On the Design of Finite-State Shaping Encoders for Partial-Response Channels

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Motivation

- **Consider binary-input ISI channels with AWGN**
  - *Capacity can be achieved with Markov sources (Chen and Siegel, 2004)*

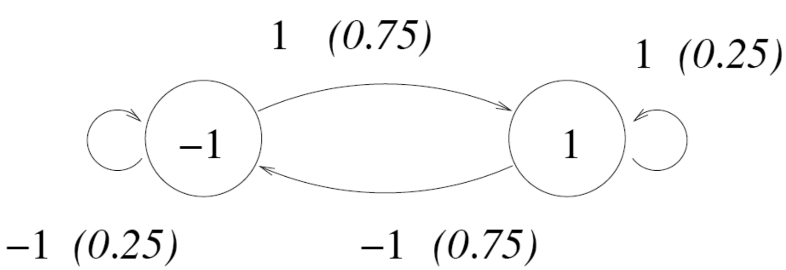
$$Y_t = \sum_{i=0}^{\nu} h_i X_{t-i} + N_t$$

- **Serially-concatenated codes can approach capacity**
  - **Inner finite-state "shaping" encoder modulates** i.i.d. equiprobable inputs to give an *optimal distribution for channel*
  - **Outer parity-check code ensures reliable communication**
  - *(Kavcic, Varnica, Ma, 2002; Doan and Narayanan, 2003; Soriaga and Siegel, 2003/04)*

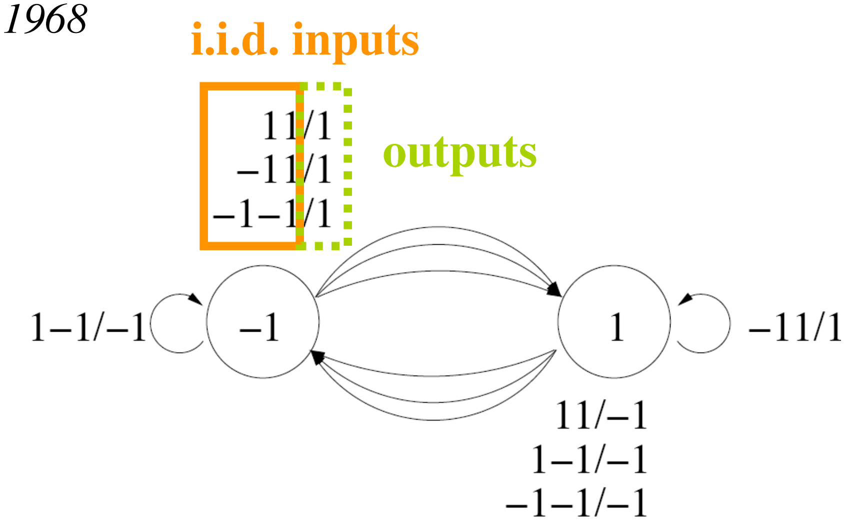
General Design of Shaping Encoders

Example: Gallager Construction

Information Theory and Reliable Communication, 1968



Target Process



Rate 2:1 Shaping Encoder

- Rate *k:1* encoder design can approximate any source
- **Using large *k* is inefficient**

Overview

Limits for designing *shaping encoders* to **minimize divergence rate** are related to the **capacity of cost-constrained channels**

- **Introduction**
  - Minimizing divergence rate between Markov sources
  - Capacity of cost-constrained noiseless channels
- **Analysis of divergence rate for shaping encoders**
  - *Method 1:* Encoders from a **modified Gallager construction**
  - *Method 2:* Encoders from **constraint graphs for typical sequences**
  - Numerical results

Minimizing Divergence Rates **and** *Coding for Channels with Cost Constraints*

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Designing Shaping Encoders to Minimize Kullback-Leibler Divergence Rate

- Consider a **target process** with **distribution** *P*
- Let the **encoder output process** have **distribution** *Q*
- Design encoder to **minimize divergence rate** *D(Q||P)*
  Encoder output process "resembles" the optimal distribution P

