The VC-Dimension of s-Intersecting Curves

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Abstract

In this work we present a collection of results in Combinatorial Geometry, aggregated in three chapters. The first one deals with the VC-dimension of a family of subsets of points in the plane, surrounded by simple closed curves. The second deals with sweeping of an arrangement of curves. In the last chapter presented is a special case of a problem that was first introduced by S. Fekete and G.J Woeginger and deals with polygonal paths with no small angles.

Given a set X, a *set-system on the ground set* X is a collection of subsets of X. The *Vapnik-Chervonenkis dimension*, or *VC-dimension* for short, is a parameter assigned to a set-system on a ground set X, $\mathcal{F} = \{A_i | i \in I, A_i \subseteq X\}$. An important result related to the VC-dimension is the Shatter Function Lemma. This Lemma enables one to obtain an upper bound on the cardinality of a set-system on a finite ground set via the VC-dimension. In this thesis we deal with the following collection of geometric objects in the plane: Let P be a set of n points in the plane and let $\mathcal{C}$ be a family of simple closed curves in the plane each of which avoids the points of P. We show that if every two curves $C, C' \in \mathcal{C}$ intersect at most s times and the intersection of the regions in the plane that are bounded by C, C' is either empty or a connected set, then the set system $\mathcal{F}$ on the ground set P, consists of all the subsets of P that are surrounded in curves in $\mathcal{C}$, has VC-dimension at most $s+1$.

Chapter 3 deals with the notion of *sweeping an arrangement of curves.* J. Snoeyink and J. Hershberger proved that any collection of pseudo circles surrounding a common point can be swept by a ray. We have generalized this result for a collection of simple close curves that are surrounding a common points and satisfy the connected intersection property.

The last chapter of this thesis deals with a special case of a problem of S. Fekete and G.J Woeginger. Given a finite set X of points in the plane, and $\alpha > 0$. Is it possible to find a polygonal path P on X such that all the angles that are formed by consecutive edges of P are at least α? Fekete and Woeginger conjectured that it is possible for $\alpha \leq \frac{\pi}{6}$. Bárány Pór and Valtr proved the existence of such path for $\alpha \leq \frac{\pi}{9}$. We show that if the points are in convex position, then for every $\alpha \leq \frac{\pi}{5}$, there exists a path on X with no angle smaller than α, and that this upper bound on α is tight.

List of Symbols

$\mathbb{N}$ - The set of natural numbers
$\mathbb{R}^d$ - The real Euclidean d-dimensional space
and in particular
$\mathbb{R}^2$ - The real Euclidean plane
$\{A_1, A_2, ..., A_n\}$ - The set whose members are A_1,A_2,...,A_n
$\{a|\Phi(a)\}$ - The set of members which specify the condition Φ
When the context is clear we sometimes write only the condition - For instance we write $\{z \geq 0\}$ for $\{(x, y, z) \in \mathbb{R}^3|\ z \geq 0\}$
$U \subseteq V$ - U is a subset of V
$V \setminus U$ - The set of members which are in V but not in U
ϕ - The empty set
$V \cup U$ - The union set of V and U
$\cup A$ - The union of all members in A
$V \cap U$ - The intersection set of V and U
$\cap A$ - The intersection of all members in A
$a \in A$ - a is a member of A
$a \notin A$ - a is not a member of A
$G = (V, E)$ - A graph G whose vertex set is V and whose edge set is E
$p = (a, b)$ - A point p whose cartesian coordinates are a and b
$[a, b]$ - The closed real interval $\{t|\ a \leq t \leq b\}$
(a, b) - The open real interval $\{t|\ a < t < b\}$
(a, b) can also mean the edge of a directed graph corresponding to the vertices a and b. The distinction between the above two meanings is specified in each relevant place in the text.
Let A and B be two different points in R^d for some $d \in \mathbb{N}$
$\overrightarrow{A}$ - The vector pointing from the origin to A
$\overline{AB}$ - The straight line segment from A to B
θ_A - The angle created between the positive ray of the x-axis and the vector $\overrightarrow{A}$
$\measuredangle AOB$ - The angle created between the segments $\overline{AO}$ and $\overline{OB}$

$|E|$ - The size of the set E
$\lfloor t \rfloor$ - The maximum integer which is not bigger than the real number t
∞ - infinite

