On Codes with Spectral Nulls at Rational Submultiples of the Symbol Frequency

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Abstract—In digital data transmission (respectively, storage systems), line codes (respectively, recording codes) are used to tailor the spectrum of the encoded sequences to satisfy constraints imposed by the channel transfer characteristics or other system requirements. For instance, pilot tone insertion requires codes with zero mean and zero spectral density at tone frequencies. Embedded tracking/focus servo signals produce similar needs. Codes are studied with spectral nulls at frequencies $f = kf_s/n$, where f_s is the symbol frequency and k, n are relatively prime integers with $k \le n$; in other words, nulls at rational submultiples of the symbol frequency. A necessary and sufficient condition is given for a null at f in the form of a finite discrete Fourier transform (DFT) running sum condition. A corollary of the result is the algebraic characterization of spectral nulls which can be simultaneously realized. Specializing to binary sequences, we describe canonical Mealy-type state diagrams (directed graphs with edges labeled by binary symbols) for each set of realizable spectral nulls. Using the canonical diagrams, we obtain a frequency domain characterization of the spectral null systems obtained by the technique of time domain interleaving.

I. Introduction

Several applications in the realm of digital data transmission and storage require the use of codes which impart specific properties to the spectrum of the encoded symbol sequence. For example, pulse position modulation (PPM) digital magnetic recording channels employ run-length-limited (RLL) codes [14] to shape the signal spectrum, limiting low-frequency content to provide self-clocking and high-frequency content to reduce intersymbol interference.

Line codes used in baseband pulse-amplitude modulation (PAM) digital transmission systems and recording codes for certain magnetic and optical channels require, in addition to RLL constraints, a zero-mean and vanishing spectral density at zero frequency (dc). We refer to this as a *spectral null at dc*.

A necessary and sufficient condition for the existence of a null at dc is that the running digital sum at dc (denoted RDS_0) for codestrings $\boldsymbol{a} = a_0, a_1, \cdots, a_N$, defined by

$$\mathrm{RDS}_0(\boldsymbol{a}) = \sum_{i=0}^{N} a_i, \qquad (1.1)$$

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takes values in a finite range [20], [13]. We call this the finite RDS_0 condition.

In this paper, we investigate codes with spectral nulls at arbitrary rational submultiples kf_s/n of the symbol frequency f_s. These codes have applications in transmission systems employing pilot tones for synchronization and recording systems with embedded tracking, focus, or timing servos [10], [11].

In Section II, we give background terminology and results on Markov models for constrained channel sequences, spectral analysis of digital signals, and codes with spectral nulls at dc. In Section III, we extend the result for null at $f = 0$ (dc) to $f = kf_s/n$ by introducing a generalization of RDS_0, namely the running digital sum at f (denoted RDS_f) based on the discrete Fourier transform (DFT). The RDS at $f = kf_s/n$ for a sequence $\boldsymbol{a} = a_0, \cdots, a_N$ is

$$\mathrm{RDS}_f(\boldsymbol{a}) = \sum_{m=0}^{N} a_m e^{-i2\pi km/n} \qquad (1.2)$$

where $i = \sqrt{-1}$.

We show that a necessary and sufficient condition for a spectral null at $f = kf_s/n$ is that the encoded symbol sequences have a finite range of values of RDS_f. The condition also will be stated in the form of a "coboundary" equation for the encoded sequences (see Definition 4).

Some applications, such as [10], require simultaneous spectral nulls at distinct frequencies. In Section IV, using the finite RDS_f condition, we give an algebraic characterization of finite sets of spectral null frequencies which can be simultaneously realized. This generalizes a result of Gorog [9] for block codes.

For codes with rational symbol values, the characterization is as follows. Associate the root of unity $\omega = e^{-i2\pi k/n}$ to the null frequency $f = kf_s/n$. The spectral nulls at frequencies $f_1, f_2, \cdots, f_l$ are simultaneously realizable if and only if the corresponding set of roots of unity $\omega_1, \cdots, \omega_l$ forms a full set of roots of a polynomial with integer coefficients.

Code construction techniques (see, for example, [1]) often make use of a Mealy-type finite state transition diagram (FSTD) to represent the system of constrained sequences. In Section V, we describe canonical diagrams G_p^f for the set of binary sequences having a spectral null at frequency f. The diagrams are locally finite countable state transition diagrams (CSTD) with the properties:

1) every FSTD contained in G_p^f generates a spectral null at f;
2) every FSTD H, with period p, which generates a spectral null at f collapses to an FSTD in G_p^f via a label-preserving directed graph homomorphism.

These canonical diagrams are based on states which lie naturally on an integer lattice with a simple algebraic description. Canonical diagrams for sets of simultaneous nulls are also described.[1]

The paper concludes in Section VI with an application of the canonical diagrams to the study of constraints obtained by interleaving sequences having a null at specified frequency f. Reference has been made in communication theory applications to the fact that a null at f_s/m can be produced by interleaving m sequences, each having a null at dc [6], [7]. We determine the set of nulls produced by interleaving a null at $f = kf_s/n$, m times, and prove that the spectrum of an FSTD includes nulls at these frequencies if and only if the strings it generates are obtained by m-way interleaving of sequences with a null at f. This result provides an equivalent frequency domain characterization of the spectral null systems obtained by the technique of time domain interleaving.

II. Spectral Analysis of Markov Driven Signals

In this section, background terminology and results are established.

A. Markov Models for Constrained Channel Sequences

We are interested in the spectral analysis of digital sequences which represent a finite memory function of a Markov chain. It will be convenient to describe the sequences in terms of a finite state transition diagram.

Definition 1: An FSTD G is a directed graph with state set $\mathcal{S} = \{\sigma_0, \cdots, \sigma_n\}$ and edges labeled with symbols from an alphabet $\mathcal{A}$ (usually an alphabet consists of numbers). Without loss of generality, we can assume at most one edge connects σ_i to σ_j, and we denote the edge label by $a(\sigma_i, \sigma_j)$.

Paths through the graph underlying G give rise to symbol sequences obtained by reading off the labels of edges as they are traversed. By associating to G a matrix of state transition probabilities, $P = (p_{ij})$, the state sequences $\{(s_n)\}$ of paths through G become a Markov chain Γ, and the symbol sequences $\{(a_n)\}$ are a function of the Markov chain $a_n = a(s_n, s_{n+1})$. We remark that this description encompasses digital encoders based on a Mealy finite-state sequential machine (FSSM) model, when driven by independent identically distributed input symbols. The output sequences can also be represented as a memoryless function of a Markov chain by defining an FSTD with state set composed of triples $\{(\sigma_i, \sigma_j, a(\sigma_i, \sigma_j))\}$, and label $a(\sigma_i, \sigma_j)$ on the state $\{(\sigma_i, \sigma_j, a(\sigma_i, \sigma_j))\}$. This representation includes digital encoders based on a Moore FSSM model (see [20]). The Moore model is referred to as the *edge graph* of the Mealy model. For example, we represent the binary sequences with RDS_0 in the range [0,3] by the FSTD shown in Fig. 1. By going to the edge graph, one obtains the Moore-type presentation of the same constraint, shown in Fig. 2.

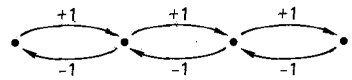


Fig. 1. FSTD for channel with RDS_0 values in range [0,3] (Mealy-type).

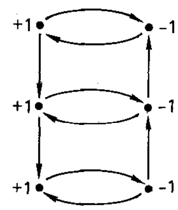


Fig. 2. Moore-type presentation of channel in Fig. 1.

B. Spectral Analysis of Digital Signals

There is a substantial literature on the spectral analysis of digital signals which are a finite-memory function of a Markov chain. For example, see [2]–[8]. Let G be an FSTD describing a constrained channel, with underlying Markov chain Γ.

Definition 2: An FSTD G is *irreducible* if for every pair of states σ_i and σ_j there is a path in G from σ_i to σ_j. The *period* of G is the greatest common divisor (gcd) of cycle lengths in G. Here the length of a cycle is the number of edges traversed. G is said to be *aperiodic* if it is irreducible and has period 1. We say that (G, Γ) is irreducible (respectively, aperiodic) if G is irreducible (respectively, aperiodic) and the transition probability p_{ij} is positive if and only if G contains an edge from state σ_i to state σ_j.

Assume that (G, Γ) is irreducible in the following definition.

