Perfect Permutation Codes with the Kendall's τ -Metric

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Abstract-The rank modulation scheme has been proposed for efficient writing and storing data in non-volatile memory storage. Error-correction in the rank modulation scheme is done by considering permutation codes. In this paper we consider codes in the set of all permutations on nelements, S_n , using the Kendall's τ -metric. We prove that there are no perfect single-error-correcting codes in S_n , where n > 4 is a prime or $4 \le n \le 10$. We also prove that if such a code exists for n which is not a prime then the code should have some uniform structure. We define some variations of the Kendall's τ -metric and consider the related codes and specifically we prove the existence of a perfect single-error-correcting code in S_5 . Finally, we examine the existence problem of diameter perfect codes in S_n and obtain a new upper bound on the size of a code in S_n with even minimum Kendall's τ -distance.

I. INTRODUCTION

Flash memory is a non-volatile technology that is both electrically programmable and electrically erasable. It incorporates a set of cells maintained at a set of levels of charge to encode information. While raising the charge level of a cell is an easy operation, reducing the charge level requires the erasure of the whole block to which the cell belongs. For this reason charge is injected into the cell over several iterations. Such programming is slow and can cause errors since cells may be injected with extra unwanted charge. Other common errors in flash memory cells are due to charge leakage and reading disturbance that may cause charge to move from one cell to its adjacent cells. In order to overcome these problems, the novel framework of rank modulation codes was introduced in [15]. In this setup the information is carried by the relative ranking of the cells charge levels and not by the absolute values of the charge levels. This allows for more efficient programming of cells, and coding by the ranking of the cells' levels is more robust to charge leakage than coding by their actual values. In this model codes are subsets of S_n , the set of all permutations on n elements, where each permutation corresponds to a ranking of n cells' levels. Permutation codes were mainly studied in this context using two metrics, the infinity metric and the Kendall's τ -metric.

Codes in S_n under the infinity metric were considered in [19], [28], [30], [32]. Anticodes in S_n under the infinity metric were considered in [18], [29], [31].

In this paper we consider codes using the Kendall's τ -metric and some variation of the Kendall's τ -metric. Under the Kendall's τ -metric, codes in S_n with minimum distance d should correct up to $\lfloor \frac{d-1}{2} \rfloor$ errors that are caused by charge leakage and read disturbance. A comprehensive work on error-correcting codes in S_n using the Kendall's τ -metric [17], is given in [16]. In that paper [16] there is also a construction of single-errorcorrecting codes using codes in the Lee metric. This method was generalized in [3] for the construction of t-error-correcting codes that are of optimal size, up to a constant factor, where t is fixed. In [33], systematic-errorcorrecting codes were proposed. In particular, they constructed a systematic single-error-correcting code in S_n of size (n-2)!, which is of optimal size, assuming that a perfect single-error-correcting code does not exist. But, they only prove the nonexistence of perfect single-errorcorrecting codes for n = 4.

The first part of this paper is devoted to perfect singleerror-correcting codes in S_n , using the Kendall's τ -metric and related structures. Perfect codes is one of the most fascinating topics in coding theory. A perfect code in a given metric is a code in which the set of spheres with a given radius R around its codewords forms a partition of the space. These codes were mainly considered for the Hamming scheme, e.g. [10], [22], [23], [24], [25]. They were also considered for other schemes such as the Johnson scheme, e.g. [7], [9], [27], the Grassmann scheme [5], [21], and to a larger extent in the Lee and the Manhattan metrics, e.g. [8], [12], [13], [26]. Perfect codes were also considered on Cayley graphs [14] and for distance-transitive graphs [4].

The rest of this work is organized as follows. In Section II we define the basic concepts for the Kendall's τ -metric and for perfect codes. In Section III we prove the nonexistence of a perfect single-error-correcting code in S_n , using the Kendall's τ -metric, where n > 4 is a prime or $4 \le n \le 10$. We also show that perfect singleerror-correcting codes must have a uniform structure. If we slightly modify the Kendall's τ -distance to define a cyclic Kendall's τ -distance, then we have at least one perfect single-error-correcting code in S_5 . This code and more variations of the Kendall's τ -metric are discussed in Section IV. In Section V we examine diameter perfect codes in S_n , using the Kendall's τ -metric, and improve some known upper bounds on the size of a code in S_n with even minimum Kendall's τ -distance. We conclude in Section VI where we also present some questions for future research.