- $\Theta(n^2)$ function f - A function f on $\mathbb{N}$ that satisfies the following: there exists $N \in \mathbb{N}$ and there exist positive constants C_1 and C_2, such that for every $n > N$, $C_1 \cdot n^2 \leq f(n) \leq C_2 \cdot n^2$

- $\Omega(n^2)$ function f - A function f on $\mathbb{N}$ that satisfies the following: there exists $N \in \mathbb{N}$ and there exists a positive constants C_1, such that for every $n > N$, $C_1 \cdot n^2 \leq f(n)$

- $f(*) : A \to B$ - A function f (which receives $*$ as an argument) from the Domain set A into the Range set B

- $f \restriction A$ - A function f restricted to A - a subset of the Domain of f

S^2 - The unit sphere in $\mathbb{R}^3$; that is the boundary of the unit ball (having radius 1) in $\mathbb{R}^3$ which is centered at the origin
S^1 - The unit circle in $\mathbb{R}^2$; that is the boundary of the unit disc (having radius 1) in $\mathbb{R}^2$ which is centered at the origin
∂A - The boundary of A (for a set $A \subseteq \mathbb{R}^d$)
Area(A) - The two-dimensional area of A (for a set $A \subseteq \mathbb{R}^2$)
$conv(A)$ - The smallest convex set that contains A (for a set $A \subseteq R^d$)
$\binom{n}{k}$ - The number of ways for choosing k objects out of n objects $\big($it is equal to $\frac{n!}{k!(n-k)!}\big)$
$\max(n, m)$ - The maximum of the two real numbers n and m
$\max A$ - The maximum of all members of A.
Any other symbol used in the text is explicitly defined next to the first appearance of the symbol.

Chapter 1

Introduction

In this work we introduce the following results:

i. Theorem 2.1.1- Given a finite set of points in the plane, and a collection of s-intersecting simple closed curves that satisfy the connected intersection property, the VC-dimension of the family of all subsets of the points which are surrounded by the curves is at most $s + 1$, and this bound is tight.

ii. Theorem 3.1.1- An arrangement of simple closed curves in the plane that has the connected intersection property, can be swept by a ray.

iii. Theorem 4.0.5- Given a finite set of points in the plane, in a convex position, there exists a polygonal path that goes through the points such that the angle between every pair of consecutive edges of the path is at least $\frac{\pi}{5}$, and this bound is tight.

In chapter 2 we use the notion of set system on a ground set X. Given a set X, a *set system on the ground set* X is a set $\mathcal{F} = \{A_i | i \in I\}$. The elements of $\mathcal{F}$, A_i, are subsets of X. In our set system, the ground set is a finite set P of points in the plane and the set system consists of the subsets of P surrounded by curves from a given collection $\mathcal{C}$ of curves. An important example to this type of set system is the collection of all k-sets of a given set of points P in the plane. A k-point subset $S \subseteq P$ is a *k-set* of P if there exists an open half-plane H such that $S = P \cap H$. This set-system has been investigated intensively in the last four decades. The most interesting question in the context of k-sets, which is still an open question, is finding upper and lower bounds on the number of k-sets of a set of n points in the plane or in higher dimensional space. The best known upper bound on the number of k-sets of a set of n points in the plane is due to Dey [6], and stands on $O(nk^{\frac{1}{3}})$. A lower bound was obtained by Erdős, Lovátz, Simmons and Straus in [7], which is around $\Omega(n \log k)$. This bound was further improved by Tóth [16] to $\Omega(n \exp(\sqrt{\log n}))$. The set system that we introduce consists of all the subsets $S \subseteq P$ that are surrounded by closed curves in a given collection of curves. These set systems can be though of as an extension of the set systems that consists of all the k-sets of P, since their elements are subsets of P that are separated by a closed curve and k-sets are subsets of P that are separated by a line.

The curves that we consider in this work are simple closed Jordan curves. By Jordan's theorem a simple closed Jordan curve C divides the plane into two regions, only one of which is bounded. We call the bounded region the *disc* bounded by C and we denote this region by $\text{disc}(C)$, we denote $C \cup \text{disc}(C)$ by $\overline{\text{disc}(C)}$. Any point p in $\text{disc}(C)$ is said to be *surrounded* by C and C is said to be *surrounding* p. The curves in this work have the following two intersection restrictions: The *s-intersection property* and the *connected intersection property.*

Definition 1.0.1. We say that a collection of curves $\mathcal{C}$ has the *s-intersection property* if s is the minimum integer such that any two closed curves in $\mathcal{C}$ intersect properly in at most s points. We say that $\mathcal{C}$ has the *connected intersection property* if for each pair of curves $C, C' \in \mathcal{C}$ the set $\text{disc}(C) \cap \text{disc}(C')$ is either empty or connected.