Definition 3: The average power spectral density $\Phi(f)$ associated to (G, Γ) is defined as

$$\Phi(f) = \lim_{M \to \infty} E\left[\frac{1}{M}\left|\sum_{m=0}^{M} a_m e^{-i2\pi m f/f_s}\right|^2\right], \quad (2.1)$$

where the expected value is over the set of sequences (a_i), $i \geq 0$, generated by paths through G, with respect to the measure induced by Γ. The limit should be interpreted in the distribution sense. The frequency f_s is the symbol rate. Closed-form expressions for $\Phi(f)$ are given in [2], [4], [8]. We say that (G, Γ) has a *null* at frequency f if $\Phi(f) = 0$. In particular, if $\Phi(0) = 0$, there is a *null at dc*.

[1] We are indebted to Jonathan Ashley for contributions to this section.

C. RDS_0 Coboundary Conditions and Spectral Null at DC

The following well-known result gives a sufficient condition for a spectral null at dc.

Lemma 1: Let (G, Γ) be an FSTD G with underlying Markov chain Γ. If G generates sequences which assume only a finite number of RDS_0 values, then the spectral density of (G, Γ) vanishes at dc. That is, $\Phi(0) = 0$.

Proof: For any sequence $\{a_m\}$ generated by G, we have

$$\left| \sum_{m=0}^{M} a_m \right| \leq B, \text{ for some } B < \infty.$$

So,

$$\frac{1}{M} \left| \sum_{m=0}^{M} a_m \right|^2 \leq \frac{B^2}{M}$$

and

$$\lim_{M \to \infty} E\left[\frac{1}{M} \left| \sum_{m=0}^{M} a_m \right|^2 \right] = 0$$

for all Markov measures Γ.

Pierobon analyzed the closed form matrix expression for $\Phi(f)$, to prove the interesting converse to Lemma 1.

Theorem 1 [20], [13, Theorem 3]: If (G, Γ) is aperiodic and has vanishing average power spectrum at dc, $\Phi(0) = 0$, then the range of values of RDS_0 produced by G is finite.

In the course of his proof, Pierobon actually shows that (G, Γ) satisfies what we will call a "coboundary condition," implying the finite RDS_0 result. In fact, the condition is equivalent to the finite RDS_0 property. Because of its relevance to later sections, we formalize the definition of the coboundary condition and prove this equivalence.

Definition 4: The FSTD G satisfies a *coboundary condition at dc* if there is a function χ: $\mathcal{S} \to \mathbb{C}$ from the state set $\mathcal{S}$ of G to the complex numbers such that

$$a_{ij} = \chi(\sigma_j) - \chi(\sigma_i)$$

where $a_{ij} = a(\sigma_i, \sigma_j)$.

Remark 1: The terminology "coboundary condition" derives from the theory of cohomology of simplicial complexes, in particular, graphs. Many references can be found in the ergodic theory literature to similar cohomological concepts (see, for example, [17, p. 13]). Note that the function χ, if it exists, is unique only up to an additive constant.

Remark 2: Theorem 1 was first stated by Justesen in [13] with reference to a coboundary formula for the output symbols. The approach given there is very natural, but the proof does not seem to be complete. Theorem 1 can also be proved by applying Theorem 3 and Theorem 1 of Leonov [15] to obtain a coboundary formula for the output symbols

$$a(s_n, s_{n+1}) = \varphi(n+1) - \varphi(n),$$

where φ is a function of the state sequence generated by the Markov chain. It can then be shown that $\varphi(n)$ depends only on the state at time n, by application of [17, Corollary 42, p. 29].[2]

Remark 3: Theorem 1 requires G to be a *finite* state-transition diagram for the conclusion to hold. The countably infinite state transition diagram (CSTD) G' shown in Fig. 3 has the property that for every infinite sequence $\boldsymbol{a} = a_0, a_1, \cdots, a_N, \cdots$ generated by G', there exists an integer $M(\boldsymbol{a})$ such that for $N > M(\boldsymbol{a})$,

$$N > |RDS_0(a_0, a_1, \cdots, a_N)|^3.$$

This inequality implies that for each sequence $\boldsymbol{a}$,

$$\lim_{N \to \infty} \frac{1}{N} \left| \sum_{m=0}^{N} a_m \right|^2 = 0.$$

With the appropriate choice of Markov measure, the power spectral density of (G', Γ') at dc is zero, $\Phi(0) = 0$. However, it is easy to see that the range of RDS_0 values is not finite.

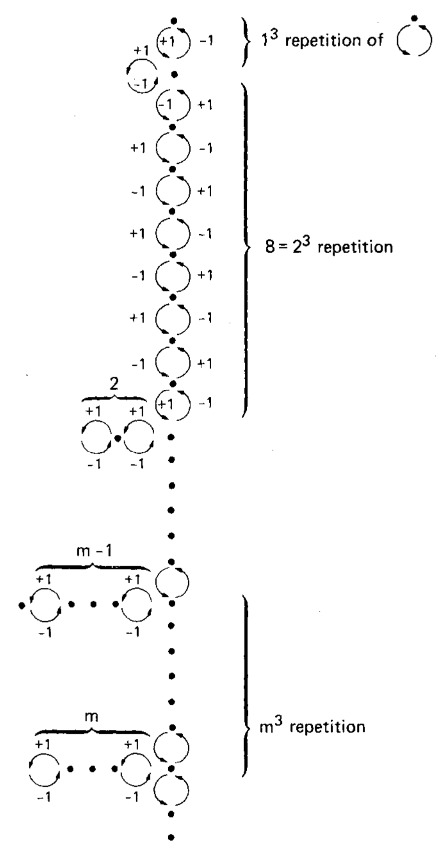


Fig. 3. CSTD with spectral null at dc, but unbounded RDS_0 values.

Theorem 2: Let G be an irreducible FSTD with underlying Markov chain Γ. The following are equivalent:

1) every cycle in G has $RDS_0 = 0$;
2) G satisfies the finite RDS_0 condition;

[2] We are indebted to M. Ratner for pointing out the relevance of results in [15].

3) G satisfies a coboundary condition;
4) (G, Γ) has a spectral null at dc.

Proof (3) ⇒ (2): Let $\boldsymbol{a} = (a_0, \cdots, a_M)$ be a sequence produced by following the state sequence $s_0, s_1, \cdots, s_{M+1}$. Then

$$\mathrm{RDS}_0(\boldsymbol{a}) = \sum_{m=0}^{M} a_m = \sum_{m=0}^{M} [\chi(s_{m+1}) - \chi(s_m)] = \chi(s_{M+1}) - \chi(s_0).$$

The RDS_0 values therefore lie in the finite set $\{\chi(\sigma_j) - \chi(\sigma_i)\}, \sigma_i, \sigma_j \in \mathscr{S}$.

(2) ⇒ (1): Consider any cycle $\boldsymbol{a} = (a_0, a_1, \cdots, a_M)$, with running sum $\mathrm{RDS}_0(\boldsymbol{a}) = c$. Then the sequences obtained by r concatenations of $\boldsymbol{a}$ will have running sum rc. Since G satisfies the finite RDS_0 condition, $\{rc | r \geq 1\}$ must be a finite set. This forces $c = 0$.

(1) ⇒ (3): We construct the coboundary function as follows. Pick an arbitrary state σ and define $\chi(\sigma) = 0$. The idea is to push the definition of χ along, edge by edge, and verify that no inconsistencies arise. More precisely, for each state τ, choose a path with underlying state sequence $\boldsymbol{s} = (\sigma = s_0, s_1, \cdots, s_{n-1}, s_n = \tau)$ from σ to τ. Define $\chi(\tau)$ to be the running sum along the path,

$$\chi(\tau) = c = \sum_{m=0}^{n-1} a(s_m, s_{m+1}).$$

The resulting function χ will be well-defined. To see this, let $\boldsymbol{s}' = (\sigma = s_0', s_1', \cdots, s_p' = \tau)$ be another state sequence underlying a path from σ to τ, with running sum c'. By irreducibility of G, we can find a path back from τ to σ, say with state sequence $\boldsymbol{t} = (\tau = t_0, t_1, \cdots, t_i = \sigma)$, with running sum d. Let $\boldsymbol{st}$ denote the concatenation of sequences $\boldsymbol{s}$ and $\boldsymbol{t}$. Then $\boldsymbol{st}$ and $\boldsymbol{s't}$ both correspond to cycles beginning and ending at σ, with running sums $c + d$ and $c' + d$, respectively. Since the cycles have RDS_0 exactly 0 by hypothesis, we conclude

$$c = c' = -d,$$

so $\chi(\tau) = c$ is independent of the state sequence $\boldsymbol{s}$ from σ to τ chosen. This shows that (1) ⇔ (2) ⇔ (3). Lemma 1 shows (2) ⇒ (4). Finally, Theorem 1 proves (4) ⇒ (2) (see remark below). This completes the proof.

