Sliding-Block Decodable Encoders Between (d, k)-Constrained Systems of Equal Capacity*

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Abstract — **We determine the pairs of (d,k)-constrained systems, $S(d,k)$ and $S(\hat{d},\hat{k})$, of equal capacity, for which there exists a rate 1:1 sliding-block decodable encoder from $S(d,k)$ to $S(\hat{d},\hat{k})$. Whenever such an encoder exists, we explicitly describe one such encoder and its corresponding sliding-block decoder.**

I. Introduction

Given non-negative integers d, k, with $d < k$, we say that a binary sequence is (d,k)-constrained if every run of zeros has length at most k and any two successive ones are separated by a run of zeros of length at least d. A (one-dimensional) (d,k)-*constrained system* is defined to be the set of all finite-length (d,k)-constrained binary sequences. The above definition is also extended to the case $k = \infty$ by not imposing an upper bound on the lengths of zero-runs.

The *capacity* of a (d,k)-constrained system, $S(d,k)$, is defined as $C(d,k) = \lim_{n\to\infty} \frac{1}{n}\log_2 q_{d,k}(n)$, where $q_{d,k}(n)$ is the number of length-n sequences in $S(d,k)$. It is well known that for all $d \geq 1$, we have the identities $C(d,2d) = C(d+1,3d+1)$ and $C(d,\infty) = C(d-1,2d-1)$. Repeatedly applying these identities also yields the chain of equalities $C(1,2) = C(2,4) = C(3,7) = C(4,\infty)$. In [2], it was shown that no other equalities exist among the capacities $C(d,k)$.

Given a pair of (d,k)-constrained systems, $S(d,k)$ and $S(\hat{d},\hat{k})$, with the same capacity, a question that naturally arises in the context of constrained coding is whether or not there exists a rate 1:1, finite-state encoder from $S(d,k)$ to $S(\hat{d},\hat{k})$ that is sliding-block decodable (*cf.* [1] for the relevant definitions). The main result of this paper is the following theorem, which completely resolves this question.

Theorem 1 *Let $S(d,k)$ and $S(\hat{d},\hat{k})$ be such that $C(d,k) = C(\hat{d},\hat{k})$. Then, there exists a rate 1:1, sliding-block decodable, finite-state encoder from $S(d,k)$ to $S(\hat{d},\hat{k})$ if and only if one of the following conditions holds:*

1: $(d,k) = (0,1)$ *and* $(\hat{d},\hat{k}) = (1,\infty)$

2: $(d,k) = (d,2d)$ *and* $(\hat{d},\hat{k}) = (d+1,3d+1)$, $d \geq 1$

3: $(d,k) = (d,\infty)$ *and* $(\hat{d},\hat{k}) = (d-1,2d-1)$, $d \geq 1$

4: $(d,k) = (1,2)$ *and* $(\hat{d},\hat{k}) = (3,7)$.

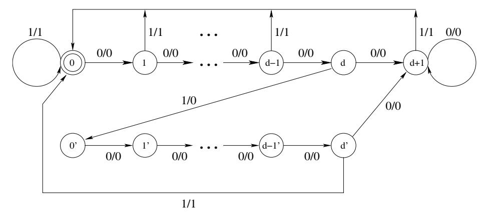
In Section II, we show the sufficiency of each of the above conditions by explicitly describing rate 1:1 finite-state encoders and sliding-block decoders in each case. The necessity of one of conditions 1–4 above follows from results from the symbolic dynamics literature, and the reader is referred to our full paper [3] for the details.

*This work was supported by Applied Micro Circuits Corporation, the UC Discovery Grant Program, and the Center for Magnetic Recording Research at UC San Diego.

II. Existence of Encoders

A rate 1:1 finite-state encoder from $S(0,1)$ onto $S(1,\infty)$ that is trivially sliding-block decodable is obtained by mapping the symbols 0 and 1 to their respective complements.

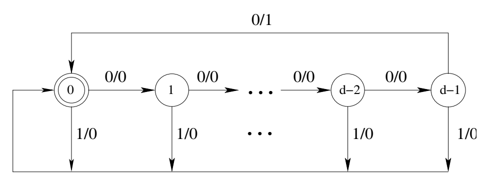
The following figure depicts a rate 1:1 encoder from $S(d,2d)$ to $S(d+1,3d+1)$.



A sliding-block decoder for the above encoder, with memory $d+1$ and anticipation d, is defined via the following rule:

$$\mathcal{D}(y_{i-(d+1)},\ldots,y_{i+d}) = \begin{cases} 1 & \text{if } y_{i-(d+1)} = 1, \text{ and} \\ & y_{i-d} = \cdots = y_{i+d} = 0 \\ y_i & \text{otherwise.} \end{cases}$$

A rate 1:1 encoder from $S(d,\infty)$ to $S(d-1,2d-1)$ is shown in the following figure.



The rule given below defines a sliding-block decoder for the above encoder, with memory d and anticipation $d-1$:

$$\mathcal{D}(y_{i-d},\ldots,y_{i+d-1}) = \begin{cases} 0 & \text{if } y_{i-d} = y_i = 1, \text{ and} \\ & y_j = 0,\ j \neq i-d,\ i \\ y_i & \text{otherwise.} \end{cases}$$

Finally, a rate 1:1, sliding-block decodable encoder from $S(1,2)$ to $S(3,7)$, is obtained by concatenating the encoders guaranteed by Condition 2 of Theorem 1 for the cases $d = 1$ and $d = 2$.

References

[1] B.H. Marcus, R.M. Roth and P.H. Siegel, "Constrained Systems and Coding for Recording Channels," in *Handbook of Coding Theory*, R. Brualdi, C. Huffman and V. Pless, Eds., Amsterdam, The Netherlands: Elsevier, 1998.

[2] N. Kashyap and P.H. Siegel, "Equalities among capacities of (d,k)-constrained systems," *SIAM J. Discrete Math.*, vol. 17, no. 2, pp. 276–297, 2003.

[3] N. Kashyap and P.H. Siegel, "Sliding-block decodable encoders between (d,k) runlength-limited constraints of equal capacity," to appear in *IEEE Trans. Inform. Theory*, June 2004.