1] are the first family of error-correcting codes that was proved to achieve the capacity of binary memoryless symmetric (BMS) channels with efficient encoding and decoding algorithms. Successive cancellation (SC) decoding was proposed in [1] as the original decoding algorithm for polar codes. To improve the performance, SC list (SCL) decoding, which maintains a list of most likely decoding paths until the final decision, was proposed in [25]. Unlike SC-based decoding, the parallelism of iterative belief propagation (BP) decoding can offer a higher throughput and a lower latency. Soft-cancellation (SCAN) [8] decoding is another iterative decoding algorithm that can reduce the number of iterations with a sequential processing schedule. However, the error-correction performance of the BP and SCAN decoders is not competitive with that of SCL decoding.

One of the methods to improve the performance of BP decoding is to utilize permuted factor graphs [13]. Furthermore, a BP list (BPL) decoding was proposed in [6] where $L$ parallel independent BP decoders are executed on different permuted factor graphs. However, how to select the optimal permutation list is still an open problem. Cyclic shift permutations were first considered in [13], and then, in [6]. In [4], the $L$ best permutations among those that permute only a few of the rightmost layers were selected. It was shown in [14] that better performance can be obtained if the Hamming distance between any two permutations among the list is larger. A genetic algorithm-based technique was used in [9] to search for an optimal solution in a relatively large sample space. In [17], the authors selected $L$ permutations from those with the smallest error probability bound. Reinforcement learning algorithms were used in [5] to select good permutations during decoding.

In the literature on polar code construction, several algorithms have been proposed in [1], [12], [24], and [26] to optimize the performance under SC decoding. Reed-Muller (RM) codes have been proved in [15] to achieve capacity on erasure channels under maximum a posteriori (MAP) decoding. Relying on the connection between their generator matrices, codes that interpolate between RM and polar codes have been proposed in [7], [11], [16], [18]. Simulation results in [18] demonstrate that such an interpolated code can offer better performance under BP and SCL decoding. In [6] and [17], the RM-polar codes proposed in [16] were shown to provide superior performance under BPL decoding. Despite this progress, code constructions optimized for decoding methods beyond SC require further study.

In this paper, we first propose a new code construction method that interpolates between RM and polar codes. We demonstrate the method by constructing a new family of half-rate codes with explicitly identified information sets. We also analyze the complexity of the new polar code construction problem, which reduces to the problem of ordering the bit-channels with a fixed Hamming weight. We then define a special family of factor-graph layer permutations called stable permutations (SPs) that preserve a specified information set when the corresponding bit permutations are applied to message bit indices. The cardinalities of the SP sets of the new half-rate hybrid RM-polar codes are discussed. We show simulation results confirming the superior performance of the new codes compared to 5G polar codes under SCL decoding. We then show that SPs provide a good set of permutations for BPL decoding. In particular, a permutation list selected from the set of SPs improves the block error rate (BLER) performance and reduces the time complexity for the new hybrid RM-polar codes.

## II. Preliminaries

### A. Polar and RM Codes

For $N = 2^n$, let $G_N = G_2^{\otimes n}$, where $G_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, and $\otimes$ stands for the Kronecker product. An $(N, K)$-polar code of

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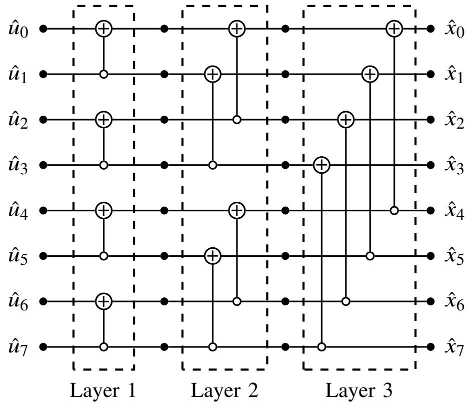


Fig. 1. The factor graph of polar code with $N = 8$.