Remark: The proof of (4) ⇒ (2) given in Pierobon assumes aperiodicity of G. The periodic case can be reduced to the aperiodic case as follows. First, we need to introduce the pth power of an FSTD.

Definition 5: Let G be an FSTD. The *pth power of* G, denoted G^p, is the FSTD with state set identical to that of G, and an edge from σ_i to σ_j for each path of length p in G from σ_i to σ_j. The corresponding edge label will be the p-tuple of symbols generated by following the path in G. See Fig. 4. To satisfy the property that at most one edge connects σ_i to σ_j in G^p, it may be necessary to replace G^p by the edge graph.

If G has period p, each irreducible component of G^p will be aperiodic, with edge labels consisting of p-tuples of symbols $a = [a_0, \cdots, a_{p-1}]$. Fix a component of G^p. Construct a new FSTD H with the same underlying directed graph as this component but with edge labels obtained by adding the symbols in the p-tuple labels, say

$$b = \sum_{i=0}^{p-1} a_i. \tag{2.2}$$

Then H is aperiodic and has a Markov chain structure Γ' induced by Γ. From Definition 2 and the fact that $E\{b\} = pE\{a\}$, we conclude that if (G, Γ) has a spectral null at dc, then so does (H, Γ'). By Theorem 1, the RDS_0 for sequences produced by H takes values in a finite range. From (2.2), it follows that the values of RDS_0 for sequences generated by G also fall into a finite range.

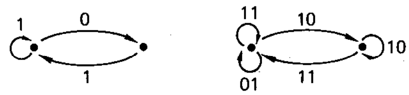


Fig. 4. Channel S and second power of S, S^2.

III. Spectral Nulls at $f = kf_s/n$

We now extend the results of Section II to the general case of a spectral null at $f = kf_s/n$, where gcd $(k, n) = 1$. The key tool will be the running digital sum at f, RDS_f, defined in (1.2). We have the following generalization of Lemma 1.

Lemma 2: Let (G, Γ) be an FSTD G with underlying Markov chain Γ. If G generates sequences which assume a finite number of RDS_f values for $f = kf_s/n$, then the power spectral density of (G, Γ) vanishes at $f, \Phi(f) = 0$.

Proof: The proof is a straightforward generalization of the proof of Lemma 1.

As in the case of a spectral null at dc, we will see that this sufficient condition is also necessary. First we define a coboundary condition at f in analogy to Definition 4.

Definition 6: The FSTD G satisfies a *coboundary condition at* $f = kf_s/n$ if there is a function χ: $\mathscr{S} \to \mathbf{C}$ from the state set $\mathscr{S}$ of G to the complex numbers such that

$$a_{ij} = e^{-i2\pi k/n}\chi(\sigma_j) - \chi(\sigma_i)$$

where $a_{ij} = a(\sigma_i, \sigma_j)$.

Theorem 3: Let (G, Γ) be an irreducible FSTD G with underlying Markov chain Γ. The following are equivalent:

1) every cycle in G of length a multiple of n has $\mathrm{RDS}_f = 0$;
2) G has a finite range of RDS_f values;
3) G satisfies a coboundary condition at f;
4) (G, Γ) has a spectral null at f.

Proof: As in the proof of Theorem 2, we first show (3) ⇒ (2) ⇒ (1) ⇒ (3). We then use Theorem 1 to prove (4) ⇒ (3), which, along with Lemma 2, completes the proof of equivalence.

(3) ⇒ (2): Let $\boldsymbol{a} = a_0, \cdots, a_{M-1}$ be a sequence produced by the state sequence $\boldsymbol{s} = s_0, s_1, \cdots, s_M$. Let $\omega = e^{-i2\pi k/n}$. Then

$$\mathrm{RDS}_f(\boldsymbol{a}) = \sum_{m=0}^{M-1} a_m \omega^m = \sum_{m=0}^{M-1} \omega^m [\omega \chi(s_{m+1}) - \chi(s_m)]$$
$$= \omega^M \chi(s_M) - \chi(s_0).$$

Since $\{\omega^M | M \geq 0\}$ is a finite set (ω is an nth root of unity), the RDS_f values lie in a finite set $\{\omega^M \chi(\sigma_j) - \chi(\sigma_i) | \sigma_i, \sigma_j \in \mathscr{S}, M \in \{0,1,2,\cdots\}\}$.

(2) ⇒ (1): Let $\boldsymbol{a} = a_0, a_1, \cdots, a_{M-1}$ be a sequence generated by a cycle of length M, a multiple of n, with $\mathrm{RDS}_f(\boldsymbol{a}) = c$. Then the sequence obtained by r concatenations of $\boldsymbol{a}$ will have running sum at f equal to $r\mathrm{RDS}_f(\boldsymbol{a})$. The finite RDS_f condition implies $\mathrm{RDS}_f(\boldsymbol{a}) = 0$.

(1) ⇒ (3): We consider 2 cases corresponding to possible cycle lengths in G.

Case 1: All cycles in G are of length a multiple of n.

The proof is very similar to the proof of (1) ⇒ (3) in Theorem 2. Pick a state σ and define $\chi(\sigma) = 0$. Now extend the definition of χ by "pushing along" paths in G. Specifically, if $\boldsymbol{s} = (\sigma = s_0, \cdots, s_M = \tau)$ is the state sequence associated to a path from σ to τ, define

$$\chi(\tau) = \sum_{m=0}^{M-1} a(s_m, s_{m+1}) \omega^{m-M}. \tag{3.1}$$

The proof that χ is well defined is identical to the argument used in Theorem 2, and one can see that if there is an edge from σ_i to σ_j, then $a(\sigma_i, \sigma_j) = \omega\chi(\sigma_j) - \chi(\sigma_i)$.

Case 2: Some cycle $\boldsymbol{a}$ has length not a multiple of n (of course, this implies $f \neq 0$).

By irreducibility, every state σ has a cycle which has length q not a multiple of n, corresponding to state sequence $\boldsymbol{s} = (\sigma = s_0, \cdots, s_q = \sigma)$, and symbol sequence $\boldsymbol{b} = b_0, \cdots, b_{q-1}$. Define

$$\chi(\sigma) = (\mathrm{RDS}_f(\boldsymbol{b}))/(\omega^q - 1). \tag{3.2}$$

To see that χ is well defined, suppose $\boldsymbol{c}$ is another symbol sequence associated to a cycle at σ of length r not a multiple of n. We must show that

$$\frac{\mathrm{RDS}_f(\boldsymbol{b})}{\omega^q - 1} = \frac{\mathrm{RDS}_f(\boldsymbol{c})}{\omega^r - 1}. \tag{3.3}$$

Choose positive integers k_1, k_2 such that $n | k_1 q$ and $n | k_2 r$. Consider the cycle concatenations

$$\boldsymbol{x} = \underbrace{\boldsymbol{b} \cdots \boldsymbol{b}}_{k_1 \text{ times}} \underbrace{\boldsymbol{c} \cdots \boldsymbol{c}}_{k_2 \text{ times}}$$

and

$$y = \underbrace{\boldsymbol{b} \cdots \boldsymbol{b}}_{k_1 - 1 \text{ times}} \; \boldsymbol{cb} \; \underbrace{\boldsymbol{c} \cdots \boldsymbol{c}}_{k_2 - 1 \text{ times}}.$$

Both $\boldsymbol{x}$ and $\boldsymbol{y}$ have length equal to a multiple of n, so

$$\mathrm{RDS}_f(\boldsymbol{x}) = \mathrm{RDS}_f(\boldsymbol{y}) = 0.$$

By cancellation of the terms in $\mathrm{RDS}_f(\boldsymbol{x})$ and $\mathrm{RDS}_f(\boldsymbol{y})$ corresponding to the first $k_1 - 1$ subcycles and the last $k_2 - 1$ subcycles, we get

$$\omega^{(k_1-1)q}\mathrm{RDS}_f(\boldsymbol{b}) + \omega^{k_1 q}\mathrm{RDS}_f(\boldsymbol{c}) = \omega^{(k_1-1)q}\mathrm{RDS}_f(\boldsymbol{c}) + \omega^{((k_1-1)q+r)}\mathrm{RDS}_f(\boldsymbol{b})$$

or, dividing by $\omega^{(k_1-1)q}$,

$$\mathrm{RDS}_f(\boldsymbol{b})[1 - \omega^r] = \mathrm{RDS}_f(\boldsymbol{c})[1 - \omega^q].$$

Since n divides neither q nor r, then the last equation yields (3.3).

