Error Event Characterization on 2-D ISI Channels

Ismail Demirkan, Paul H. Siegel, and Jack K. Wolf [1]
Dept. of Electrical and Computer Engineering
University of California, San Diego
9500 Gilman Dr, La Jolla, CA 92093-0401 USA
e-mail: {demirkan, psiegel, jwolf}@ucsd.edu

***Abstract*— For one-dimensional (1-D) recording channels, the detector performance can be assessed by the distance properties of the channel, which are defined in terms of the maximum-likelihood decoding trellis. Error events with minimum and near minimum distances play an important role in the performance of the recording system particularly at moderate to high signal-to-noise ratio (SNR). In this paper, we analyze the distance properties of two-dimensional (2-D) inter-symbol interference channels, in particular the 2-D PR1 channel which is an extension of the 1-D PR1 channel. The minimum distance of this channel is proved to be 2 and a complete characterization of the distance-2 error events is provided. Also, the error events with squared-Euclidean distance 6 are partially characterized. Analogous to 1-D channels, error-state diagrams for 2-D channels can be constructed to characterize the error events. We propose an efficient error event search algorithm operating on the error-state diagram that is applicable to any 2-D channel.**

I. Introduction

Detection and coding for 2-D inter-symbol interference (ISI) channels have been the subject of much research recently because of advances in holographic storage technology. A generalization of 1-D detection and coding methods to 2-D channels is not trivial due to the lack of convenient graph-based descriptions of such channels. In particular, there is no simple trellis-based maximum-likelihood detection algorithm analogous to the 1-D Viterbi algorithm. Signal processing and coding aspects of holographic storage systems have been studied by several authors [1], [2], [3].

However, there are suboptimal detection techniques such as the iterative multi-strip (IMS) algorithm for 2-D ISI channels that demonstrate very good error-rate performance and appear to approximate the performance of a maximum-likelihood (ML) detector [4]. The IMS algorithm is a message-passing algorithm operating on soft-input, soft-output detectors, such as the maximum a posteriori (MAP) detector. It is therefore of interest to identify the dominant 2-D error events, where we define an error event as the difference between the recorded and the decoded data arrays. In the 2-D setting, error events can be classified as closed or open depending on whether the area of the smallest square region containing nonzero differences is bounded or unbounded. Empirical evidence has shown that data arrays forming dominant error events for the IMS algorithm generate channel outputs with small squared-Euclidean distance. Therefore, it is important to characterize the 2-D error events with small squared-Euclidean distance, so that 2-D distance-enhancing constrained codes can be designed to improve system performance [3]. The design of 2-D distance-enhancing constrained codes will not be elaborated upon this paper.

Chugg investigated the performance of an ML page detector in the presence of ISI and additive white Gaussian noise [5]. If the channel impulse response has finite support size, then the channel output can be characterized as a Markov random field. The bit error rate performance of an ML page detector can be bounded from above by a union bound, which is computed by using the fundamental error patterns in the channel input arrays. In this work, we refer to fundamental error events as connected error events. In Section V, we compute some terms of the union bound for the 2-D PR1 channel.

Karabed *et al.* introduced an analytic method to characterize the distance properties of some 1-D partial-response channels [6]. In this paper, we extend this method to characterize the closed error events of some 2-D ISI channels. In particular, we study the 2-D PR1 channel whose impulse response is given by

$$h = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}. \tag{1}$$

This impulse response is observed to be a good ISI model for holographic storage systems when there is a half-period sampling shift between the read-back signal and detector.

The analytic method used for characterizing error events for the 2-D PR1 channel is tedious to apply for most 2-D channels, particularly for the channels whose impulse responses span $p \times q$ arrays where $p, q > 2$. For 1-D ISI channels, efficient search algorithms working on error-state diagrams have been developed to characterize error events for high-order partial response channels [7], [8]. Error-state diagrams for 2-D ISI channels can be generated by fixing the size of the error event in the horizontal or vertical direction. In this paper, we propose a bounded depth-search algorithm for finding closed and connected error events for any 2-D ISI channel. The complexity of the algorithm solely depends on the underlying 2-D ISI channel.