II. BASIC CONCEPTS

Let S_n be the set of all permutations on the set of n elements $[n] = \{1, 2, \dots, n\}$. We denote a permutation $\sigma \in S_n$ by $\sigma = [\sigma(1), \sigma(2), \dots, \sigma(n)].$ For two permutations $\sigma, \pi \in S_n$, their multiplication $\pi \circ \sigma$ is defined as the composition of σ on π , namely, $\pi \circ \sigma(i) = \sigma(\pi(i))$, for all $1 \le i \le n$. Note that this operation on S_n is not commutative. Given a permutation $\sigma \in S_n$, an adjacent transposition, (i, i + 1), is an exchange of two adjacent elements $\sigma(i), \sigma(i+1)$ in σ , for some $1 \le i \le n-1$. The result is the permutation $\pi =$ $[\sigma(1),\ldots,\sigma(i-1),\sigma(i+1),\sigma(i),\sigma(i+2),\ldots,\sigma(n)].$ The permutation π can also be written as $\pi = (i, i+1) \circ \sigma,$ where (i, i)+1)is the cycle decomposition of the permutation $[1, 2, \ldots, i - 1, i + 1, i, i + 2, \ldots, n].$

For two permutations $\sigma, \pi \in S_n$, the Kendall's τ distance between σ and π , $d_K(\sigma, \pi)$, is defined as the minimum number of adjacent transpositions needed to transform σ into π [17]. The following expression for $d_K(\sigma, \pi)$ is well known (e.g. [16], [20]).

$$d_K(\sigma,\pi) = |\{(i,j) : \sigma^{-1}(i) < \sigma^{-1}(j) \land \pi^{-1}(i) > \pi^{-1}(j)\}|.$$

Given a metric space, one can define codes. We say that $C \subset S_n$ is an (n, M, d) code if |C| = M and $d_K(\sigma, \pi) \ge d$ for every two permutations σ, π in C (d is called the *minimum distance* of the code C).

For a given space \mathcal{V} with a distance measure $d(\cdot, \cdot)$, a subset C of \mathcal{V} is a *perfect code* with *radius* R if for every element $x \in \mathcal{V}$ there exists a unique codeword $c \in C$ such that $d(x,c) \leq R$. For a point $x \in \mathcal{V}$, the *sphere* of radius R centered at x, S(x,R), is defined by $S(x,R) \stackrel{\text{def}}{=} \{y \in \mathcal{V} : d(x,y) \leq R\}$. In all the spaces and metrics considered in this paper the size of a sphere does not depend on the center of the sphere. This is a consequence from the fact that right multiplication of permutations is an isometric operation related to the distance. If C is a code with minimum distance 2R + 1and S is a sphere with radius R then it is readily verified that

Theorem 1. For a code C with minimum distance 2R+1and a sphere S with radius R we have $|C| \cdot |S| \leq |\mathcal{V}|$.

Theorem 1 known as the sphere packing bound. In a code C which attains the sphere packing bound, i.e. $|C| \cdot |S| = |\mathcal{V}|$, the spheres with radius R around the codewords of C form a partition of \mathcal{V} . Hence, such a code is a perfect code. A perfect code with radius R is also called a *perfect R-error-correcting code*.

III. NONEXISTENCE OF SOME PERFECT CODES

In this section we prove that there are no perfect singleerror-correcting codes in S_n , where n is a prime greater than 4 or $4 \le n \le 10$. For each $i, 1 \le i \le n$, let $S_{n,i}$ be the subset of S_n which consists of all the permutations $\sigma \in S_n$ for which $\sigma(i) = 1$, i.e., one is in the *i*th position in σ . Clearly we have that $|S_{n,i}| = (n-1)!$.

Assume that there exists a perfect single-errorcorrecting code $C \subset S_n$. For each $i, 1 \leq i \leq n$, let

$$C_i = C \cap S_{n,i}$$
 and $x_i = |C_i|$.