Finally, we check that $b = a(\sigma_i, \sigma_j) = \omega\chi(\sigma_j) - \chi(\sigma_i)$, if there is an edge from σ_i to σ_j. Let $\boldsymbol{c}$ be the symbol sequence of a path of length m from σ_j to σ_i, where $m+1$ is not a multiple of n. Then $\boldsymbol{bc}$ is the symbol sequence of a cycle of length $m+1$ based at σ_i, and $\boldsymbol{cb}$ is the symbol sequence for such a cycle at σ_j. See Fig. 5. By the definition of $\chi(\sigma)$, (3.2),

$$\chi(\sigma_i) = \frac{\mathrm{RDS}_f(\boldsymbol{bc})}{\omega^{m+1} - 1} = \frac{b + \omega\mathrm{RDS}_f(\boldsymbol{c})}{\omega^{m+1} - 1}$$

and

$$\chi(\sigma_j) = \frac{\mathrm{RDS}_f(\boldsymbol{cb})}{\omega^{m+1} - 1} = \frac{\mathrm{RDS}_f(\boldsymbol{c}) + \omega^m b}{\omega^{m+1} - 1}.$$

Therefore,

$$\omega\chi(\sigma_j) - \chi(\sigma_i) = \frac{b(\omega^{m+1} - 1) - \mathrm{RDS}_f(\boldsymbol{c})[\omega - \omega]}{\omega^{m+1} - 1}$$
$$= b.$$

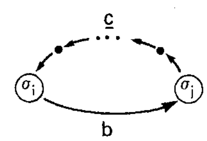


Fig. 5. Cycle $\boldsymbol{bc}$ at σ_i and cycle $\boldsymbol{cb}$ at σ_j.

Remark: When the period of G is a multiple of n (in particular when $f = 0$), then there is one parameter of freedom in the definition of χ, namely the choice of $\chi(\sigma_0)$ in the proof of (1) ⇒ (3), Case 1. Otherwise, χ is uniquely determined.

(4) ⇒ (3): The proof follows from Theorem 1. In the case of aperiodic G, construct the nth power of G, G^n, having edges labelled by n-tuples $\boldsymbol{a} = [a_0, \cdots, a_{n-1}]$. Define FSTD H with the same underlying graph as G^n, but with edge labels

$$b = \sum_{m=0}^{n-1} a_m \omega^m = \mathrm{RDS}_f(\boldsymbol{a}). \tag{3.4}$$

Then H is aperiodic, and with its induced Markov structure, it has a spectral null at dc, as seen from Definition 2. Therefore, the values of RDS_0 for G^n fall in a finite range, by Theorem 1. It follows from (3.4) that for G the values of RDS_f fall in a finite range. The periodic case reduces to the aperiodic case as it did for $f = 0$.

IV. Simultaneous Nulls

Certain applications, such as embedded timing and track-following servos in digital data recording, require simultaneous nulls in the coded spectrum at a set of frequencies $f_1, f_2, \cdots, f_l$.

We assume in this section that the symbol alphabet of the FSTD is a subset of the rational numbers. Theorem 3, (1) $\Leftrightarrow$ (4), permits an algebraic characterization of the set of frequencies $\{k_i f_s / n_i\}$ $i = 1, \cdots, l$ at which nulls can be achieved simultaneously by a constrained system (G, Γ).

Lemma 3: Let $f_1 = k_1 f_s / n_1, \cdots, f_l = k_l f_s / n_l$ be the set of rational submultiple null frequencies $< f_s$ in the spectrum Φ of the constrained channel (G, Γ), assuming Φ is not identically 0. Let $\omega_j = e^{-i2\pi k_j / n_j}$. Then $\{\omega_1, \cdots, \omega_l\}$ forms a full set of roots of an integer polynomial; that is, if we define $p(x) = (x - \omega_1)(x - \omega_2) \cdots (x - \omega_l)$, then $p(x)$ is a polynomial with integer coefficients.

Proof: It suffices to prove that if (G, Γ) has a null at $f = k f_s / n$, with gcd $(k, n) = 1$, then it must have nulls at all frequencies $f = q f_s / n$ where gcd $(q, n) = 1$. Then $p(x)$ will be a product of cyclotomic polynomials (that is, irreducible integer polynomials whose roots are complex roots of unity).

By Theorem 3, (4) $\Rightarrow$ (1), a null at $f = k f_s / n$ implies that every cycle $\boldsymbol{a} = a_0, \cdots, a_{mn-1}$ of length a multiple of n satisfies

$$\mathrm{RDS}_f(\boldsymbol{a}) = 0;$$

i.e.,

$$\sum_{j=0}^{mn-1} a_j \omega^j = 0$$

where $\omega = e^{-i2\pi k / n}$. Clearing denominators of the rational coefficients a_j yields an integer polynomial $q(x) = \Sigma q_j x^j$ with ω as a root; $q(\omega) = 0$.

The polynomial must be divisible by the minimal integral polynomial $g(x)$ belonging to ω, that is, the polynomial $g(x)$ of smallest degree such that $g(\omega) = 0$ (see, for example, [12, p. 120]). Since ω is a primitive nth root of unity, the polynomial $g(x)$ is a cyclotomic polynomial with roots $\mathscr{R} = \{\omega^q | 1 \le q < n, \text{ gcd } (q, n) = 1\}$, that is, the set of all primitive nth roots of unity. This implies that the elements of $\mathscr{R}$ are roots of $q(x)$ also, so

$$\sum_{j=0}^{mn-1} a_j \omega^{qj} = 0, \qquad \text{all } \omega^q \in \mathscr{R}.$$

Theorem 3, (1) $\Rightarrow$ (4), then shows that (G, Γ) has a spectral null at $f = q f_s / n$, for all $1 \le q < n$, gcd $(q, n) = 1$.

Remark: Gorog [9] mentions this result in the context of memoryless block codes, which correspond to (G, Γ) in which G consists of distinct cycles of fixed length a multiple of n, based on one state σ, each cycle corresponding to a codeword. See Fig. 6.

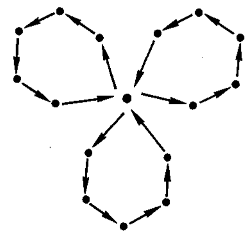


Fig. 6. Structure of underlying graph for a block code with three codewords of length 6 symbols.

Definition 7: A null in the power spectral density $\Phi(f)$ at frequency $f_0, 0 \le f_0 < f_s$, is called *unique* if $\Phi(f) \ne 0$ for $f \ne f_0, 0 \le f < f_s$.

Corollary 1: The only possible unique nulls in the spectrum of (G, Γ) are $f_0 = 0$ and $f_0 = f_s/2$.

Proof: By Lemma 3, a unique null corresponds to a root of unity ω whose minimal polynomial is of degree 1; that is, a *rational* root of unity. Therefore, $\omega = 1$ or $\omega = -1$, corresponding to $f_0 = 0$ or $f_0 = f_s/2$, respectively.

Example 1: This example illustrates a unique null at $f = 0$. Let G be as shown in Fig. 7, with associated Markov chain Γ of maximal entropy, that is, the Markov chain supported on the given graph having maximal entropy.

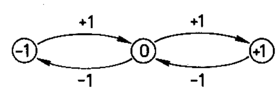


Fig. 7. FSTD which has unique null at $f = 0$, with measure of maximal entropy.