A collection of lines or pseudo-lines in the plane has the 1-intersection property, circles and pseudo-circles have both the 2-intersection property and the connected intersection property. The latter are special cases of the curves we deal with.

In Chapter 2, we investigate the VC-dimension of the above mentioned set system. The VC-dimension of a set system is defined as follows:

Definition 1.0.2. Let $\mathcal{F}$ be a set system on X and let $Y \subseteq X$. We say that Y is *shattered* by $\mathcal{F}$ if for every $A \subseteq Y$ there exists $S \in \mathcal{F}$ such that $A = Y \cap S$. The *VC-dimension* of $\mathcal{F}$, denoted by $VCdim(\mathcal{F})$ is the supremum of the sizes of all the subsets of X with finite size that are shattered by $\mathcal{F}$.

An important result related to the VC-dimension is the Shatter Function Lemma which was proved independently by Sauer [12], Perles and Shelah [14] and by Vapnik and Chervonenkis [17]. We need to define the *shatter function* notion in order to state this lemma.

Definition 1.0.3. The *shatter function* of a set system $\mathcal{F}$ on a ground set X is:

$$\pi_{\mathcal{F}}(m) = \max_{Y \subseteq X, |Y|=m} |S \cap Y : S \in \mathcal{F}|$$

Lemma 1.0.4. *Shatter Function Lemma: For any set system $\mathcal{F}$ on a ground set X that has VC-dimension at most d, $\pi_{\mathcal{F}}(m) \leq \binom{m}{0} + \binom{m}{1} + ... + \binom{m}{d}$. In particular, if $|X| = n$, then $|\mathcal{F}| = \pi_{\mathcal{F}}(n) \leq \binom{n}{0} + \binom{n}{1} + ... + \binom{n}{d}$.*

VC-dimension and the Shatter Function Lemma play an important role in several mathematical fields such as statistic, computational learning theory, combinatorics of hypergraphs, and discrepancy theory. The importance of the VC-dimension is that it enables one to obtain an upper bound on the size of a set-system via the Shatter Function Lemma. Although the obtained bound is not necessarily the best upper bound, in our case, the upper bound on the set-system which will be achieved via the VC-dimension is also a tight upper bound.

In Theorem 2.1.1 we provide an upper bound on the VC-dimension of the set-system $\mathcal{F} = \{P \cap \text{disc}(C) | C \in \mathcal{C}\}$ of a set P of points in the plane, where $\mathcal{C}$ is a collection of simple closed curves in the plane that has both the s-intersection property and the connected intersection property. We show that the VC-dimension of $\mathcal{F}$ is at most $s + 1$. In particular, we use the Shatter Function Lemma in Corollary 2.1.2 to conclude that if $|P| = n$ and every curve in $\mathcal{C}$ surrounds a unique subset of P, then $|\mathcal{C}| = O(n^{s+1})$. In Theorem 2.1.3 we introduce a specific set of points in the plane and a collection of $O(n^{s+1})$ bi-infinite and x-monotone curves (the graphs of continues real functions) such that every curve lies above a unique subset of the points, and every pair of curves in that collection properly cross at most s times. This collection can be easily realized as a collection of simple closed curves that satisfy the conditions of Theorem 2.1.1, thus Theorem 2.1.3 implies that the upper bounds on the VC-dimension in Theorem 2.1.1 is the best possible.

Chapter 3 deals with sweeping an arrangement of curves by a ray. The curves we consider are simple closed curves in the plane that have both the s-intersection property and the connected intersection property.