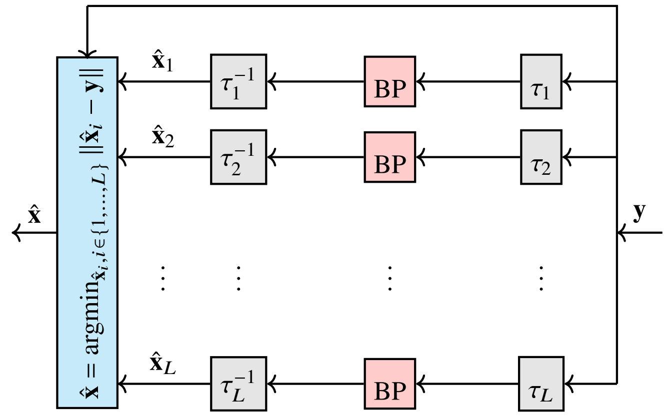


Fig. 2. Block diagram of BPL decoding with a list size $L$.

length $N$ and dimension $K$ and a RM code of length $2^n$, dimension $K = \sum_{i=0}^{k} \binom{n}{i}$, and minimum distance $2^{n-k}$, denoted $RM(k, n)$, can both be encoded as

$$\mathbf{x} = \mathbf{u}G_N,$$

where $\mathbf{u} = [u_0, u_1, \ldots, u_{N-1}]$ is the word of message bits and $\mathbf{x} = [x_0, x_1, \ldots, x_{N-1}]$ is the codeword. We denote the indices carrying information in $\mathbf{u}$ as the information set $\mathcal{I}$. Polar codes select the $K$ indices corresponding to the highest bit-channel reliability. RM codes use the indices of the $K$ rows of $G_N$ with the largest Hamming weights. In particular, for $RM(k, n)$, the rows with Hamming weight $2^{n-k}$ or more are selected.

### B. BP Decoding

We consider BP decoding of polar codes on the $G_N$-based (Forney-style) factor graph [2]. The log-likelihood ratio (LLR) messages are updated over the factor graph in a round-trip fashion [27]. After each iteration, the decoder can make a decision on the LLR values of the leftmost and rightmost nodes to obtain the estimates $\hat{\mathbf{u}}$ and $\hat{\mathbf{x}}$, respectively. Furthermore, the BP decoder can stop after a predefined maximum number of iterations or if the $G_N$-based early stopping condition [28], i.e., $\hat{\mathbf{x}} = \hat{\mathbf{u}}G_N$, is satisfied.

### C. BPL Decoding

Factor graph permutations provide multiple representations of a single code. It was observed in [13] that there exists $n!$ different ways to represent a polar code by permuting the factor-graph layers. The three layers in a factor graph for $N = 8$ are illustrated in Fig. 1.

In [6], the authors proposed a BPL decoder by performing $L$ parallel BP decoders over different permuted factor graphs, with $L$ as the list size. Therefore, a list of $L$ candidate codewords is available. Finally, the codeword among those candidates satisfying the $G_N$-based early stopping condition that is closest, in terms of Euclidean distance, to the channel output $\mathbf{y}$ is chosen to be the final output.

According to [4, Theorem 1], there is a one-to-one mapping between the layer permutations on the factor graph and the bit-index permutations on both the codeword vector $\mathbf{x}$ and the message vector $\mathbf{u}$. This allows the use of the same factor graph for all of the parallel BP decoders by simply permuting the incoming channel LLRs and message LLRs, accordingly, thereby significantly reducing hardware cost and simplifying the practical decoder implementation. Fig. 2 shows an abstract view of the BPL decoder used in our work, where $\tau_i$ $(i = 1, \ldots, L)$ represent bit-index permutations corresponding to the factor-graph layer permutations. Recently, automorphism ensemble decoding of RM codes was proposed [10], in which the constituent BP decoders in the BPL decoding architecture are replaced by other types of decoders (e.g., SC, SCL).

## III. Hybrid RM-Polar Code Construction

In this section, we propose a new hybrid RM-polar code construction. There are two cases to consider when constructing half-rate codes, depending on the value of $n = \log_2 N$. These are presented in Sections III-B and III-C, respectively. In Section III-D, we discuss the complexity of the new construction method.