We say that a codeword $\sigma \in C$ covers a permutation $\pi \in S_n$ if $d_K(\sigma, \pi) \leq 1$. Since C is a perfect singleerror-correcting code, it follows that every permutation in $S_{n,1}$ must be at distance at most one from exactly one codeword of C and this codeword must belong either to C_1 or C_2 . Every codeword $\sigma \in C_1$ covers exactly n-1permutations in $S_{n,1}$. It covers itself and the n-2 permutations in $S_{n,1}$ obtained from σ by exactly one adjacent transposition (i, i+1), 1 < i < n. Each codeword $\sigma \in C_2$ covers exactly one permutation $\pi \in S_{n,1}, \pi = (1, 2) \circ \sigma$. Therefore, we have the following equation

$$(n-1)x_1 + x_2 = (n-1)! . (1)$$

Similarly, by considering how the permutations of $S_{n,n}$ are covered by codewords of C, we have that

$$x_{n-1} + (n-1)x_n = (n-1)! .$$
⁽²⁾

For each $i, 2 \leq i \leq n-1$, each permutation in $S_{n,i}$ is covered by exactly one codeword that belongs to either C_{i-1} , C_i , or C_{i+1} . Each codeword $\sigma \in C_i$ covers exactly n-2 permutations in $S_{n,i}$. It covers itself and the n-3 permutations in $S_{n,i}$ obtained from σ by exactly one adjacent transposition (j, j+1), where j < i-1 or j > i. Each codeword in $C_{i-1} \cup C_{i+1}$ covers exactly one permutation from $S_{n,i}$. Therefore, for each i, $2 \leq i \leq n-1$, we have the equation

$$x_{i-1} + (n-2)x_i + x_{i+1} = (n-1)! .$$
(3)

Let $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and let 1 denote the all-ones vector. Equations (1), (2), and (3) can be written in matrix form as

$$A\mathbf{x} = (n-1)! \cdot \mathbf{1},\tag{4}$$

where $A = (a_{i,j})$ is defined by

	$\begin{pmatrix} n-1\\ 1\\ 0 \end{pmatrix}$	$\begin{array}{c}1\\n-2\\1\end{array}$	$\begin{array}{c} 0\\ 1\\ n-2 \end{array}$	$\begin{array}{c} 0 \\ 0 \\ 1 \end{array}$	 	0 0 0	0 0 0	···· ····	$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$	
A =	÷	÷	÷	÷	·.	÷	:	÷	÷	
	:	÷	:	:	·	:	:	:	:	
		···· ···	0 0 0	0 0 0	· · · · · · ·		n-2 1 0	$n - 2 \\ 1$	$\begin{pmatrix} 0 \\ 1 \\ n-1 \end{pmatrix}$	

Since the sum of every row in A is equal to n it follows that the linear equation system (4) has a solution $\mathbf{y} = \frac{(n-1)!}{n} \cdot \mathbf{1}$. We will show that if n > 3 then A is a nonsingular matrix and hence \mathbf{y} is the unique solution of (4), i.e., $\mathbf{x} = \mathbf{y}$. To this end, we need the following lemma, that can be easily verified, and is also an immediate conclusion of the well known Gerschgorin circle theorem [11].

Lemma 1. Let $B = (b_{i,j})$ be an $n \times n$ matrix. If $|b_{i,i}| > \sum_{j \neq i} |b_{i,j}|$ for all $i, 1 \le i \le n$, then B is nonsingular.

For n > 4, we have that for each i, $1 \le i \le n$, $a_{i,i} \ge n-2 > 2 \ge \sum_{j \ne i} a_{i,j}$. By Lemma 1 it follows that A is nonsingular. For n = 4 it can be readily verified that the matrix A is nonsingular. Hence, for all $n \ge 4$, $x_i = \frac{(n-1)!}{n}$ for all $1 \le i \le n$. If n = 4 or n is a prime greater than 4, then x_i is not an integer and therefore, a perfect single-error-correcting code does not exist for these parameters.

By using similar methods, we prove the nonexistence of perfect single-error-correcting codes in S_n for $n \in$ $\{6, 8, 9, 10\}$. For each of these cases, we obtained a system of linear equations, which we solved by computer. For n > 11, the system of linear equations is very large and the computer failed to solve it. We summarize our result in the following theorem.

Theorem 2. There is no perfect single-error-correcting code in S_n , where n > 4 is a prime or $4 \le n \le 10$.