The organization of the paper is as follows. In Section II, we present a 2-D ISI channel model and define its distance properties. In Section III, the characterization of minimum and near-minimum distance error events of the 2-D PR1 channel is

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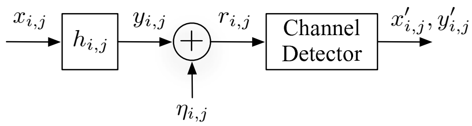

Fig. 1. A 2-D ISI channel model.

investigated by studying the channel impulse response in the spectral domain. By generalizing this concept, some distance properties of the channels other than the 2-D PR1 channel can be found.

The method of precoding is commonly used in 1-D recording channels to invert the ISI effect of the channel. In Section IV, we discuss the effect of a precoding scheme on error events for the 2-D PR1 channel. Section V discusses the union bounds for the unprecoded and precoded 2-D PR1 channel. Error-state diagrams and a bounded depth-search algorithm are introduced in Section VI. Analytical results are compared with the computed results for the 2-D PR1 channel.

II. The 2-D ISI Channel

Consider a 2-D ISI channel with bipolar input array $x = \{x_{i,j}\}$, channel impulse response $h = \{h_{i,j}\}_{i=0,j=0}^{p-1,q-1}$, and output $y = x * h$ (see Fig. 1). Additive white Gaussian noise $\eta = \{\eta_{i,j}\}$ with zero mean and variance σ^2 is added to the channel output array to obtain the received array $r = \{r_{i,j}\}$.

For a channel output array y and its estimated array y', the normalized squared-Euclidean distance is defined as

$$d^2(y, y') \triangleq \sum_{i,j} [(y_{i,j} - y'_{i,j})/2]^2$$

which is taken to be ∞ if the sum is unbounded. Normalized squared-Euclidean distances will be referred as *squared distances*. The quantity $d^2(y, y')$ can be expressed in terms of the corresponding input arrays x and x', respectively,

$$d^2(y, y') = d^2(\epsilon * h, 0)$$

where $\epsilon = (x - x')/2$ is the normalized channel input error array. The normalized channel input error arrays are called *error events*, whose elements are commonly represented by the symbols $\{0, +, -\}$. The input arrays x and x' are called *supporting arrays* of ϵ. The *distance* of an error event ϵ is defined as $d(\epsilon) \triangleq \sqrt{d^2(\epsilon * h, 0)}$.

Analogous to the 1-D channels, the error events for 2-D channels are classified as either open or closed. An error event is *closed* if the area of the smallest square region containing nonzero differences is bounded. Error events which are not closed are called *open*. Let $\mathcal{E}_{\texttt{closed}}$ be the set of closed error events, $\mathcal{E}_{\texttt{open}}$ be the set of open error events, and $\mathcal{E} = \mathcal{E}_{\texttt{closed}} \cup \mathcal{E}_{\texttt{open}}$ be their union. We define the *minimum closed event distance*

$$d_{<>} = \min_{\epsilon \neq 0, \epsilon \in \mathcal{E}_{\texttt{closed}}} d(\epsilon)$$

and the *minimum event distance*

$$d_{<} = \min_{\epsilon \neq 0, \epsilon \in \mathcal{E}} d(\epsilon).$$

1-D sequences are often represented in the D-transform domain, which is equivalent to the z-transform where $D = z^{-1}$. Likewise, 2-D arrays can be represented in the (D, E)-transform domain. For an array x, the (D, E)-transform of x is defined as

$$x(D, E) \triangleq \sum_{i,j} x_{i,j} D^i E^j.$$

In this representation, the impulse response of the 2-D PR1 channel is given by

$$h(D, E) = 1 + D + E + DE$$

and the channel input-output relationship becomes

$$y(D, E) = x(D, E)h(D, E). \tag{2}$$

The minimum closed event distance of a 2-D ISI channel can be expressed as

$$d_{<>} = \min_{\epsilon(D,E) \neq 0,\ \epsilon \in \mathcal{E}_{\texttt{closed}}} \|h(D, E)\epsilon(D, E)\|$$

where

$$\epsilon(D, E) = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \epsilon_{i,j} D^i E^j, \quad \epsilon_{i,j} \in \{-1, 0, +1\} \tag{3}$$

is the polynomial corresponding to an error event $\epsilon = \{\epsilon_{i,j}\}_{i=0,j=0}^{m-1,n-1}$ such that the edges of ϵ, $\epsilon_{0,*}$, $\epsilon_{*,0}$, $\epsilon_{m-1,*}$, $\epsilon_{*,n-1}$ contain at least one non-zero entry.