### A. Notations and Definitions

Denote $\{0, 1, \ldots, N-1\}$ by $[N]$. Given $i \in [N]$, let $b_n b_{n-1} \ldots b_1$ be its binary expansion over $n$ bits, where the first bit, $b_n$, is the most significant bit. The Hamming weight of the index $i \in [N]$ is denoted as $w_H(i)$, which is the number of ones in the binary expansion of $i$. Let $\mathcal{A}_w = \{i \in [N] | w_H(i) = w\}$, for $w = 0, 1, \ldots, n$. The complement of index $i \in [N]$ is denoted as $\bar{i}$; their binary expansions are the bitwise complement of each other. Accordingly, we use $\bar{\mathcal{P}}$ to denote the complement index set of $\mathcal{P}$, i.e., for any $i \in \mathcal{P}$, $\bar{i} \in \bar{\mathcal{P}}$ and vice versa.

### B. Half-rate Codes with $n = 2k + 1$

In this case, the half-rate $RM(k, n)$ has information set

$$\mathcal{I} = \mathcal{A}_{2k+1} \cup \mathcal{A}_{2k} \cup \cdots \cup \mathcal{A}_{k+1}.$$

The new codes are constructed by selectively exchanging indices in $\mathcal{A}_k$ with their complements in the information set

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of $RM(k,n)$. We call each exchange a *complement transposition*. The new half-rate code is denoted as $\mathscr{C}(N,\mathcal{D})$, where $N$ is the code blocklength and $\mathcal{D} \subset \mathcal{A}_k$ represents the transposition index set. Then, the information set of $\mathscr{C}(N,\mathcal{D})$ is defined as

$$\mathcal{I} = \mathcal{A}_{2k+1} \cup \mathcal{A}_{2k} \cup \cdots \cup (\mathcal{A}_{k+1} \setminus \bar{\mathcal{D}}) \cup \mathcal{D}. \qquad (1)$$

In this work, we choose the transposition set $\mathcal{D}$ using the bit-channel ordering based on the $\beta$-expansion method proposed in [12]. Specifically, the polarization weight (PW) of an index $i \in [N]$ is defined as

$$\text{PW}(i) = \sum_{j=1}^{n} b_j \beta^j,$$

where $b_n b_{n-1} \ldots b_1$ is the binary expansion of $i$ and $\beta$ is chosen to be $2^{1/4} \approx 1.1892$ according to [12]. Then, given the chosen cardinality of $\mathcal{D}$, the information set is generated by performing complement transposition on the $|\mathcal{D}|$ indices with the highest PWs in $\mathcal{A}_k$ and their complements in $\mathcal{A}_{k+1}$.

### C. Half-rate Codes with $n = 2k$

In this case, there are no half-rate RM codes. Instead, the information set of the new constructed code is composed of the information set of $RM(k-1,n)$ and an appended index set, whose elements are chosen from $\mathcal{A}_k$. We still denote the new half-rate code as $\mathscr{C}(N,\mathcal{D})$, where $N$ is the code blocklength and $\mathcal{D} \subset \mathcal{A}_k$ is the appended index set. The information set of $\mathscr{C}(N,\mathcal{D})$ can be written as

$$\mathcal{I} = \mathcal{A}_{2k} \cup \mathcal{A}_{2k-1} \cup \cdots \cup \mathcal{A}_{k+1} \cup \mathcal{D}, \qquad (2)$$

where $\mathcal{D} \subset \mathcal{A}_k$ and $|\mathcal{D}| = |\mathcal{A}_k|/2$. The set $\mathcal{D}$ is chosen to include the $|\mathcal{A}_k|/2$ indices from $\mathcal{A}_k$ with the highest PWs.

### D. Sublinear Construction Complexity

To construct the new family of hybrid RM-polar codes, we essentially need to find the best $|\mathcal{D}|$ indices among those in $\mathcal{A}_k$ according to some reliability measure. We used PW in our examples since it can be obtained by easy and fast computation. However, some other reliability measures, such as Bhattacharyya parameter or MAP error probability, are often considered in conventional polar code construction problem.

Here we consider the hybrid code construction problem using the Bhattacharyya parameter, stated as follows.

- Given a threshold $\gamma \in (0,1)$ and $0 \leqslant w \leqslant n$, output all the indices in $\mathcal{A}_w$ corresponding to the bit-channels with Bhattacharyya parameter smaller than $\gamma$.