By similar methods we can also prove the following property of perfect single-error-correcting codes.

Theorem 3. Assume that there exists a perfect singleerror-correcting code $C \subset S_n$, where n > 11. If $r < \frac{n}{4}$, then for every sequence of r distinct elements of [n], i_1, i_2, \ldots, i_r , and for every set of r positions $1 \le j_1 < j_2 < \ldots < j_r \le n$, there are exactly $\frac{(n-r)!}{n}$ codewords $\sigma \in C$, such that $\sigma(j_\ell) = i_\ell$, for each ℓ , $1 \le \ell \le r$.

Theorem 3 implies that perfect single-error-correcting codes must have some kind of symmetry. This might be useful to rule out the existence of these codes for other parameters as well.

IV. The Cyclic Kendall's τ -metric

In this section we discuss a new metric which naturally risen in the context of the Kendall's τ -metric.

Given a permutation $\sigma \in S_n$, a *c-adjacent transposition* is either an adjacent transposition or the exchange of the elements $\sigma(1)$ and $\sigma(n)$.

For two permutations $\sigma, \pi \in S_n$, the cyclic Kendall's τ -distance between σ and π , $d_{\kappa}(\sigma, \pi)$, is defined as the minimum number of c-adjacent transpositions needed to transform σ into π .

For example, if $\sigma = [1, 2, 3, 4]$ and $\pi = [4, 3, 2, 1]$, then $d_{\kappa}(\sigma, \pi) = 2$, since two c-adjacent transpositions are enough to change the permutation from σ to π : $[1, 2, 3, 4] \rightarrow [4, 2, 3, 1] \rightarrow [4, 3, 2, 1]$, and we need at least two c-adjacent transpositions for this purpose. Clearly, $d_{\kappa}(\sigma, \rho) \leq d_{K}(\sigma, \rho)$ and therefore, if C has minimum cyclic Kendall's τ -distance d then C also has minimum Kendall's τ -distance at least d.

By Theorem 2 there is no perfect single-errorcorrecting code in S_5 , using the Kendall's τ -distance. However, there exists a perfect single-error-correcting code in S_5 , using the cyclic Kendall's τ -distance. For example, the following 20 codewords form such a code.

	[0, 1, 2, 3, 4],	[0, 2, 4, 1, 3],	[0, 3, 1, 4, 2],	[0, 4, 3, 2, 1]
	[1, 2, 3, 4, 0],	[2, 4, 1, 3, 0],	[3, 1, 4, 2, 0],	[4, 3, 2, 1, 0]
	[2, 3, 4, 0, 1],	[4, 1, 3, 0, 2],	[1, 4, 2, 0, 3],	[3, 2, 1, 0, 4]
	[3, 4, 0, 1, 2],	[1, 3, 0, 2, 4],	[4, 2, 0, 3, 1],	[2, 1, 0, 4, 3]
[[4, 0, 1, 2, 3],	[3, 0, 2, 4, 1],	[2, 0, 3, 1, 4],	[1, 0, 4, 3, 2].

Note that the permutations in each column are cyclic shifts of the first permutation in the column. Moreover, the permutations in the first row are of the form $[0, \alpha, 2\alpha, 3\alpha, 4\alpha]$, where $1 \le \alpha \le 4$, and multiplication is taken modulo 5. A similar code of size $n \cdot (n-1)$ can be formed for each prime n > 5. These codes can be represented by another related distance measure. We consider the following equivalence relation Eon S_n . For two permutations $\sigma = [\sigma_1, \sigma_2, \ldots, \sigma_n]$ and $\pi = [\pi_1, \pi_2, \ldots, \pi_n]$ we have that $(\sigma, \pi) \in E$ if there exist an integer $i, 1 \leq i \leq n$, such that $\sigma = [\pi_i, \pi_{i+1}, \dots, \pi_n, \pi_1, \dots, \pi_{i-1}].$ Clearly, E is an equivalence relation on S_n with (n-1)! equivalence classes, each one of size n. Let S_n^c denote the set of these (n-1)! equivalence classes. Two elements of S_n^c are at distance one if there exist two representatives of the two equivalence classes whose Kendall's τ -distance in one. If \mathcal{C} is a code with minimum Kendall's τ -distance (of S_m^c) d then there exists a code \mathcal{C}' in S_n , of size $n|\mathcal{C}|$, with minimum cyclic Kendall's τ -distance d. Moreover, there exists also a code of size $|\mathcal{C}|$ in S_{n-1} with minimum cyclic Kendall's τ -distance d.