The state name represents the RDS_0 value beginning from state 0; that is, the values of the function χ referred to in the coboundary condition of Theorem 2. The maximal entropy of this system is $C = \log_2 \lambda = 1/2$, with transition probabilities $p_{0,+1} = p_{0,-1} = 1/2$, $p_{+1,0} = p_{-1,0} = 1$. With these probabilities, the spectrum of (G, Γ) is given by the Fourier transform of the autocorrelation function $R(\tau)$, which is easily computed to be

$$R(\tau) = \begin{cases} 1, & \text{if } \tau = 0 \\ -\dfrac{1}{2}, & \text{if } \tau = \pm 1 \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, $S(f) = 1 - \cos 2\pi f / f_s$. Note $S(0) = 0$ and $S(f) \ne 0$, for $0 < f < f_s$. See Fig. 8.

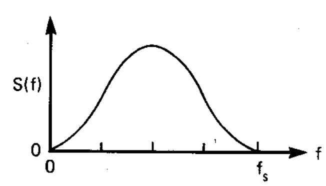


Fig. 8. Spectrum with unique null at $f = 0$. $S(f) = 1 - \cos(2\pi f / f_s)$.

Example 2: This example shows a unique null at $f = f_s/2$. Let (G, Γ) be the constrained channel corresponding to the FSTD in Fig. 9. It has the same transition probabilities for the measure of maximal entropy as the previous example. The state names correspond to the values of the function χ in the coboundary of Theorem 3. Here

$$R(\tau) = \begin{cases} 1, & \text{if } \tau = 0 \\ \dfrac{1}{2}, & \text{if } \tau = \pm 1 \\ 0, & \text{otherwise} \end{cases}$$

and $S(f) = 1 + \cos 2\pi f/f_s$. Note $S(f_s/2) = 0$, and $S(f) \neq 0$, $0 \leq f \leq f_s$ where $f \neq f_s/2$. See Fig. 10.

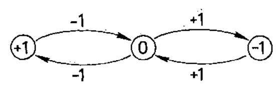


Fig. 9. FSTD which has unique null at $f_s/2$, with measure of maximal entropy.

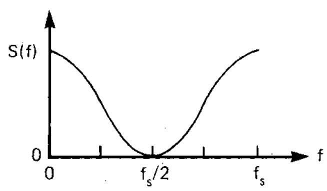


Fig. 10. Spectrum with unique null at $f = f_s/2$. $S(f) = 1 + \cos(2\pi f/f_s)$.

Remark: The graph in Fig. 7 describes the constrained sequences generated by the "biphase" modulation code, also known as Manchester code, which has been used in data recording and transmission applications. The binary code power spectra shown in Figs. 8 and 10 are also encountered in pseudoternary line coding using partial response with system polynomials $(1 - D)$ and $(1 + D)$, respectively. See, for example, Croisier [6].

V. Canonical State Diagrams For Spectral Null Constraints: Binary Alphabet $\{\pm 1\}$

Applications in magnetic and optical digital recording often utilize the binary signal alphabet $\mathscr{A} = \{\pm 1\}$. In this section we present a result on the structure of a constrained channel (G, Γ) with output alphabet $\mathscr{A}$ which produces a spectral null at $f = kf_s/n$.

Specifically, for each $p = 0, 1, \cdots, n-1$, we define a *countable* state diagram G_p^f which has the property that any finite state diagram contained in G_p^f generates a null at f, and, moreover, *any* finite-state diagram of period $q \equiv p \pmod n$ which produces a null at f collapses to a subgraph of G_p^f *via* a label-preserving directed graph homomorphism. In this section, graph homomorphisms are taken to be directed.

The set of states $\mathscr{L}^f$ of G_p^f is defined to be the ring $\mathbb{Z}[\omega]$ where $\omega = e^{-2\pi i k/n}$. Elements of $\mathbb{Z}[\omega]$ are polynomials in ω with integer coefficients of degree less than $\phi(n)$, the Euler ϕ-function, which is defined as the number of positive integers less than and relatively prime to n. This is a consequence of the fact that the minimal polynomial of ω has degree $\phi(n)$ (see [12, p. 208]). When $n = 1$ or 2, then $\mathscr{L}^f = \mathbb{Z}$. When $n = 4$, $\mathscr{L}^f = \mathbb{Z}[i]$, the Gaussian integers, which form a square planar lattice, and when $n = 3$ or 6,

$$\mathscr{L}^f = \mathbb{Z}\left[\frac{1}{2} + i\frac{\sqrt{3}}{2}\right],$$

a hexagonal planar lattice. For other values of n, $\mathscr{L}^f$ has dimension > 2.

The state transition rule for G_p^f is defined by means of a modified coboundary formula. Specifically, for each $\sigma \in \mathscr{L}^f$ and $b \in \mathscr{A} = \{\pm 1\}$, we have a unique transition which starts at state σ, is labeled b, and ends at state $h_p(\sigma, b)$ defined as follows:

$$h_p(\sigma, b) = \bar{\omega}(\sigma + b), \qquad \text{for } p = 0 \tag{5.1}$$

and

$$h_p(\sigma, b) = \bar{\omega}(\sigma + (\bar{\omega}^p - 1)b), \qquad \text{for } p = 1, \cdots, n-1. \tag{5.2}$$

Here $\bar{\omega}$ denotes the complex conjugate of ω, namely, $\bar{\omega} = e^{i2\pi k/n}$. The transition rules ((5.1) and (5.2)) correspond to translations (by $(\bar{\omega}^p - 1)b$ in (5.2), for example) followed by rotation by $\bar{\omega}$ about the origin. For $\omega \neq 1$, any such motion can be expressed as a pure rotation about some point. When $b = +1$, this point is $c_+ = \bar{\omega}(\bar{\omega}^p - 1)/(1 - \bar{\omega})$ for $p \neq 0$ and $c_+ = \bar{\omega}/(1 - \bar{\omega})$ for $p = 0$. When $b = -1$, this point is $c_- = -c_+$. (This interpretation was pointed out to us by J. Ashley.)

Definition 8: A state-transition diagram G is *period p canonical for a spectral null at f* if

1) every FSTD contained in G generates a spectral null at f;
2) every FSTD H with period p which generates a spectral null at f collapses to a subgraph of G via a label-preserving graph homomorphism.

The next proposition shows that the graph G_p^f previously defined is in fact period p canonical for spectral null at f.

Proposition 1: G_p^f is period q canonical for a spectral null at f, for all $q \equiv p \pmod n$. In particular, the *n-fold trellis presentation* of any constrained binary system generated by an FSTD H and having a spectral null at f collapses to a subdiagram of G_0^f *via* a label-preserving graph homomorphism.

Remark: The *n-fold trellis presentation* is the FSTD with states $\{(\sigma, i) | \sigma \text{ is a state in } H, 0 \leq i \leq n-1\}$ and edges from (σ, i) to $(\tau, (i+1) \bmod n)$ corresponding to each edge from σ to τ in H.

Before giving the proof, we illustrate the concept of canonical graphs with three examples, for a null at dc, a null at $f_s/2$, and a null at $f_s/6$. In the figures, solid arrows have label $+1$ and dotted arrows have label -1.

Example 3: Canonical graph for null at dc.

In this case, $f = 0$, and $\omega = 1$ and $\mathscr{L}^f = \mathbb{Z}[\omega] \cong \mathbb{Z}$. The canonical graph G_0^0 with labeled state transitions given by (5.1) is shown in Fig. 11.

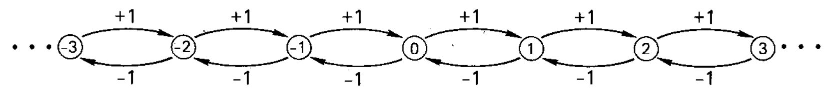


Fig. 11. G_0^0.

It is not hard to see that any FSTD contained in G_0^0 has bounded RDS, and so produces a spectral null at dc. Moreover, if H is an irreducible FSTD with binary labels and a null at dc, Theorem 2 shows that H satisfies a coboundary condition. Namely, there is a function ϕ: $\mathcal{S} \to \mathbb{C}$ from the states $\mathcal{S} = \{\sigma_0, \cdots, \sigma_l\}$ of H such that, for any transition from σ_i to σ_j,

$$a_{ij} = \varphi(\sigma_j) - \varphi(\sigma_i). \tag{5.3}$$

Since φ is defined only up to an additive constant, we can assume that $\varphi(\sigma) \in \mathbb{Z}$ for some and hence all states σ. Now, φ can be viewed as a map from the states $\mathcal{S}$ of H to the states of G_0^0. Since (5.3) is consistent with the transition rule (5.1) for G_0^0, the map φ naturally extends (by defining it on edges) to a label-preserving graph homomorphism from H into G_0^0. The proof of Proposition 1 will proceed along similar lines, using the results of Theorem 3.