Definition 1.0.5. An *arrangement of curves* is a quadruple $(\mathcal{C}, V(\mathcal{C}), E(\mathcal{C}), F(\mathcal{C}))$. $\mathcal{C}$ is a finite set of simple closed curves in the plane. We denote the arrangement by $\mathcal{C}$ as well. We assume throughout this work that $\cup_{C \in \mathcal{C}} C$ is connected and that every pair of curves in $\mathcal{C}$ is either disjoint or properly cross finite number of times. A *vertex* of $\mathcal{C}$ is an intersection point of at least two curves in $\mathcal{C}$. An arrangement is *simple* if no three curves share a common point. We assume in this work that the arrangements are simple. $V(\mathcal{C})$ is the set of vertices of $\mathcal{C}$. An *edge* of $\mathcal{C}$ is a connected component of $(\cup_{C \in \mathcal{C}} C) \setminus V(\mathcal{C})$. $E(\mathcal{C})$ is the set of edges of $\mathcal{C}$. A *face* of $\mathcal{C}$ is a connected component of $\mathbb{R}^2 \setminus \cup_{C \in \mathcal{C}} C$. $F(\mathcal{C})$ is the set of faces of $\mathcal{C}$.

Sweeping is an important tool used in mathematical proofs. The underlying idea is to determine properties of a collection of objects in a space of dimension d by looking at a series of consecutive $(d-1)$-dimensional slices. Sweeping converts a static problem into a dynamic problem of lower dimension. Sweep algorithms can refer to an arrangement of closed curves, bi-infinite x-monotone curves, lines segment and various other geometric objects in the plane or in higher dimensional spaces. There are different types of sweeps, depending on the curve used to sweep the arrangements. For example, this curve may be a line, a pseudo-line, a closed curve or a ray. J. Snoeyink and J. Hershberger [15] introduced sweeping process by a ray, a pseudo-line and a pseudo circle.

As an example of sweep algorithms in the literature, consider the problem of finding the intersections of n lines or segments in the plane. Shamos and Hoey [13] showed how to detect an intersection in $O(n \log(n))$ time by sweeping the plane with a line. Chazelle and Edelsbrunner [5] developed an algorithm to report all K segments intersection in $O(n \log(n) + K)$ time by sweeping the plane with a pseudo-line.

In Chapter 3 we consider an arrangement of curves $\mathcal{C}$ that are surrounding a common point in the plane. We are interested in sweeping the arrangement by a ray, that is a Jordan arc connecting a point in the plane to infinity. We now define the notion of sweeping an arrangement of curves by a ray.

Let $\mathcal{C}$ be an arrangement of simple closed curves in the plane. Assume that all the curves in $\mathcal{C}$ surround a common point O and there exists a ray γ, starting from O and intersecting every curve in $\mathcal{C}$ exactly once. A sweep of $\mathcal{C}$ by the ray γ is a continuous rotation of γ around O, say, counter-clockwise, until it returns to its starting position, such that γ always intersects each of the curves in $\mathcal{C}$ exactly once. Although sweeping is a continuous process, we will see in Chapter 3 that we can carry it out in discrete steps.

J. Snoeyink and J. Hershberger [15] proved the following:

Theorem 1.0.6. *Any family of pseudo circles surrounding a common point can be swept by a ray.*

In Theorem 3.1.1 we generalize this result. We show the existence of a sweep by a ray of an arrangement $\mathcal{C}$ of s-intersecting simple closed curves that are surrounding a common point and has the connected intersection property. Theorem 3.1.1 implies that $\mathcal{C}$ can be realized, after one-to-one and continuous transformation of the plane, as a collection of s-intersecting bi-infinite x-monotone curves, and thus we can implement results related to bi-infinite x-monotone curves on $\mathcal{C}$. One example is the following result from [?]:

Theorem 1.0.7. *Let P be a set of n points in the plane and let $\mathcal{C}$ be a family of s-intersecting bi-infinite x-monotone curves, where $s \geq 0$ is an even integer. Let $\mathcal{F}$ be the set-system on the ground set P, consists of all the subsets $S \subseteq P$ of which there exists a curve $C \in \mathcal{C}$, such that all the points in S lie below C and all the points in $P \setminus S$ lie above C. For every integer $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$ consider the subset of $\mathcal{F}$, $\mathcal{F}_{\mathcal{C},P,k} = \{S \in \mathcal{F} || S| = k\}$. Then, $|\mathcal{F}_{\mathcal{C},P,k}| = O((kn)^{\frac{s}{2}})$ and this upper bound on $|\mathcal{F}_{\mathcal{C},P,k}|$ is best possible.*

By Theorem 3.1.1, this result is also valid if $\mathcal{C}$ is a collection of s-intersecting simple closed curves that are surrounding a common point and have the connected intersection property. In that case we define $\mathcal{F}_{\mathcal{C},P,k}$ as the set-system on P, consists of all the k-points subsets $S \subseteq P$ such that there exists a curve $C \in \mathcal{C}$ for which $S = P \cap \text{disc}(C)$.