Here, we have defined error events for 2-D channels in a way analogous to 1-D channels. However, error events for 2-D channels cannot be interpreted as differences of paths in the decoding trellis, since such trellis representations do not exist for 2-D channels.

III. Error Event Characterization on the 2-D PR1 Channel

The minimum and near-minimum distance error events can be characterized by studying spectral properties of the channel transfer function and the corresponding limitations on error coefficients, $\{\epsilon_{i,j}\}$ [6, Sec. III.A]. Using this method, the minimum distance error events (distance-2) are now completely characterized for the 2-D PR1 channel. In addition, the error events with squared distance 0 and 6 are partially characterized.

A. Minimum Distance Error Events

Proposition 1: The minimum closed event distance of the 2-D PR1 channel is 2. All distance-2 closed error events are of the form

$$\epsilon = \begin{bmatrix} + & - & \cdots & \epsilon_{0,n-1} \\ - & + & & \\ \vdots & & & \vdots \\ \epsilon_{m-1,0} & & \cdots & \epsilon_{m-1,n-1} \end{bmatrix} \tag{4}$$

and their negatives. Here $\epsilon_{m-1,0} = +$ ($\epsilon_{0,n-1} = +$) if m (n) is odd; otherwise $\epsilon_{m-1,0} = -$ ($\epsilon_{0,n-1} = -$). The bottom right entry is determined as $\epsilon_{m-1,n-1} = \epsilon_{m-1,0}\epsilon_{0,n-1}$.

The proof of the proposition is based on the expansion of (2) using the definition of closed error arrays (3). The details of the proof will not be given here.

Example 1: Flipping one bit in the input array changes the four adjacent locations at the channel output by 1. Therefore one of the minimum distance error events is $[+]$ (or $[-]$). Some other distance-2 error events are

$$[+-], \begin{bmatrix} + \\ - \end{bmatrix}, [+-+], \begin{bmatrix} + \\ - \\ + \end{bmatrix}, \begin{bmatrix} + & - \\ - & + \end{bmatrix}$$

and their negatives.

There are only two supporting arrays of a distance-2 error event. Therefore, the larger the error event area, the less probable it is to encounter that error event. More precisely, the probability of having a minimum distance error event of size $m \times n$ is 2^{-mn+1}.

B. Open Error Events

Proposition 1 implies that all error events with $d < 2$ are open error events. This class of open error events is particularly important for an ML detector using a finite window size which is smaller than the decoding page size. In this case, the distance between some ambiguous arrays can be less than 2, which makes the detector performance worse. Here we give some examples of open error events with squared distance 0, 1, 2 and 3.

Example 2: The channel output error array corresponding to an open error event is given by

$$y(D,E) = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} (\epsilon_{i,j}+\epsilon_{i-1,j}+\epsilon_{i,j-1}+\epsilon_{i-1,j-1}) D^i E^j.$$

In order to obtain a distance-0 error event, each term in this sum has to be zero. Therefore the error coefficients for each term have to be one of the following combinations

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} + & 0 \\ 0 & - \end{bmatrix}, \begin{bmatrix} + & 0 \\ - & 0 \end{bmatrix}, \begin{bmatrix} + & + \\ - & - \end{bmatrix}, \begin{bmatrix} + & - \\ - & + \end{bmatrix}$$

and their rotations and reflections.

Some distance-0 error events can be obtained by tiling the plane with the same pattern vertically and horizontally. Examples of such patterns include

$$\begin{bmatrix} + & 0 \\ 0 & - \end{bmatrix}, \begin{bmatrix} + & - \\ - & + \end{bmatrix}, \begin{bmatrix} + & + & - & + \\ - & - & + & - \end{bmatrix}. \tag{5}$$

One of the supporting arrays of the left-most error event in (5) can be obtained by tiling the plane with the following pattern

$$\begin{bmatrix} +1 & +1 \\ +1 & -1 \end{bmatrix}.$$

Example 3: A distance-1 open error event can be obtained by tiling one quadrant of the plane with the following pattern

$$\begin{bmatrix} + & - \\ - & + \end{bmatrix}. \tag{6}$$

Similarly, an error event with squared distance 2 can be obtained by tiling the region $[a, \infty) \times [b_1, b_2]$ with the same pattern where $a, b_1, b_2 \in \mathbb{Z}$, $b_2 > b_1$ and $b_2 - b_1$ is even. Note that tiling a bounded rectangle with this pattern gives a minimum distance closed error event. Also note that tiling a half plane with this pattern gives a distance-0 open error event.