Example 4: Canonical graphs for null at $f = f_s/2$. Here $\omega = -1$. Therefore, $\mathcal{L}_0^{f_s/2} \cong \mathbb{Z}[\omega] \cong \mathbb{Z}$. The two canonical graphs $G_0^{f_s/2}$ and $G_1^{f_s/2}$ are shown in Figs. 12 and 13. Note that $G_0^{f_s/2}$, the canonical graph for FSTD's with even period, is isomorphic to a subdiagram of $G_1^{f_s/2}$, shown by the heavy edges. We will discuss the general nesting relationship among canonical graphs G_p^f in Proposition 2.

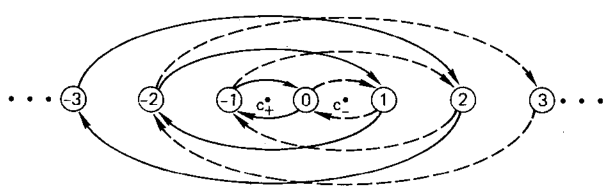


Fig. 12. $G_0^{f_s/2}$.

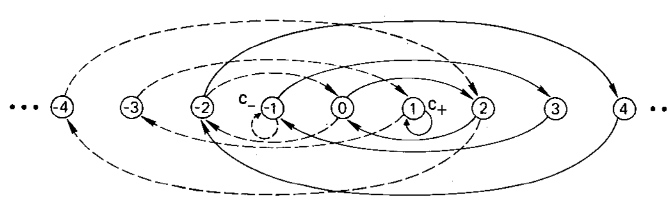
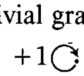


Fig. 13. $G_1^{f_s/2}$.

Note also that the trivial graph



has a null at $f_s/2$ but cannot collapse onto a subgraph of $G_0^{f_s/2}$ since $G_0^{f_s/2}$ has no self-loop.

Example 5: Canonical graph for null at $f = f_s/6$.

Here $\omega = (1/2) - i(\sqrt{3}/2)$. Figs. 14–16 show regions of the canonical graphs $G_p^{f_s/6}$, $p = 0, 1, 2$. Again, there are nesting relationships, with $G_0^{f_s/6}$, $G_1^{f_s/6}$, and $G_5^{f_s/6}$ isomorphic to a subdiagram of each of the canonical graphs with $p \geq 1$. In fact, these three diagrams are isomorphic. Also, $G_2^{f_s/6}$ and $G_4^{f_s/6}$ are isomorphic to subdiagrams of each other.

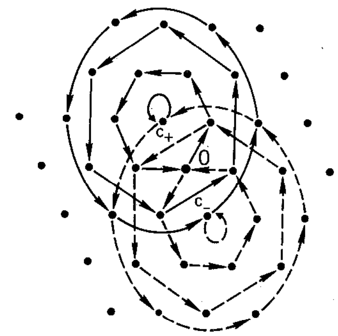


Fig. 14. $G_0^{f_s/6}$.

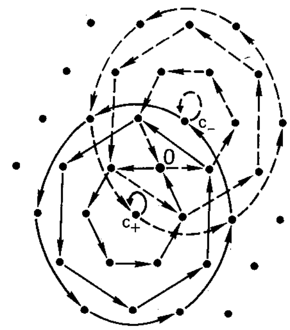


Fig. 15. $G_1^{f_s/6}$.

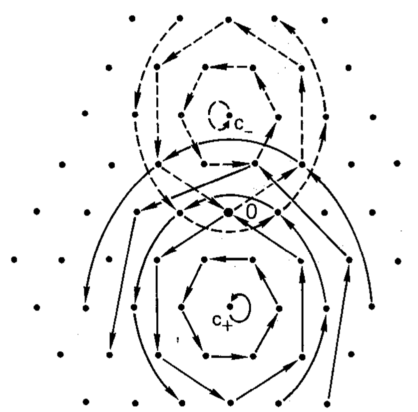


Fig. 16. $G_2^{f_s/6}$.

Proof of Proposition 1: 1) We define functions φ_p: $\mathcal{L}^f \to \mathbb{C}$ for $p = 0, \cdots, n-1$. For $p = 0$,

$$\varphi_0(\sigma) = \sigma \tag{5.4}$$

while for $p \neq 0$,

$$\varphi_p(\sigma) = \frac{\sigma}{\overline{\omega}^p - 1}. \tag{5.5}$$

For any FSTD G contained in G_p^f, the transition rules (5.1) and (5.2), together with (5.4) and (5.5), yield a coboundary condition at f:

$$b = \omega\varphi_p(h_p(\sigma, b)) - \varphi_p(\sigma) \tag{5.6}$$

for every state σ in G. By Theorem 3, G has a spectral null at f.

2) Let H be any FSTD which generates a null at f. The period of H is congruent, modulo n, to some p, with $0 \le p < n$. First, suppose $p = 0$. The solution to the coboundary equation $\chi(\sigma)$ defined in Theorem 3, (1) ⇒ (3), case 1, can be viewed as a map from states of H into states of G_0^f. The map χ naturally extends to a label-preserving directed graph homomorphism (by defining it on edges) from H into G_0^f because the coboundary equation is consistent with the transition rule (5.1). Now, if $p \neq 0$, then we use the mapping

$$\sigma \mapsto (\bar{\omega}^p - 1)\chi(\sigma)$$

(where χ is the solution to the coboundary equation in Theorem 3, (1) ⇒ (3), case 2) to define a graph homomorphism from H into G_p^f. This works because the coboundary is consistent with the transition rule (5.2).

Since the n-fold trellis presentation of any constrained system with a spectral null at f has period $p \equiv 0 \bmod n$, it follows from the above that the trellis collapses to a subdiagram of G_0^f.

The graphs G_p^f satisfy a natural nesting relationship, a special case of which we saw in Example 4. The general relationship is given by the following.

Proposition 2: G_p^f is label-preserving graph isomorphic to a subdiagram of G_q^f in either of the following situations: 1) $p = 0$ and q is arbitrary; 2) $q \neq 0$ and $q \equiv mp \bmod n$ for some m.

Proof: 1) The state mapping

$$\varphi: \sigma \mapsto (\bar{\omega}^q - 1)\sigma$$

extends to a graph homomorphism of G_0^f into G_q^f because the G_0^f transition rule (5.1) is translated into the G_q^f transition rule (5.2) by multiplication by $(\bar{\omega}^q - 1)$:

$$\begin{array}{ccc} \sigma & \xrightarrow{\varphi} & (\bar{\omega}^q - 1)\sigma \\ G_0^f - \text{transition} \downarrow & & \downarrow G_q^f - \text{transition} \\ \bar{\omega}(\sigma + b) & \xrightarrow[\varphi]{} & \bar{\omega}((\bar{\omega}^q - 1)\sigma + (\bar{\omega}^q - 1)b). \end{array}$$

Since we may assume $q \neq 0$ (otherwise the result is trivial), we have $\omega^q \neq 1$ (recall that $q < n$), and thus the homomorphism is $1-1$. Thus the mapping of G_0^f into G_q^f is actually a label-preserving graph isomorphism between G_0^f and a subdiagram of G_q^f.

2) If $q \equiv mp \bmod n$, then $\bar{\omega}^p - 1$ divides $\bar{\omega}^q - 1$ in $\mathbb{Z}[\omega]$. Specifically,

$$\begin{aligned} \bar{\omega}^q - 1 &= \bar{\omega}^{mp} - 1 \\ &= (\bar{\omega}^p - 1)(v_m(\bar{\omega}^p)) \end{aligned} \tag{5.7}$$

where $v_m(x) = x^{m-1} + x^{m-2} + \cdots + 1$.

The state map

$$\varphi: \sigma \mapsto \sigma v_m(\bar{\omega}^p)$$

extends to a graph homomorphism of G_p^f into G_q^f as before. The G_p^f transition rule (5.2), as the reader can check, is translated into the G_q^f transition rule (5.2) by multiplication by $v_m(\bar{\omega}^p)$. Since $\bar{\omega}^q \neq 1$ by assumption, we must also have that $v_m(\bar{\omega}^p) \neq 0$ by (5.7) and hence the mapping of G_p^f into G_q^f defines a label-preserving graph isomorphism of G_p^f into G_q^f.