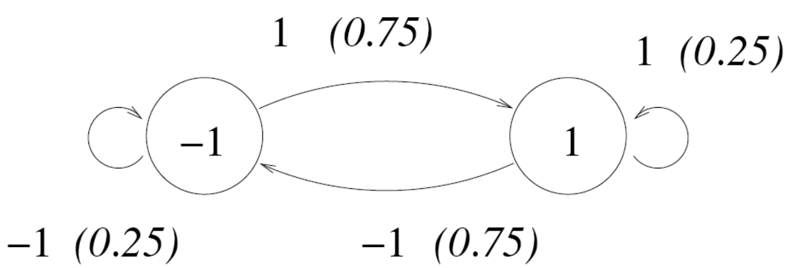
$$D(Q||P) = \limsup_{N \to \infty} \sum_{\mathbf{x}_1^N \in \mathbb{X}_1^N} q(\mathbf{x}_1^N) \log_2 \frac{q(\mathbf{x}_1^N)}{p(\mathbf{x}_1^N)}$$

Cost Assignments from a Target Process

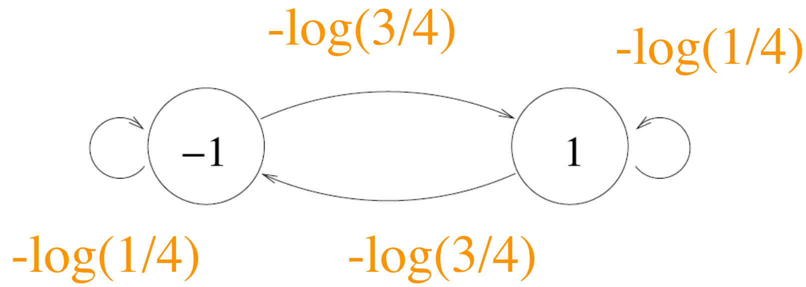
- Consider deriving a **cost function** using **target distribution** P

$$w(x_i|\mathbf{x}_{i-\mu}^{i-1}) = -\log_2 P(X_i = x_i|\mathbf{X}_{i-\mu}^{i-1} = \mathbf{x}_{i-\mu}^{i-1})$$

Target process



Finite-memory cost function



- ***More probable*** *branches* ***have less cost***
- ***Limiting average cost*** *leads to* **Cost-Constrained Noiseless Channel**

Information-Theoretic Value of Average Cost

$$w(x_i|\mathbf{x}_{i-\mu}^{i-1}) = -\log_2 P(X_i = x_i|\mathbf{X}_{i-\mu}^{i-1} = \mathbf{x}_{i-\mu}^{i-1})$$

- For a *Markov source with distribution Q*, the **average cost** equals:

$$\sum_{x_i, \mathbf{x}_{i-\mu}^{i-1}} \pi_q(\mathbf{x}_{i-\mu}^{i-1}) q(x_i|\mathbf{x}_{i-\mu}^{i-1}) \log_2 \frac{1}{p(x_i|\mathbf{x}_{i-\mu}^{i-1})}$$

$$= H(Q) + D(Q||P)$$

- **Average cost =** *entropy rate + divergence rate*
  (property extends to other distributions Q)

Capacity of Cost-Constrained Noiseless Channels

$$\mathbf{A}(s) = \left[\begin{array}{ccc} s^{w_{1,1}} & s^{w_{1,2}} & \cdots \\ \vdots & & \ddots \end{array} \right]$$