Example 4: An error event with squared distance 3 is given by the following form:

$$\epsilon = \begin{bmatrix} + & - & + & - & \cdots \\ - & 0 & 0 & 0 & \cdots \\ + & 0 & & & \\ - & 0 & & \epsilon_1 & \\ \vdots & \vdots & & & \end{bmatrix}$$

where ϵ_1 is a distance-1 error event.

C. Error Events with Squared Distance 6

We now describe two classes of error events with squared distance 6. The first class is obtained by combining two distance-2 error events ϵ_1 and ϵ_2 in the following way

$$\begin{bmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{bmatrix} \tag{7}$$

where the bottom right corner of ϵ_1 is the negative of the top left corner of ϵ_2. The second error event class has the following form

$$\begin{bmatrix} \epsilon_1 & \epsilon_2 \\ 0 & \epsilon_3 \end{bmatrix} \tag{8}$$

where ϵ_1, ϵ_2 and ϵ_3 are distance-2 error events such that there are no adjacent $+$'s and no adjacent $-$'s at the error event boundaries, horizontally and vertically.

The following patterns are two examples of error events in the first and the second classes, respectively:

$$\begin{bmatrix} + & 0 \\ 0 & - \end{bmatrix}, \begin{bmatrix} + & - \\ 0 & + \end{bmatrix}.$$

D. Extension to Other 2-D Channels

The method mentioned for the 2-D PR1 channel can be extended to prove the following results for general 2-D ISI channels.

Proposition 2: A general 2-D channel with impulse response $h = \{h_{i,j}\}_{i=0,j=0}^{1,1}$ achieves the matched-filter bound; i.e., the minimum closed event distance of this channel is given by

$$d_{<>} = \sqrt{h_{0,0}^2 + h_{0,1}^2 + h_{1,0}^2 + h_{1,1}^2}.$$

Proposition 3: For a general 2-D channel with impulse response $h = \{h_{i,j}\}_{i=0,j=0}^{p-1,q-1}$, the minimum closed event distance can be bounded from below by

$$d_{<>} \geq \sqrt{h_{0,0}^2 + h_{p-1,0}^2 + h_{0,q-1}^2 + h_{p-1,q-1}^2}.$$

The proofs of Propositions 2 and 3 will not be given here.

IV. The Effect of Precoding on the 2-D PR1 Channel

The precoded 2-D ISI channel model is shown in Fig. 2. Let u be a binary unconstrained user data array at the input to the precoder. A precoder complementing the effect of the 2-D PR1 channel is given by

$$a_{i,j} = u_{i,j} \oplus a_{i-1,j} \oplus a_{i,j-1} \oplus a_{i-1,j-1}.$$

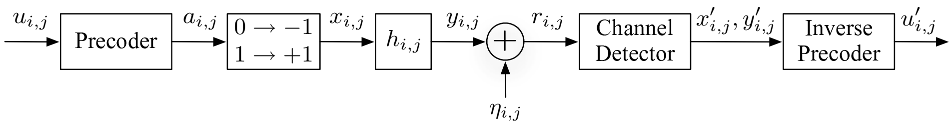

Fig. 2. Precoded 2-D ISI channel model.

The channel input array is obtained from a using the binary-to-bipolar conversion $x_{i,j} = 2a_{i,j} - 1$. We now discuss threshold and the IMS detectors for the precoded 2-D PR1 channel.

The user data array u is directly related to the channel output array as follows

$$\frac{y_{i,j}}{2} \equiv u_{i,j} \pmod 2. \tag{9}$$

Threshold detection provides y' using the received array r. The user data array can be estimated by using (9) as follows

$$u'_{i,j} = \begin{cases} 0, & y'_{i,j} = -4, 0, 4 \\ 1, & y'_{i,j} = -2, 2. \end{cases}$$

If the threshold detector makes an error, i.e., $y' \neq y$, then it is directly reflected in u'.