Remark: Fix $f = kf_s/n$. If n is prime, then Proposition 2 implies that for each p, G_p^f imbeds in G_1^f. So, in this case, G_1^f is period p canonical for all p. From Example 5, the reader can check that this is not true for $n = 6$.

However, for general n, one can construct a *natural* graph G^f which is period p canonical for all p. Namely, the state set of the graph G^f is again $\mathbb{Z}[\omega]$. The transition rule is given by

$$h_p(\sigma, b) = \bar{\omega}(\sigma + \eta \cdot b)$$

where η is the least common multiple of $\{\bar{\omega}^p - 1\}_{1 \le p < n}$ in the ring $\mathbb{Z}[\omega]$. The point here is that each G_p^f imbeds in G^f. We leave it to the reader to check this. (This was pointed out to us by J. Ashley.)

The graph G_0^f can be used to investigate the maximum possible code rate for binary codes having RDS at f which is bounded by a constant c. Specifically, let $G_{0,c}^f \subset G_0^f$ be the subgraph obtained by restricting to the set of states $\mathcal{L}_{0,c}^f = \{v \in \mathcal{L}_0^f \,|\, |v| < c\}$, where the states v are viewed as complex numbers.

For $f = kf_s/n$, $n = 1,2,3,4,6$, the graph $G_{0,c}^f$ is an FSTD, the Shannon capacity of which defines the maximum possible code rate. This capacity can be computed for $n = 1$ and $n = 2$ using the formula derived by Chien [5]. For other values of n and c sufficiently large, $G_{0,c}^f$ is a countable state transition diagram. The Shannon capacity of these "disk systems" has been studied by Petersen [18] and more recently by Ashley [22].

There is a straightforward extension of the definition of a period p canonical diagram for a spectral null at a frequency f (Definition 8) to the case of simultaneous nulls. To understand the state sets of canonical diagrams for simultaneous nulls, we first describe the state set $\mathcal{L}^f = \mathbb{Z}[\omega]$, above, in a slightly different manner. Namely, $\mathbb{Z}[\omega] \approx \mathbb{Z}[x]/(g(x))$, where the latter is the "quotient ring" of $\mathbb{Z}[x]$, the ring of polynomials with integer coefficients in one indeterminate x, by the "ideal" generated by $g(x)$, the minimal polynomial of ω. Elements of $\mathbb{Z}[x]/(g(x))$ are equivalence classes of polynomials, where two polynomials are regarded as equivalent if they differ by a multiple (as polynomials) of $g(x)$. Letting $[u(x)]$ denote the equivalence class of a polynomial $u(x)$, the identification of $\mathbb{Z}[\omega]$ with $\mathbb{Z}[x]/(g(x))$ is given by $[u(x)] \leftrightarrow u(\bar{\omega})$. Each class $[u(x)]$ is represented by a polynomial $u(x)$ of degree less than $\phi(n) =$ order of ω.

From this point of view, the transition rule for G_p^f given in (5.1) and (5.2) takes the form

$$h_0([u(x)], b) = [x(u(x) + b)], \qquad \text{for } p = 0, \tag{5.8}$$

$$h_p([u(x)], b) = [x(u(x) + (x^p - 1)b)], \qquad \text{for } p = 1, \cdots, n-1. \tag{5.9}$$

Now, $G_p^{f_1, f_2}$, the period p canonical graph for nulls at both $f_1 = k_1 f_s / n_1$ and $f_2 = k_2 f_s / n_2$, has state set

$$\mathscr{L}^{f_1, f_2} = \mathbb{Z}[x]/\mathrm{lcm}(g_1(x), g_2(x));$$

that is, the quotient ring of $\mathbb{Z}[x]$ by the ideal generated by the least common multiple of $g_1(x)$ and $g_2(x)$, the minimal polynomials of $e^{-i2\pi k_1/n_1}$ and $e^{-i2\pi k_2/n_2}$. Setting $n = \mathrm{lcm}(n_1, n_2)$, we define state transition rules for $G_p^{f_1, f_2}$ exactly as in (5.8) and (5.9). Namely, use the rules (5.8) and (5.9) where $[u(x)]$ is the equivalence class of $u(x)$ defined by the relation that regards two polynomials as equivalent if they differ by a multiple (as polynomials) of $\mathrm{lcm}(g_1(x), g_2(x))$.

There are natural label-preserving graph homomorphisms π_1: $G_p^{f_1, f_2} \to G_p^{f_1}$ and π_2: $G_p^{f_1, f_2} \to G_p^{f_2}$, which coincide on the state sets with the natural ring homomorphisms $\mathbb{Z}[x]/\mathrm{lcm}(g_1(x), g_2(x)) \to \mathbb{Z}[x]/(g_1(x))$ and $\mathbb{Z}[x]/\mathrm{lcm}(g_1(x), g_2(x)) \to \mathbb{Z}[x]/(g_2(x))$. For a period p FSTD H with nulls at f_1, f_2, we can construct a commutative diagram of labeled graph homomorphisms as shown in Fig. 17, where ψ_1, ψ_2 are labeled graph homomorphisms defined in Proposition 1.

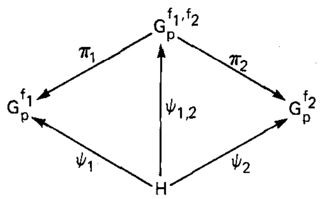


Fig. 17. Commutative diagram.

The generalization to more than two nulls is similar. By the universal map extension property of fibered products [16, p. 117] and by Proposition 1, we conclude the following.

Corollary 2: A period p canonical graph $G_p^{f_1, \cdots, f_l}$ for spectral nulls at $f_1, \cdots, f_l$ is the diagram with states $\mathscr{L}^{f_1, \cdots, f_l} = \mathbb{Z}[x]/\mathrm{lcm}(g_1(x), \cdots, g_l(x))$ and transition rules given by (5.8) or (5.9). (Here $g_i(x)$ is the minimal polynomial for the root of unity corresponding to f_i.)

Example 6: Canonical graph $(G_0^{0, f_s/2})$ for nulls at dc and $f/2$.

For $f_1 = 0$, $f_2 = f_s/2$, we have

$$\begin{aligned}\mathscr{L}^{0, f_s/2} &= \mathbb{Z}[x]/(x-1)(x+1)\\ &= \mathbb{Z}[x]/(x^2-1),\end{aligned}$$

which is a square planar lattice. The state transitions are shown in Fig. 18, with states labeled by corresponding elements of $\mathscr{L}^{0, f_s/2}$, and transition rule (5.8).

The representation shown in Fig. 18 points out that any string generated by a finite subdiagram of $G_0^{0, f_s/2}$ consists of two interleaved sequences, each having finite RDS at dc, and therefore a null at dc. In other words, by Corollary 2, any FSTD which has a spectral null at dc and $f/2$ must produce sequences which consist of two interleaved sequences, each having a null at dc. This observation is generalized in the following section.

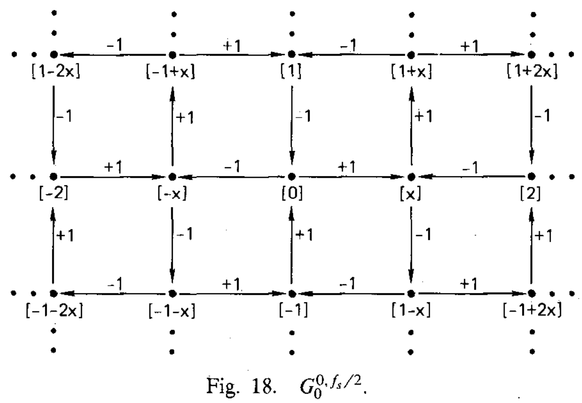


Fig. 18. $G_0^{0, f_s/2}$.

Remark: The developments here have concentrated on the binary alphabet $\{\pm 1\}$ because of its practical significance. The construction and results may be generalized in a straightforward manner to any finite alphabet contained in the rational numbers.

VI. Spectral Nulls Obtained by Interleaving

The possibility of generating sequences with spectral nulls by interleaving m sequences each with a spectral null at dc has been mentioned in [6], [7], and [19]. Generation of spectral nulls by interleaving sequences with spectral nulls at frequencies other than dc is described in [21].