The authors in [19] show that, by taking advantage of the "addition" partial order in [20] and the "left-swap" partial order in [3], [22], the reliability of only a fraction $\sim 1/\log^{3/2} N$ of the bit-channels needs to be computed to construct a polar code. However, the result in [19] is not immediately applicable to the construction problem on $\mathcal{A}_w$.

Instead, we relate the construction problem to some classical combinatorial problems by only using the "left-swap" partial order in [3], [22]. The idea is to relate the partial ordering of indices in $\mathcal{A}_w$ to the inclusion ordering of Ferrers diagrams contained in an $(n-w) \times w$ rectangle. The rank structure of the Ferrers diagrams, given by coefficients of a corresponding Gaussian polynomial, is used to prove a theorem similar to [19, Theorem 1], based on an upper bound on the maximum cardinality of an antichain in $\mathcal{A}_w$.

**Theorem 1.** *(Complexity of Hybrid Code Construction)*

- *Upper bound:* it suffices to compute the Bhattacharyya parameters of at most
$$C(n) \log \left( \frac{2\binom{n}{w}}{C(n)} \right)$$
bit-channels, for any $\gamma \in (0,1)$.
- *Lower bound:* it is necessary to compute the Bhattacharyya parameters of at least $C(n)$ bit-channels, for some $\gamma \in (0,1)$.

Here $C(n)$ is the integer sequence A277218 found in [23]. An asymptotic formula for $C(n)$ is also given as

$$C(n) \sim \frac{\sqrt{3}}{\pi} \cdot \frac{2^{n+2}}{n^2}.$$

To solve the construction problem for $\mathcal{A}_w$, we need to compute the Bhattacharyya parameters of roughly $C(n) \sim N/\log^2 N$ bit-channels. Since the total number of bit-channels (indices) in $\mathcal{A}_w$ is bounded by $\binom{n}{n/2} \sim N/\log^{1/2} N$, we conclude that the reliability of the same fraction $\sim 1/\log^{3/2} N$ of bit-channels as in [19] needs to be computed to solve the construction problem for $\mathcal{A}_w$.

## IV. Stable Permutations

In this section, we define a special family of layer permutations that keep the information set $\mathcal{I}$ unchanged. From the perspective of polar code analysis, such permutations preserve the upper bound on error probability, which is the basis for permutation selection in [17]. A similar set of permutations for information sets in decreasing monomial codes, called *stabilizers*, were studied in [11] and used to extend the known automorphism group of polar codes.

Let $\Pi(n)$ be the set of all permutations on $\{1, 2, \ldots, n\}$.

**Definition 1.** [22] Let $\pi \in \Pi(n)$. The permutation $B_\pi$ on $2^n$ letters is called a **bit significance permutation (BSP)** and is defined as the mapping $k \longrightarrow k'$ where $k$ is the integer with binary expansion $k_n k_{n-1} \ldots k_1$ and $k'$ is the integer with binary expansion $k'_n k'_{n-1} \ldots k'_1 = k_{\pi(n)} k_{\pi(n-1)} \ldots k_{\pi(1)}$.

Depending on the choice of the information set $\mathcal{I}$, there may exist some permutations $\pi \in \Pi(n)$ such that the information set remains unchanged when the codeword position is permuted by the corresponding $B_\pi$. To explore this, we define the general concept of stable permutations.

**Definition 2.** Given a subset $\mathcal{P} \subset [N]$, a permutation $\pi \in \Pi(n)$ is called a **stable permutation (SP)** for $\mathcal{P}$ if $B_\pi(i) \in \mathcal{P}$, for any $i \in \mathcal{P}$. The set including all SPs for a given $\mathcal{P}$ is denoted by $\mathcal{S}(n,\mathcal{P})$.

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TABLE I
$|\mathcal{S}(7,\mathcal{I})|$ OF $\mathscr{C}(128,\mathcal{D})$.

| $\|\mathcal{D}\|$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $\|\mathcal{S}(7,\mathcal{I})\|$ | 5040 | 144 | 24 | 24 | 4 | 4 | 2 | 4 |

The properties of the SP set and a methodology for determining it will be described in the full version of this paper, but are omitted due to space limitations. These results can be used to calculate the SP set cardinality for the new half-rate hybrid RM-polar codes, as illustrated in the following examples.