The IMS algorithm directly provides soft information for x'. After hard decisions are made for x', u' is given by

$$u'_{i,j} = a'_{i,j} \oplus a'_{i-1,j} \oplus a'_{i,j-1} \oplus a'_{i-1,j-1} \tag{10}$$

where $a'_{i,j} = (1 + x'_{i,j})/2$.

For a precoded system, a *user data error event* ϵ_u is defined as the difference between a given and estimated user data array; i.e., $\epsilon_u = u - u'$. In fact, ϵ_u and $\epsilon = (x - x')/2$ are related to each other according to the following relationship.

Proposition 4: $\epsilon_u = \mathrm{sgn}(\epsilon * h)[\epsilon * h \pmod 2]$, where the multiplication between $\mathrm{sgn}(\epsilon * h)$ and $[\epsilon * h \pmod 2]$ is element-wise.

The proof of the proposition is based on (10) and will not be discussed here.

A $m \times n$ distance-2 error event ϵ in a channel input array corresponds to the following $(m+1) \times (n+1)$ error event in the user data array

$$\epsilon_u = \begin{bmatrix} \epsilon_{0,0} & 0 \cdots & \epsilon_{0,n-1} \\ 0 & 0 & \\ \vdots & & \vdots \\ \epsilon_{m-1,0} & \cdots & \epsilon_{m-1,n-1} \end{bmatrix}.$$

The Hamming weight of ϵ_u is always 4 due to the one-to-one relationship between y and u given in (9). Likewise, for all error events with squared distance 6, the corresponding user data error events have constant Hamming weight 6.

Example 5: The following are two examples of user data error events corresponding to error events with squared distance 4 and 6, respectively.

$$\epsilon_u = \begin{bmatrix} + & 0 & - \\ 0 & 0 & 0 \\ - & 0 & + \end{bmatrix} \leftrightarrow \epsilon = \begin{bmatrix} + & - \\ - & + \end{bmatrix},$$

$$\epsilon_u = \begin{bmatrix} + & + & 0 \\ + & 0 & - \\ 0 & - & - \end{bmatrix} \leftrightarrow \epsilon = \begin{bmatrix} + & 0 \\ 0 & - \end{bmatrix}.$$

V. The Probability of Error

Chugg [5] proved that if the entries of input arrays x (or u for the precoded case) are equally probable and independent, the bit error probability under ML detection can be bounded from above by the union bound:

$$P_b \leq \sum_d K_d Q\left(\frac{d}{2\sigma}\right).$$

Here $Q(\cdot)$ is the complementary distribution function of a zero-mean and unit variance Gaussian random variable, and K_d is the average multiplicity of bit errors of distance d, given by

$$K_d = \sum_{\epsilon : d(\epsilon) = d} w(\epsilon) 2^{-w(\epsilon)},$$

where $w(\epsilon)$ is the Hamming weight of ϵ. The error events counted in this upper bound have to be *connected*, as defined below, and only one of the different shifts of ϵ should be taken into account.

Definition 1: An error event ϵ is *connected* if the error event cannot be divided into two separate error events ϵ_1 and ϵ_2 such that $d^2(\epsilon) = d^2(\epsilon_1) + d^2(\epsilon_2)$. The error events which are not connected are called *disconnected*.

Connected error events are referred to as *fundamental error patterns* in [5]. A condition for connected error events is also given in that paper.

Example 6: The following error events are two examples of connected and disconnected error events for the 2-D PR1 channel, respectively:

$$\begin{bmatrix} + & - & + \\ 0 & 0 & 0 \\ + & - & + \end{bmatrix}, \begin{bmatrix} + & 0 & + \\ 0 & - & 0 \\ + & 0 & + \end{bmatrix}.$$

For a precoded system, K_d becomes

$$K_d = \sum_{\epsilon : d(\epsilon) = d} w(\epsilon_u) 2^{-w(\epsilon)},$$

where ϵ_u is the user data error event corresponding to ϵ.

The bit error multiplicity generating function is defined as

$$g_b(z) = \sum_{d \in \mathcal{D}} K_d z^{d^2}$$

where $\mathcal{D}$ is the set of all distances, and K_d and z take non-negative values.