The computations of related distances, constructions of codes, and the structure of the two related graphs are intriguing research topics. Some breaking records codes are derived from these graphs. The results will be presented only in the full version of this work.

V. DIAMETER PERFECT CODES

In all the perfect codes the minimum distance of the code is an odd integer. If the minimum distance of the code C is an even integer then C cannot be a perfect code. The reason is that for any two codewords $c_1, c_2 \in C$ such that $d(c_1, c_2) = 2\delta$, there exists a word x such that $d(x, c_1) = \delta$ and $d(x, c_2) = \delta$. For this case another concept is used, a diameter perfect code, as was defined in [1]. This concept is based on the code-anticode bound presented by Delsarte [6]. An *anticode* \mathcal{A} of *diameter* D in a space \mathcal{V} is a subset of words from \mathcal{V} such that $d(x, y) \leq D$ for all $x, y \in \mathcal{A}$.

Theorem 4. If a code C, in a space V of a distance regular graph, has minimum distance d and in an anticode A of the space V the maximum distance is d - 1 then $|C| \cdot |A| \leq |V|$.

Theorem 4 which is proved in [6] is a generalization of Theorem 1 and it can be applied to the Hamming scheme since the related graph is distance regular. It cannot be applied to the Kendall's τ -metric since the related graph is not distance regular if n > 3. This can be easily verified by considering the three permutations $\sigma = [1, 2, 3, 4, 5, ..., n]$, $\pi = [3, 1, 2, 4, 5, ..., n]$, and $\rho = [2, 1, 4, 3, 5, ..., n]$ in S_n . Clearly, $d_K(\sigma, \pi) =$ $d_K(\sigma, \rho) = 2$ and there exists exactly one permutation α for which $d_K(\sigma, \alpha) = 1$ and $d_K(\alpha, \pi) = 1$, while there exist exactly two permutations α, β for which $d_K(\sigma, \alpha) = 1$, $d_K(\alpha, \rho) = 1$, $d_K(\sigma, \beta) = 1$, and $d_K(\beta, \rho) = 1$. Fortunately, an alternative proof which was given in [1] and was modified in [8] will work for the Kendall's τ -metric.

Theorem 5. Let $C_{\mathcal{D}}$ be a code in S_n with Kendall's τ distances between codewords taken from a set \mathcal{D} . Let $\mathcal{A} \subset$ S_n and let $C'_{\mathcal{D}}$ be the largest code in \mathcal{A} with Kendall's τ distances between codewords taken from the set \mathcal{D} . Then

$$\frac{|\mathcal{C}_{\mathcal{D}}|}{n!} \le \frac{|\mathcal{C}_{\mathcal{D}}'|}{|\mathcal{A}|}$$

Corollary 1. Theorem 4 holds for the Kendall's τ -metric, i.e. if a code $C \subset S_n$, has minimum Kendall's τ -distance d and in an anticode $A \subset S_n$ the maximum Kendall's τ -distance is d - 1 then $|C| \cdot |A| \leq n!$.

If there exists a code $C \subset S_n$ with minimum Kendall's τ -distance d = D + 1, and an anticode A with diameter D such that $|C| \cdot |A| = n!$, then C is called a *diameter* perfect code with diameter D. In that case, A must be an anticode with maximum distance (diameter) D of largest size, and A is called an *optimal* anticode of diameter D. Thus, it is important to determine the optimal anticodes in S_n and their sizes. Using the size of such optimal anticodes we can obtain by Corollary 1 an upper bound on the size of the related code in S_n .

One can verify that any permutation $\sigma = [\sigma(1), \sigma(2), \ldots, \sigma(n)]$ in S_n and its reverse $[\sigma(n), \ldots, \sigma(2), \sigma(1)]$ form a diameter perfect code with diameter $\binom{n}{2} - 1$. An optimal anticode with diameter $\binom{n}{2} - 1$ consists of $\frac{n!}{2}$ permutations, one permutation from each pair of permutations, π and its reverse in S_n .