For $n = 2k+1$ and $n = 2k$, the information sets of $\mathscr{C}(N,\mathcal{D})$ are given in (1) and (2), respectively. Utilizing the properties of the SP set, we can show that

$$\mathcal{S}(n,\mathcal{I}) = \mathcal{S}(n,\mathcal{D}).$$

Thus, for the new family of codes $\mathscr{C}(N,\mathcal{D})$, the SP set is determined by $\mathcal{D}$. Table I lists the cardinality of $\mathcal{S}(n,\mathcal{I})$ for $\mathscr{C}(128,\mathcal{D})$ with $|\mathcal{D}|$ up to 7, illustrating the case $n = 2k+1$. When $n = 2k$, one can deduce that $|\mathcal{D}|$ is fixed, i.e., $|\mathcal{D}| = \binom{2k}{k}/2$. Thus, the new half-rate code construction for the case of $n = 2k$ does not have as many variants as for $n = 2k+1$. For example, when $n = 8$, i.e., $N = 256$, we have $|\mathcal{D}| = \binom{8}{4}/2 = 35$. One can verify that $|\mathcal{S}(8,\mathcal{I})| = |\mathcal{S}(8,\mathcal{D})| = 4$.

The new family of hybrid RM-polar codes $\mathscr{C}(N,\mathcal{D})$ is considered since they have nontrivial SP sets that support BPL decoding. The SP sets of some other interpolated RM-polar codes, such as in [16] and [18], need further study.

## V. Simulation Results

In this section, simulations are performed over additive white Gaussian noise (AWGN) channels with binary phase shift keying (BPSK) modulation. We evaluate the BLER performance of 5G polar codes and the new family of codes $\mathscr{C}(N,\mathcal{D})$ under several decoding algorithms. The 5G polar codes are generated from [21]. Since the new code construction depends on $n$, we showcase the simulation results for two representative blocklengths, i.e., $N = 128$ and $N = 256$.

### A. SC-based Decoding

We first compare the performance of the new family of half-rate codes $\mathscr{C}(N,\mathcal{D})$ with that of 5G polar codes under SC-based decoding algorithms. Fig. 3 shows the BLER performance of $\mathscr{C}(256,\mathcal{D})$ and (256, 128)-5G polar code under SC, SCAN, and SCL decoding. The list size of SCL is 8 and the maximum number of iterations for SCAN is 8. One can see that $\mathscr{C}(256,\mathcal{D})$ has worse performance than the 5G polar code under SC decoding, but it has significantly better performance under SCL decoding.

### B. BPL Decoding

Fig. 4 shows the BLER performance of $\mathscr{C}(128,3)$ (i.e., $\mathscr{C}(128,|\mathcal{D}|=3)$) and (128, 64)-5G polar code under BPL decoding with different permutation selection schemes. Every constituent BP decoder uses a maximum of 200 iterations and checks the $G_N$-based early stopping condition after each

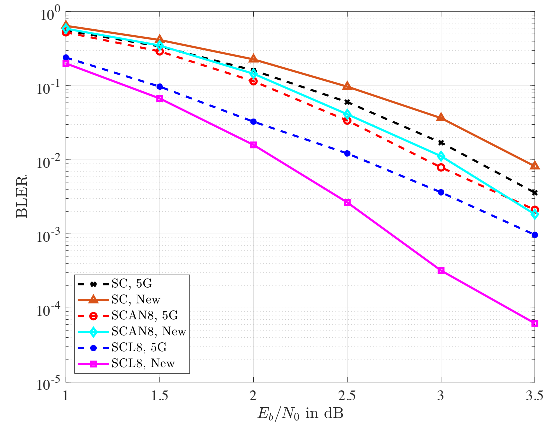


Fig. 3. BLER performance of $\mathscr{C}(256,\mathcal{D})$ and (256, 128)-5G polar code under SC-based decodings.