Example 7: For the 2-D PR1 channel without precoding and with precoding, K_2's are given by

$$K_2 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} mn 2^{-mn} \approx 4.052, \tag{11}$$

$$K_2 = 4 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} 2^{-mn} \approx 6.4264, \tag{12}$$

respectively. There are no simple closed form expressions for the K_2 expressions. The ratio test indicates that both sums converge. The approximate value of the former, shown in (11), was obtained numerically by summing over terms with $m, n \leq 100$. The latter is four times the Erdös-Borwein constant [9] and an approximate value is shown in (12). Note that, the coefficient K_2 is larger for the precoded system. Therefore, precoding increases the bit-error multiplicity of the minimum distance error events.

VI. A Bounded Depth-Search Algorithm

Error-state diagrams for 1-D channels cannot be directly generalized to 2-D channels since there are no convenient graph-based descriptions of such channels. However, when the size of the error event is fixed in either of the dimensions, error-state diagrams can be described as 1-D systems using a higher order alphabet. In this section, we propose a bounded depth-search algorithm for determining closed error events of size $m \times n$ for 2-D ISI channels with impulse response of size $p \times q$. To reduce the complexity of the algorithm, the following conditions are imposed on error events: (1) the edges of a error event contain at least one non-zero element and (2) error events are required to be connected.

A *state* σ in a error-state diagram is a sequence of $p-1$ symbols, $(e_1, \cdots, e_{p-1})$, from the alphabet Σ, which is the set of all row vectors of length n with entries $\{0, +, -\}$. Note that the memory of the equivalent 1-D channel is $p-1$. Each state can be represented as a $(p-1) \times n$ matrix whose row vectors are $(e_1, \ldots, e_{p-1})$. Therefore there are $3^{(p-1)n}$ states in the error-state diagram. An edge e has the initial state $\sigma(e) = (e_1, \cdots, e_{p-1})$ and the terminal state $\tau(e) = (e_2, \cdots, e_p)$. A closed error event of size $m \times n$ corresponds to the path $e_0, \cdots, e_{m+p-1}$ in the error-state diagram that starts and ends at the all-zero state $\sigma = (z, \cdots, z)$ without an intermediate visit to that state, where z is the all-zero row vector of length n. The algorithm searches for the connected error events of size $m \times n$ whose distances are not larger than a specified limit d_{max}.

Let $\rho^{(l-1)}$ be a path of length $l < m + p - 1$ ending with the state σ. If a row vector $e \in \Sigma$ is appended to this path, the algorithm checks the following conditions on the new path $\rho^{(l)} = (\rho^{(l-1)}, e)$:

- The distance associated with the new path satisfies $d^2(\rho^{(l)}) \leq d_{max}^2$.
- When $l = m$, the error event associated with $\rho^{(l)}$ contains at least one non-zero entry along its edges.
- $\rho^{(l)}$ gives a connected error event.

If any of these checks fails, then the algorithm will not extend the path $\rho^{(l)}$. Further details of the algorithm will be given elsewhere.

TABLE I

The Number of Error Events for the 2-D PR1 Channel

	$d^2=4$						$d^2=6$					
$m \times n$	1	2	3	4	5	6	1	2	3	4	5	6
1	2	2	2	2	2	2	0	0	0	0	0	0
2	2	2	2	2	2	2	0	12	24	36	48	60
3	2	2	2	2	2	2	0	24	48	72	96	120
4	2	2	2	2	2	2	0	36	72	108	144	180
5	2	2	2	2	2	2	0	48	96	144	192	240
6	2	2	2	2	2	2	0	60	120	180	240	300

Example 8: Table I shows the number of error events for the 2-D PR1 channel for $d^2 = 4, 6$ and $m, n \leq 6$. For $d^2 = 4$, the results of the search algorithm are consistent with the analytical results above. For $d^2 = 6$, the number of error events of the forms (7) and (8) are $4(m-1)(n-1)$ and $8(m-1)(n-1)$, respectively. The number of distinct error events that the algorithm produces is exactly $12(m-1)(n-1)$ for $m, n \leq 6$. This suggests that there may be no other error events with squared distance 6.

Using the data given in Table I, the bit error multiplicity generating function can be bounded from below by

$$g_b(z) \geq 3.800z^4 + 34.73z^6,$$

$$g_b(z) \geq 6.301z^4 + 43.30z^6$$

for the unprecoded and precoded cases, respectively. Here "$\geq$" signifies that the coefficient of each term on the right is less than the corresponding one on the left. The computed values of K_2's for both cases are close to the analytical values.

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