An intriguing question is whether a sphere with radius R in S_n , using the Kendall's τ -metric, is an optimal anticode of diameter 2R. Such types of questions for other metrics were considered in [2]. For n = 4, the sphere with radius 1 has size 4 and it is an optimal anticode of diameter 2. There exists an optimal anticode of diameter 2 in S_4 , which is not a sphere with radius 1. For example, the set $\mathcal{A} =$ $\{[1,2,3,4],[2,1,3,4],[1,2,4,3],[2,1,4,3]\}$ is an optimal anticode of diameter 2. A similar example exists for an optimal anticode of size 9 and diameter 4 in S_4 . However, for $n \geq 5$, we have the following theorem.

Theorem 6. Every optimal anticode with diameter 2 in S_n , $n \ge 5$, using the Kendall's τ -distance, is a sphere with radius 1, whose size is n.

Let S be a set of permutations in S_n and let $\pi \in S_n$. We define $S \circ \pi = \{ \sigma \circ \pi : \sigma \in S \}.$

Theorem 7. Let e = [1, 2, ..., n] be the identity permutation of S_n , $n \ge 4$. And let S(e, 1) be the sphere of radius 1 centered at e. Then the set

$$\mathcal{A} = \mathcal{S}(e, 1) \cup \mathcal{S}(e, 1) \circ (1, 2)$$

is an optimal anticode of diameter 3, whose size is 2(n-1).

Corollary 2. If $C \subset S_n$ is a code with minimum Kendall's τ -distance 4, then

$$|\mathcal{C}| \le \frac{n!}{2(n-1)}.$$

We conjecture that the largest anticode with maximum Kendall's τ -distance 2R is a sphere with radius R if $2R < \binom{n}{2}$. We conjecture that the largest anticode with maximum Kendall's τ -distance $2R + 1 < \binom{n}{2}$ is $S(e, R) \cup S(e, R) \circ (1, 2)$. The size of this anticode will be discussed in the full version of this work. It implies a new bound on the size of a code in S_n with minimum Kendall's τ -distance 2R + 2.

VI. CONCLUSIONS AND OPEN PROBLEMS

We have considered several questions regarding perfect codes in the Kendall's τ -metric. We gave a novel technique to exclude the existence of perfect codes using the Kendall's τ -metric. We applied this technique to prove that there are no perfect single-error-correcting codes in S_n , where n > 4 is a prime or $4 \le n \le 10$, using the Kendall's τ -metric. We also proved that if such a code exists for other values of n it should have some uniform structure. We showed that if we use a cyclic Kendall's τ metric then a perfect single-error-correcting code exists in S_5 . Finally, we examine the existence question of diameter perfect codes in S_n . We obtained a new upper bound on the size of a code in S_n with even Kendall's τ distance. Our discussion raises many open problems from which we choose a few as follows.

- 1) Prove the nonexistence of perfect codes in S_n , using the Kendall's τ -metric, for more values of nand/or other distances.
- 2) Do there exist more perfect codes in S_n using the cyclic Kendall's τ -metric?
- Examine the cyclic Kendall's τ-metric for its properties, find upper bounds on the size of codes with this metric, and construct codes with this metric. The same should be done if we consider the set of equivalence classes S^c_n of the relation E.
- 4) Is the sphere with radius R in S_n always optimal as an anticode with diameter $2R < \binom{n}{2}$ in S_n ? If yes, when there are other optimal anticodes with the same parameters which are not spheres?
- 5) What is the size of an optimal anticode in S_n with diameter D?
- 6) Improve the bounds on the size of codes in S_n with even minimum Kendall's τ -distance.

NOTE ADDED

Theorem 2 is a special case of Theorem 5 in [14] on perfect codes in Cayley graphs. But, the proof of Theorem 5 in [14] is wrong and some bounds on codes implied from it are false with infinite counterexamples.

ACKNOWLEDGMENT

This work was supported in part by the U.S.-Israel Binational Science Foundation, Jerusalem, Israel, under Grant No. 2012016. Sarit Buzaglo would like to thank Amir Yehudayoff for many useful discussions. The authors also thank one of the reviewers for bringing [14] to our attention.

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