cost-enumerator matrix

$$\lambda(s) = \lambda(\mathbf{A}(s))$$

maximal eigenvalue

$$(0 < s \leq 1)$$

capacity $$C(s) = \log_2 \lambda(s) - W(s) \log_2 s$$

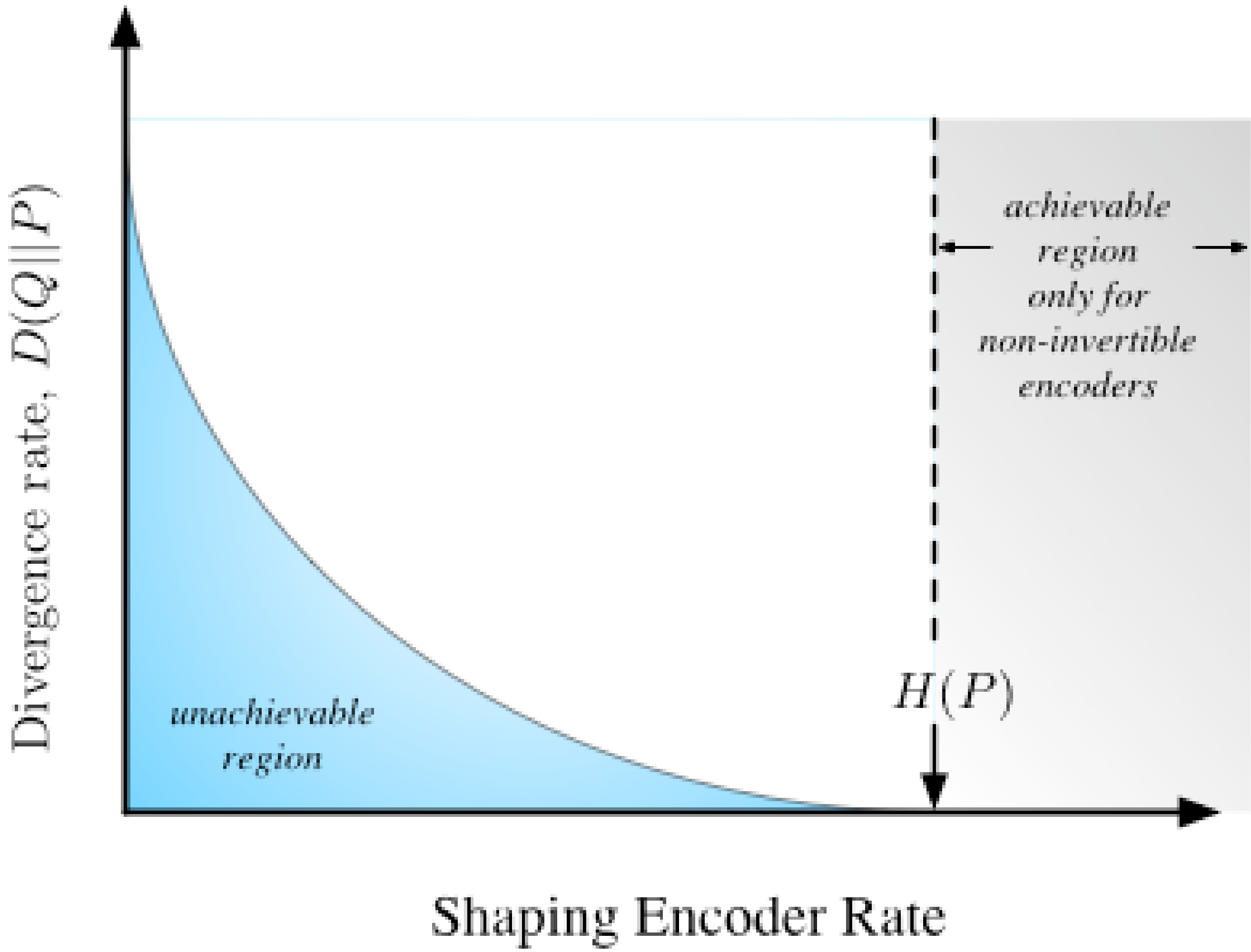
average cost-constraint $$W(s) = \frac{s}{\lambda(s)} \frac{\partial}{\partial s} \lambda(s)$$

- Justesen and Høholdt, 1984

Capacity and Minimal Divergence Rate

- C = **capacity** for an **average cost constraint** W
- W = **minimum cost** for all processes **under constraint** $H(Q)=C$
- *For the -log-probability costs according to a target process P:*
  - $W = H(Q) + D(Q||P)$ is minimized for processes with $H(Q)=C$
  - $D(Q||P) = W - C$ = **minimum divergence rate for all processes with** $H(Q)=C$

Achievable Minimum Divergence Rates



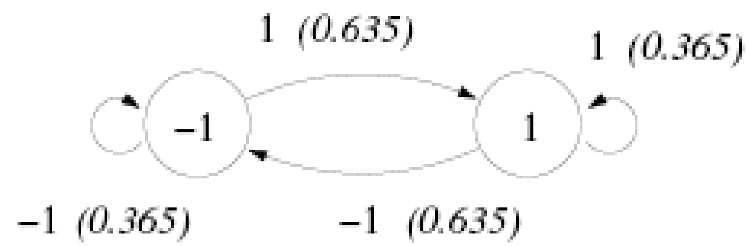
Divergence Rate Analysis of Two Encoder Construction Methods

(for finite-state Markov target processes)