In this section, we first calculate, by a simple application of the triangle inequality, the set of spectral nulls produced by interleaving m sequences, each having bounded RDS at f_s/n. We then apply the construction of canonical diagrams (and Proposition 1 of Section V) to prove that *any* FSTD which produces nulls at these frequencies must generate sequences which are obtained by interleaving m sequences each with bounded RDS at f_s/n.

Proposition 3: Suppose a sequence $\boldsymbol{a} = a_0 \cdots a_j \cdots$ consists of m interleaved sequences each with bounded RDS at f_s/n. Let $l_1, \cdots, l_{\phi(n)}$ be the $\phi(n)$ positive numbers less than and relatively prime to n. Then $\boldsymbol{a}$ has bounded RDS at frequencies

$$(pn + l_j)\frac{f_s}{mn}, \qquad 0 \le p \le m-1, 1 \le j \le \phi(n). \quad (6.1)$$

Proof: Write the sequence $\boldsymbol{a}$ as the interleaving of m sequences $\boldsymbol{b}_r$, $r = 0, \cdots, m-1$:

$$\boldsymbol{a} = b_{0,0} b_{1,0} \cdots b_{m-1,0} b_{0,1} b_{1,1} \cdots b_{m-1,1} \cdots. \quad (6.2)$$

Now

$$\mathrm{RDS}_{((pn+l_j)(f_s/mn))}(\boldsymbol{a}) = \sum_q a_q e^{-i2\pi q(pn+l_j)/mn} \quad (6.3)$$

$$= \sum_{r=0}^{m-1} \sum_s b_{rs} e^{-i2\pi(sm+r)(pn+l_j)/mn} \quad (6.4)$$

$$= \sum_{r=0}^{m-1} e^{-i2\pi r(pn+l_j)/mn} \sum_s b_{rs} e^{-i2\pi s(pn+l_j)/n} \quad (6.5)$$

$$= \sum_{r=0}^{m-1} \alpha_r \sum_s b_{rs} e^{-i2\pi s l_j/n} \quad (6.6)$$

where $\alpha_r = e^{-i2\pi r(pn+l_j)/mn}$.

By the triangle equality, we can upper bound $|\text{RDS}(\boldsymbol{a})|$ as follows:

$$|\text{RDS}(\boldsymbol{a})| \leq \sum_{r=0}^{m-1} \left| \alpha_r \sum_s b_{rs} e^{-i2\pi s l_j/n} \right|. \tag{6.7}$$

By assumption the sequences $\boldsymbol{b}_r$ have bounded RDS at f_s/n and so, by Lemma 3, they have bounded RDS at $l_j f_s/n$, $1 \leq j \leq \phi(n)$. So

$$\left| \sum_s b_{rs} e^{-i2\pi s l_j/n} \right| \leq c_r \tag{6.8}$$

for some constant c_r.

Since $|\alpha_r| = 1$, the bound on RDS($\boldsymbol{a}$) then becomes

$$|\text{RDS}_{(pn+l_j)f_s/mn}(\boldsymbol{a})| \leq \sum_{r=0}^{m-1} c_r, \tag{6.9}$$

giving our bound on the RDS.

The canonical diagram $\hat{G}_p$ for the set of nulls identified in Proposition 3 is determined by the least common multiple $\hat{g}(x)$ of minimal polynomials for the corresponding roots of unity. In fact, it is not hard to see that

$$\hat{g}(x) = g_1(x^m) \tag{6.10}$$

where $g_1(x)$ is the minimal polynomial of $e^{-i2\pi/n}$. To verify this, note that $e^{-i2\pi l_j/n}$, $1 \leq j \leq \varphi(n)$ is the full set of $\varphi(n)$ roots of $g_1(x)$, and

$$\begin{aligned}\left[e^{-i2\pi(pn+l_j)/mn}\right]^m &= e^{-i2\pi p} e^{-i2\pi l_j/n} \\ &= e^{-i2\pi l_j/n}.\end{aligned} \tag{6.11}$$

Therefore, $e^{-i2\pi(pn+l_j)/mn}$, $0 \leq p \leq m-1$, $1 \leq j \leq \varphi(n)$ are $m\varphi(n)$ distinct roots of $\hat{g}(x)$. Since

$$\begin{aligned}\text{degree } \hat{g}(x) &= m \text{ degree } g_1(x) \\ &= m\varphi(n),\end{aligned} \tag{6.12}$$

they form a full set of roots of $\hat{g}(x)$ as desired. Therefore, $\hat{G}_p$ has states $\hat{\mathscr{L}} = \mathbb{Z}[x]/(\hat{g}(x))$, and transition rule (5.8) or (5.9). Using the structure of $\hat{G}$ implied by (6.10), we now prove the converse to Proposition 3, thereby providing a frequency domain characterization of spectral null systems obtained by time-domain interleaving.

Theorem 4: If an FSTD (H, Γ) has nulls at the set of frequencies (6.1), then every sequence it generates consists of m interleaved sequences, each with bounded RDS at f_s/n.

Proof: By Corollary 2, it suffices to prove the theorem for an FSTD H contained in $\hat{G}_0$. Represent a sequence generated by H as in (6.2), namely,

$$\boldsymbol{a} = b_{0,0} b_{1,0}, \cdots, b_{m-1,0} b_{0,1} b_{1,1} \cdots b_{m-1,1} \cdots,$$

with the interleaved substrings called $\boldsymbol{b}_r$, $r = 0, \cdots, m-1$. Each state in $\hat{G}_0$ can be represented in a unique way as

$$[u(x)] = \left[u_0(x^m) + x u_1(x^m) + \cdots + x^{m-1} u_{m-1}(x^m)\right]$$

where $u(x)$ is a polynomial of degree less than $\varphi(n)m$, and each $u_i(x)$ is a polynomial of degree less than $\varphi(n)$. Since H contains a finite number of states, there are only finitely many polynomials $u_i(x)$ which can appear. The state transition corresponding to a symbol b is obtained by reference to (5.8):

$$[u(x)] \xrightarrow{b} \left[v_0(x^m) + x(u_0(x^m) + b) + x^2(u_1(x^m)) + \cdots + x^{m-1} u_{m-2}(x^m)\right]$$

where

$$v_0(x^m) = x^m u_{m-1}(x^m) \bmod g_1(x^m).$$

In other words, the polynomial representing the coefficient of x^0 is incremented by b, and the coefficient polynomials are then "shifted" cyclically, with the term $x^m u_{m-1}(x^m)$ getting reduced modulo $g_1(x^m)$ during the shift to take the position of the coefficient of x^0.

Now let $[u^0(x)], [u^1(x)], \cdots$ be a state sequence in H which generates a sequence $\boldsymbol{a}$. It then follows that, for fixed r, the sequence of coefficient polynomials

$$\{u_0^{pm+r}(x^m)\}, \qquad p = 0, 1, \cdots,$$

viewed as polynomials in $y = x^m$, describes a state sequence on $G_0^{f_s/n}$ that generates $\boldsymbol{b}_r$. Since only a finite number of states can occur in the sequence, it follows that $\boldsymbol{b}_r$ is a sequence generated by an FSTD contained in $G_0^{f_s/n}$, and therefore $\boldsymbol{b}_r$ has bounded RDS at f_s/n.

Example 7: Suppose the binary FSTD G generates a spectral null at $f_s/4$. By Lemma 3, it must generate a null at $3f_s/4$. By Theorem 4, each string produced by G must be composed of a pair of interleaved substrings, each having bounded RDS at $f_s/2$.

VII. Conclusions

This paper has investigated constrained channels with spectral nulls at rational submultiples of the channel symbol frequency f_s. For channels defined by a *finite state transition diagram* we prove that a necessary and sufficient condition for the channel to have a spectral null at $f = kf_s/n$ is that all channel sequences possess a uniformly bounded *finite running digital sum at* f (RDS_f), or, equivalently, that the channel satisfy a *coboundary condition at* f. The achievable sets of simultaneous nulls are then determined. For any set of spectral nulls, we define *period* p *canonical graphs* into which every period p channel with the prescribed set of nulls collapses via a labeled graph homomorphism induced by the coboundary condition. We use the canonical graphs to study the technique of generating spectral nulls by *interleaving* channels with a spectral null at dc. We prove that the existence of certain sets of spectral nulls is equivalent to an interleaved structure, in which the interleaved subchannels themselves possess specific spectral nulls. The canonical graphs also provide information about the Shannon capacity of channels with spectral nulls. This application is under investigation.

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