Chapter 4 deals with a special case of a problem of S. Fekete and G.J Woeginger [8] and [9]. Given a set X of n points in the plane, an ordering of the point of X, $x_1x_2...x_n$ is identified with a polygonal path P on X: its edges are the straight segments connecting x_i to x_{i+1}. The angle of P at x_i is $\measuredangle x_{i-1}x_ix_{i+1}$.

Definition 1.0.8. Let $\alpha > 0$. We call the path $x_1x_2...x_n$ *α-good* if its angle in x_i is at least α for every $2 \leq i \leq n-1$.

Fekete and Woeginger conjectured the following:

Conjecture 1.0.9. *For every finite set X of points in the plane, there exists a $\frac{\pi}{6}$-good path on X.*

In [3] Bárány, Pór and Valtr proved the following Theorems:

Theorem 1.0.10. *There exists a $\frac{\pi}{9}$-good path on every finite set of points in the plane.*

They also generalized this result for higher-dimensional spaces:

Theorem 1.0.11. *For every $d \geq 2$ there is a positive α_d such that for every finite set of points $X \in \mathbb{R}^d$ there exists a α_d-good path on X.*

In [2] Bárány and Pór generalized this problem to an infinite set of points in the plane, and proved the existence of $\frac{\pi}{20}$-good path.

In Chapter 4 we focus on a special case of this problem, where X is a set of points in the plane in convex position. A set of points X in the plane is in *convex position* if the points in X are the vertices of some convex polygon. In Theorem 4.0.5 we prove that for a set X of points in the plane in convex position, there exists a $\frac{\pi}{5}$-good path on X. We prove further that there are arbitrarily large sets of points in the plane in convex position with no α-good paths, where $\alpha > \frac{\pi}{5}$.

Chapter 2

The VC-Dimension of S-Intersecting Curves

2.1 Introduction and Basic Definitions

In this chapter we study the VC-dimension of a set-system $\mathcal{F}$ on a set of points in the plane. Given a finite set P of points in the plane, a *set-system* of P is a family of subsets of P. For a subset $S \subseteq P$, we say that $\mathcal{F}$ *shatters* S, if for every $B \subseteq S$, there exists $A \in \mathcal{F}$ such that, $B = S \cap A$. The VC-dimension of $\mathcal{F}$ is the largest cardinality of a subset of P that $\mathcal{F}$ shatters. In this chapter we will prove Theorem 2.1.1. We are interested in a particular set-system on the ground set P. Its elements are all the subsets of P that are surrounded by curves in a given collection of curves $\mathcal{C}$. By Jordan's Theorem a simple closed Jordan curve C divides the plane into two regions, only one of which is bounded. We call the bounded region the *disc* bounded by C and we denote this region by $\text{disc}(C)$, we denote $C \cup \text{disc}(C)$ by $\overline{\text{disc}(C)}$. Any point p in $\text{disc}(C)$ is said to be *surrounded* by C and C is said to be *surrounding* p. The collection $\mathcal{C}$ of the curves we will consider will have the *s-intersection property*, i.e. s is the minimum integer such that any two curves in $\mathcal{C}$ intersect properly in at most s points. We assume that $\mathcal{C}$ has also the *connected intersection property* which implies that for each pair of curves $C, C' \in \mathcal{C}$ the set $\text{disc}(C) \cap \text{disc}(C')$ is either empty or connected region (see Definition 1.0.1).

For every $C \in \mathcal{C}$ we denote $P_C = P \cap \text{disc}(C)$ and define a set system of P by $\mathcal{F}_{\mathcal{C}|P} = \{P_C | C \in \mathcal{C}\}$. We call $\mathcal{F}_{\mathcal{C}|P}$ the *restriction of* $\mathcal{C}$ *to* P.