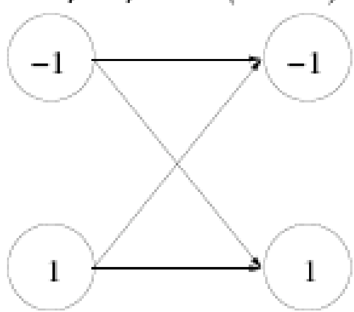
Construction Method 1

- Rate k:n encoder **approximates** ***n*-step branch probabilities** of target process **with multiples of** 2^{-k}
- *Soriaga and Siegel, 2004*

Example Target Process

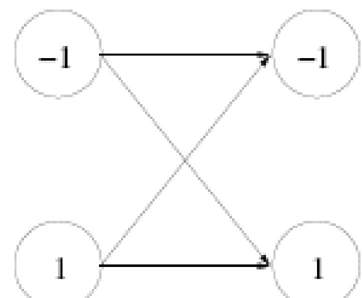


3-step Approx-imated to 1/8's



Rate 3:3 Encoder Trellis

(parenthesis for in ≠ out)

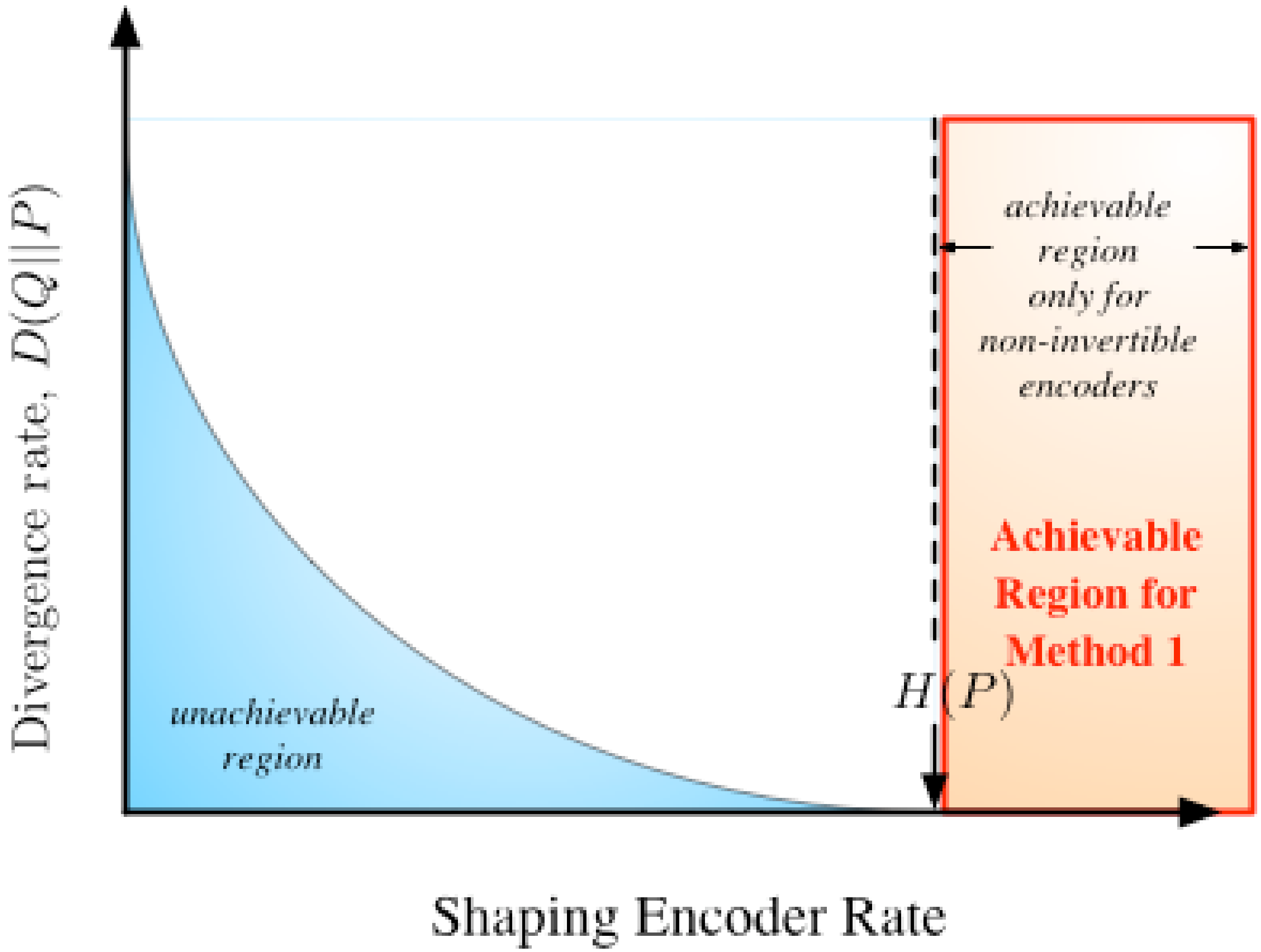


Construction Method 1 Analysis

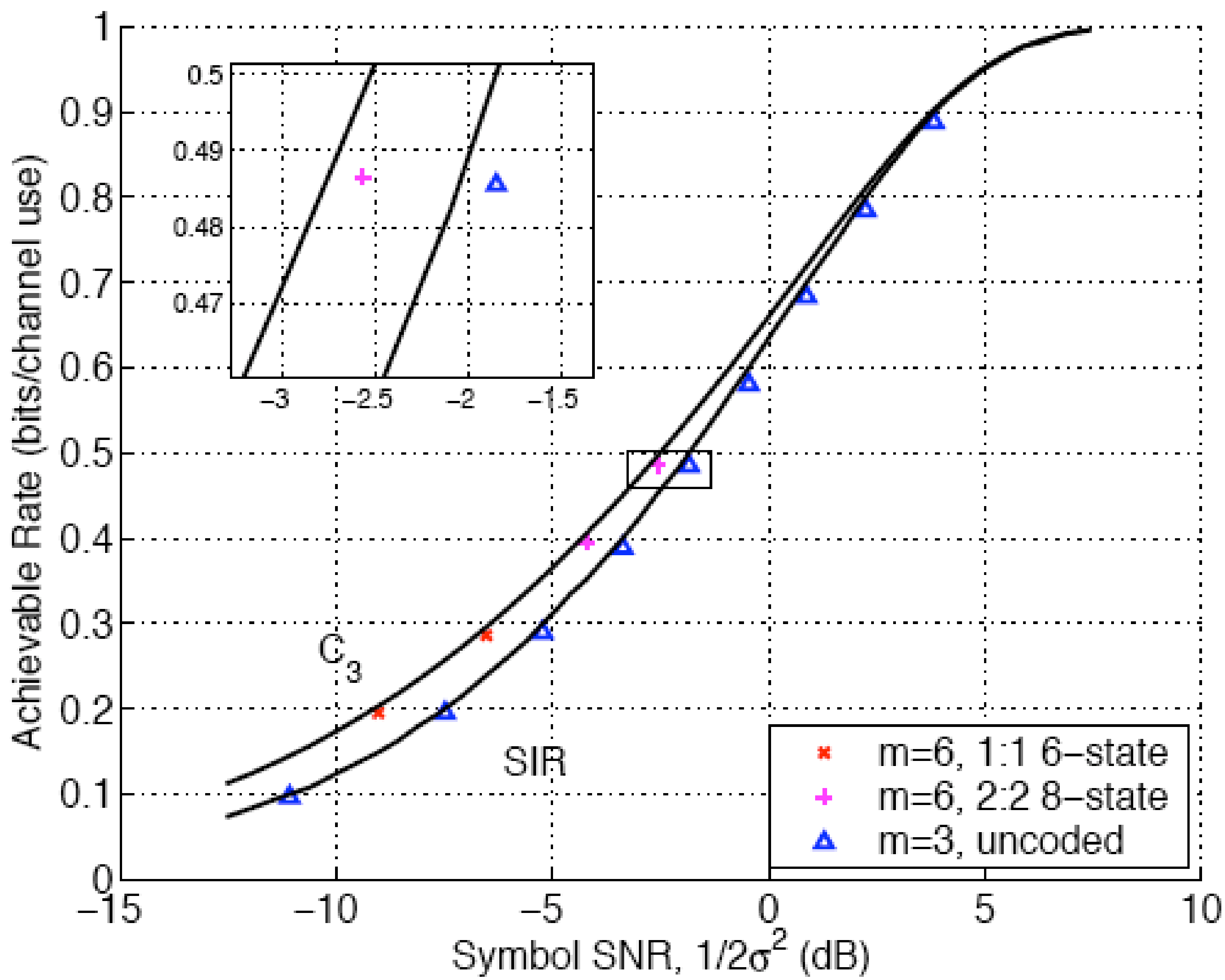
(behavior for large n)