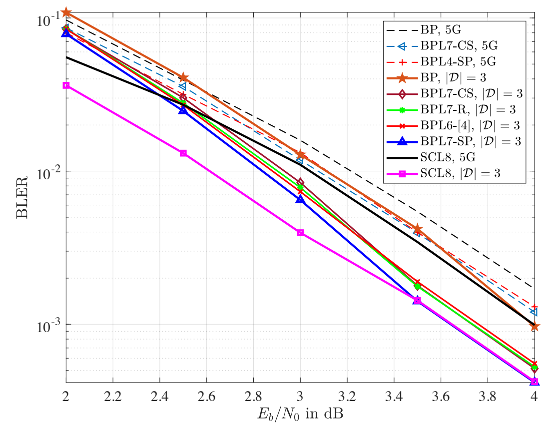


Fig. 4. BLER performance of $\mathscr{C}(128,3)$ and (128, 64)-5G polar code under BPL decoding with different selection schemes.

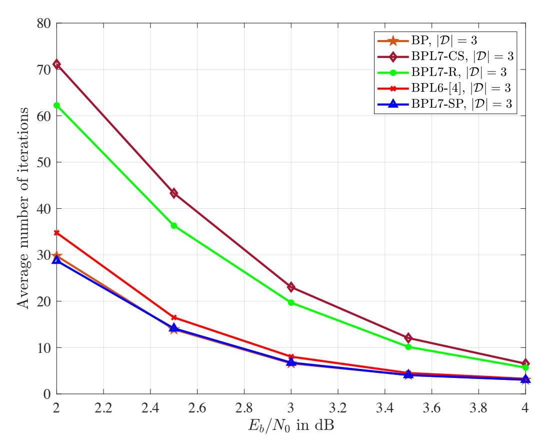


Fig. 5. The average number of iterations of $\mathscr{C}(128,3)$ under BPL decoding with different selection schemes.

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iteration. The performance of $\mathscr{C}(128,3)$ and the 5G polar code under SCL decoding with list size 8 is also provided as a benchmark. Each plot of BPL decoding is denoted as "BPL$\ell$"-"name of permutation selection scheme" where $\ell$ is the list size, and "CS" denotes cyclic shift permutations, "R" denotes randomly chosen permutations, and "SP" denotes stable permutations. We also include the selection scheme used in [4], where the 6 permutations of the three rightmost layers are used. Noting that BP decoding fails to decode with only a small probability on the original factor graph, we keep the original factor graph in the permutation list for every selection scheme. According to Table I, $|\mathcal{S}(7,\mathcal{D})| = 24$ when $|\mathcal{D}| = 3$. For the SP selection scheme, we randomly choose 6 SPs from the 23 SPs other than the identity permutation among the SP set of $\mathscr{C}(128,3)$.

As shown in Fig. 4, the performance of $\mathscr{C}(128,3)$ is better than that of the 5G polar code under BP decoding, approaching the performance of the 5G polar code under BPL and SCL decoding. Comparing the BLER performance of $\mathscr{C}(128,3)$ under BPL decoding with different selection schemes, we can see that the SP selection stands out as the best one. It can compete with SCL decoding in the high SNR region (e.g., 4dB). Furthermore, no matter which selection scheme is chosen, $\mathscr{C}(128,3)$ under BPL decoding outperforms $(128,64)$-5G polar code under SCL decoding. Finally, the SP selection for BPL decoding not only has the best BLER performance but also has the smallest average number of iterations per constituent decoder, as shown in Fig. 5, approaching that of conventional BP decoder.

## VI. Conclusion

In this paper, we consider the problem of decoding modified polar codes on permuted factor graphs. We first propose a new construction method for hybrid RM-polar codes. We then define the family of factor-graph layer permutations called SPs that preserve information sets when the corresponding BSPs are applied to message and codeword positions. We demonstrate using specific examples that the cardinalities of the SP sets of new half-rate hybrid RM-polar codes are large enough to support BPL decoding. Simulation results show that the performance of the new codes improves significantly over that of 5G polar codes under SCL decoding. It is also shown that the SP selection scheme achieves performance superior to that of other frequently-used selection schemes (e.g., cyclic shift, random, right-layer) under BPL decoding, and, moreover, it reduces the average number of iterations per constituent decoder when a $G_N$-based early stopping condition is considered.

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