- Let H = **entropy rate of target process**. At each state,
  $\approx 2^{nH}$ branches each have probability $\approx 2^{-nH}$
- Construction for *k>nH:*
  assign $2^{k-\lfloor nH \rfloor}$ input sequences to each of $2^{\lfloor nH \rfloor}$ branches
- **Divergence between** approximate and target **branch probabilities:**
  $$\leq \frac{1}{n} \log_2 \frac{2^{-k} 2^{k-\lfloor nH \rfloor}}{2^{-n(H+\epsilon)}} = H + \epsilon - \frac{1}{n} \lfloor nH \rfloor$$
- **Divergence rate can be made small** *for large n and k/n>H*

Method 1 Achievable Divergence Rates



Method 1 Results for Rate-1 Encoders on EPR4

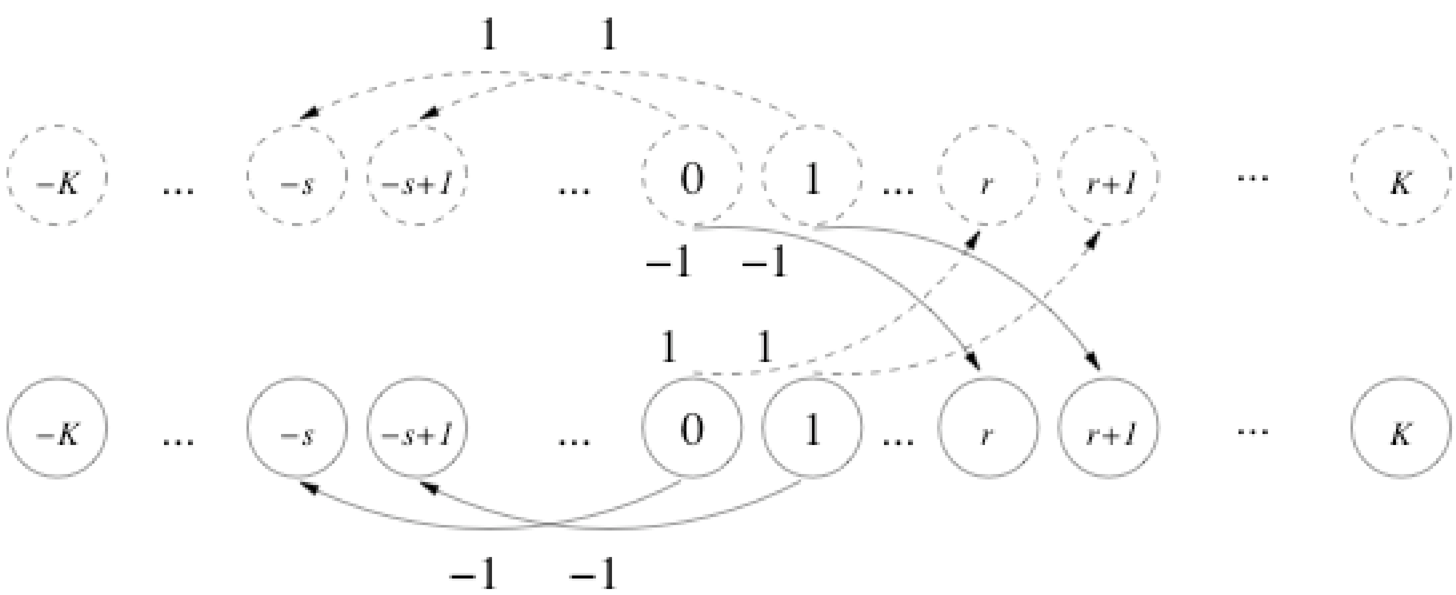


Soriaga and Siegel, 2004

Construction Method 2

- **Describe typical sequences** of target process **with a constraint graph**
- **Derive an encoder** from the constraint graph (e.g., state-splitting algorithm)
- *Soriaga and Siegel, 2003*

Example graph for a binary Markov process with $p(1|-1)=p(-1|1)=s/(r+s)$

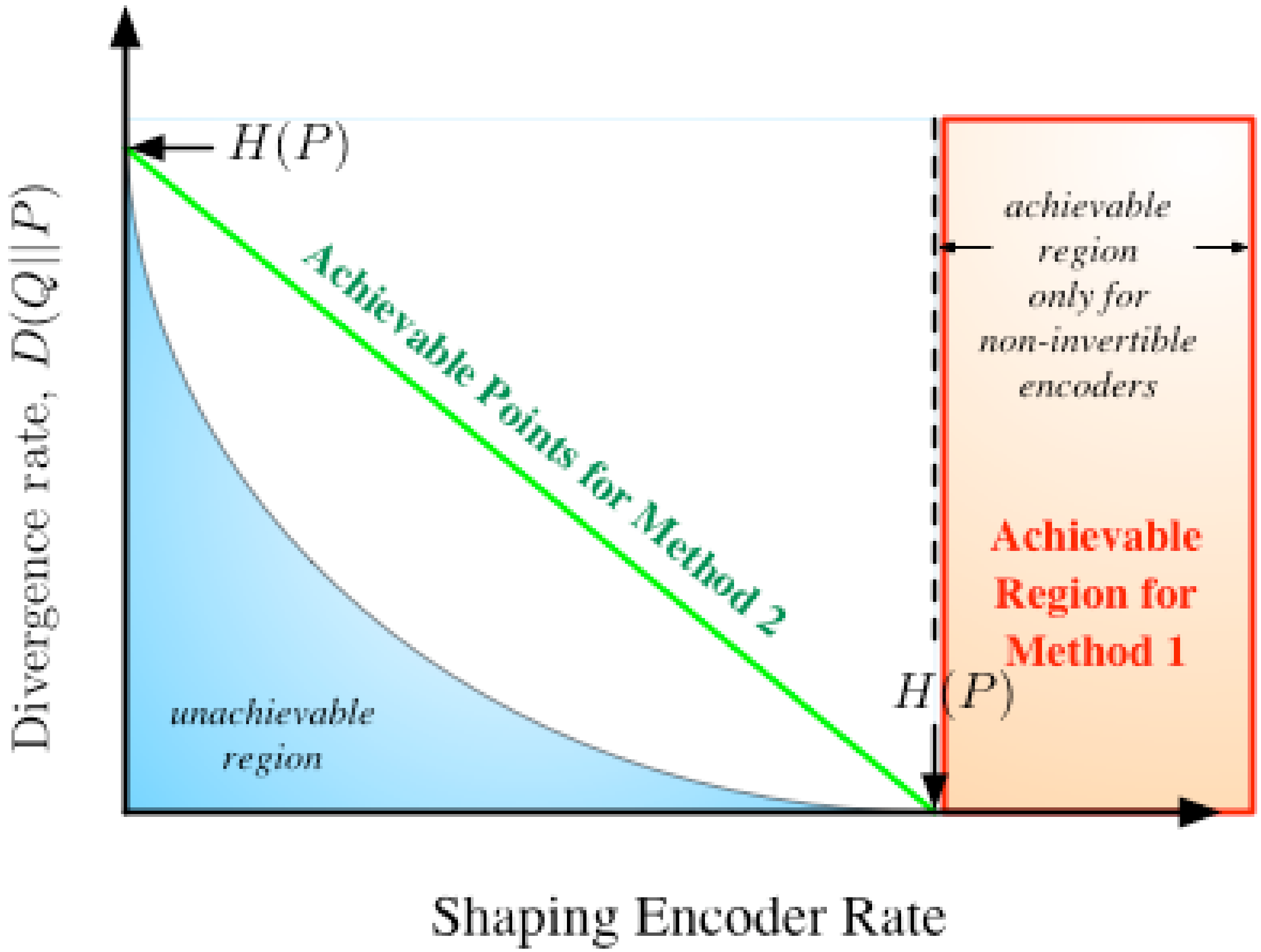


Construction Method 2 Analysis

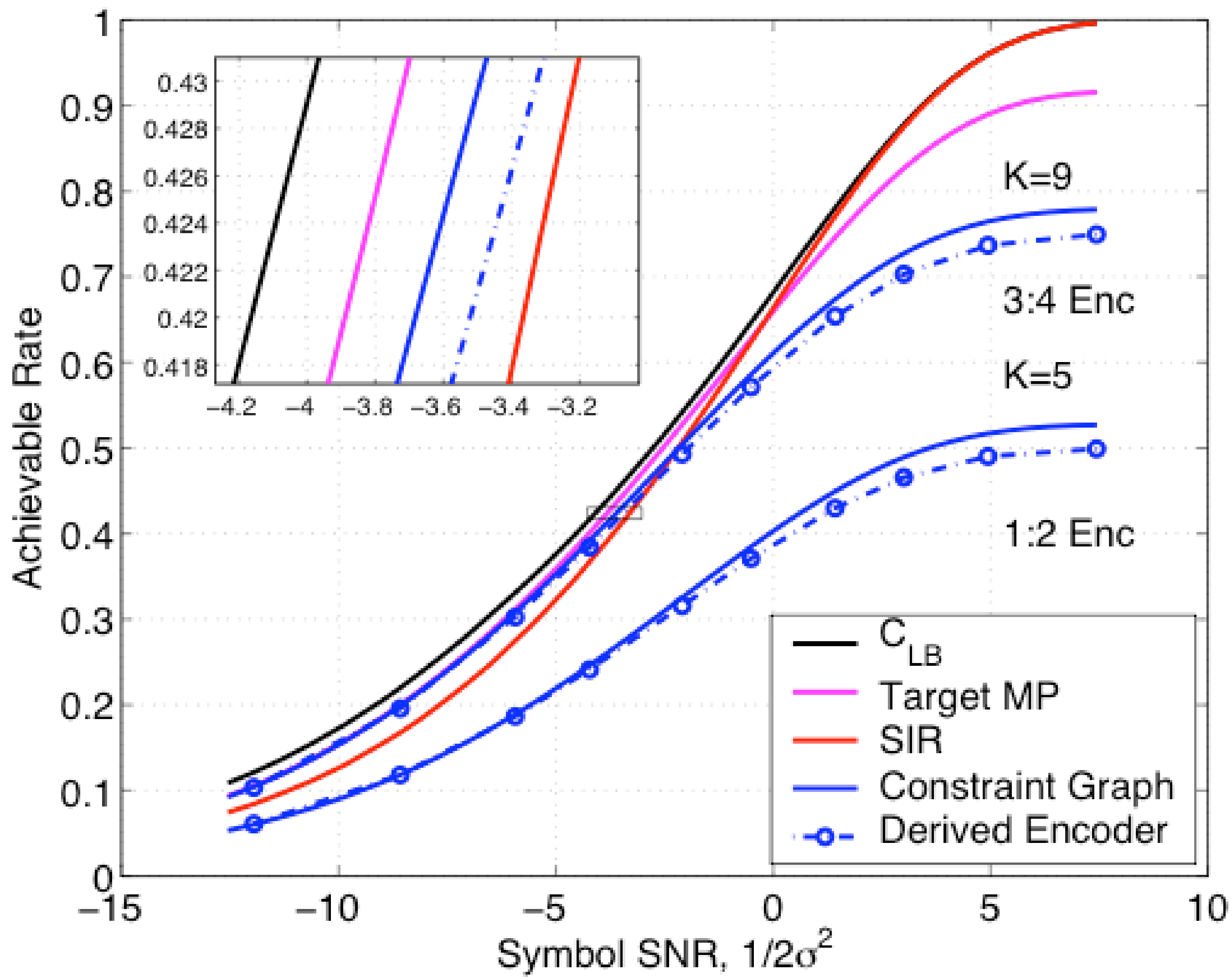
- Derived encoders **produce typical sequences**
- **Long typical sequences** have **average cost** $\approx \frac{1}{n} \log_2 \frac{1}{2^{-nH}} = H$
- Encoder **output process has overall average cost =** H
- **Divergence rate = Average Cost - Encoder Output Entropy Rate**
  - *Invertible encoder implies*

$$D(Q||P) = H - \frac{k}{n}$$

Method 2 Achievable Divergence Rates



Method 2 Numerical Results



Soriaga and Siegel, 2003

Concluding Remarks

- **Capacity of cost-constrained channels**
  - Fundamental limits are related to minimal divergence rate for shaping encoder design
- **Encoder design to minimize divergence rate**
  - Cost-minimization framework allows other approaches *e.g., Khayrallah and Neuhoff, 1996*
- **Small divergence rate criterion for inner code design**
  - Not necessary but sufficient for approaching channel capacity *See also Shamai and Verdú, 1997; Ma, Kavcic, and Varnica, 2002*