With this notations and definitions we state the main Theorem of this Chapter:

Theorem 2.1.1. *Let P be a set of n points in the plane and let $\mathcal{C}$ be a family of simple closed Jordan curves that has the s-intersecting property, for some integer $s \geq 2$, and the connected intersection property as well. Then the VC-dimension of the set system $\mathcal{F}_{\mathcal{C}|P}$ is at most $s+1$.*

The next Corollary follows immediately from Theorem 2.1.1 and the Shatter Function Lemma:

Corollary 2.1.2. *Let P be a set of n points in the plane and let $\mathcal{C}$ be a collection of simple closed curves, each of which surrounding a unique subset of P. If $\mathcal{C}$ has the s-intersecting property and the connected intersection property then $|\mathcal{F}| = |\mathcal{C}| = O(n^{s+1})$.*

We will show further that this upper bound on $|\mathcal{C}|$ is tight up to a multiplicative constant.

Theorem 2.1.3. *For all $n \in \mathbb{N}$, there exists a set of n points P and a collection of simple closed curves $\mathcal{C}$, that has the s-intersecting property and the connected intersection property, such that any curve in $\mathcal{C}$ surrounds a unique subset of P and $|\mathcal{C}|= \Omega(n^{s+1})$.*

Theorem 2.1.3 and the Shatter Function Lemma imply that the upper bound on the VC-dimension of $\mathcal{F}$ in Theorem 2.1.1 is best possible.

2.2 Preliminaries

One of the most famous results in combinatorial geometry is Helly's theorem ([10]):

Theorem 2.2.1. *Let $R_1, R_2, ..., R_n$ be convex sets in the plane. Suppose that the intersection of every three of these sets is nonempty. Then the intersection of all the sets is nonempty.*

The next lemma is a generalization of Helly's theorem proved by Molnár ([11]):

Lemma 2.2.2. *Any finite family of at least three regions in the plane has a nonempty simply connected intersection, provided that any two of its members have a connected intersection and any three have a nonempty intersection.*

We will need also the following lemma that can be found in [1]:

Lemma 2.2.3. *Let $\mathcal{C}$ be a family of closed curves that has the connected intersection property. Assume that all the curves in $\mathcal{C}$ surround a common point O. Then for every subset $\mathcal{D} \subseteq \mathcal{C}$, $\cup_{C \in \mathcal{D}} disc(C)$ is simply connected.*

Before getting to the proof of Theorem 2.1.1, we need one more crucial lemma:

Lemma 2.2.4. *Let $\mathcal{C}$ be a finite family of closed curves. Assume that the union of the closure of any number of discs bounded by curves in $\mathcal{C}$ is simply connected. Let y be an arbitrary point in $\mathbb{R}^2 \setminus \cup_{C \in \mathcal{C}} C$. Consider the family $\mathcal{C}_y \subseteq \mathcal{C}$ of all the curves in $\mathcal{C}$ that surround y. Then there exists a Jordan arc, connecting y to a point at infinity, that intersects every curve in $\mathcal{C}_y$ exactly once and avoids all the curves in $\mathcal{C} \setminus \mathcal{C}_y$.*

Proof. We shall prove the lemma by induction on $|\mathcal{C}_y|$. The case $|\mathcal{C}_y| = 0$ is easy because in this case $y \in \mathbb{R}^2 \setminus \cup_{C \in \mathcal{C}} \text{disc}(C)$. Because we assume that the union of all discs is simply connected and hence $\mathbb{R}^2 \setminus \cup_{C \in \mathcal{C}} \text{disc}(C)$ is a unbounded connected set. In particular there exists a Jordan arc, contained in $\mathbb{R}^2 \setminus \cup_{C \in \mathcal{C}} \text{disc}(C)$, that connects y to a point at infinity.

Suppose $|\mathcal{C}_y| > 0$. The induction hypothesis states that for any point $p \in \mathbb{R}^2 \setminus \cup_{C \in \mathcal{C}} C$ with $|\mathcal{C}_p| < |\mathcal{C}_y|$, there exists an arc, connecting p to a point at infinity, which intersects every curve in $\mathcal{C}_p$ exactly once and avoids all the curves in $\mathcal{C} \setminus \mathcal{C}_p$. The arrangement of curves in $\mathcal{C}$ can be viewed as a drawing of a planar graph with a vertex set V, consisting of all the intersection points of curves in $\mathcal{C}$, together with a set of edges E, consisting of all the connected components in $\cup_{C \in \mathcal{C}} C \setminus V$. There exists a face F_y in this arrangement that contains y. The face F_y must be bounded since $|\mathcal{C}_y| > 0$. An edge of F_y will be called an *inner edge* if it is a portion of a curve in $\mathcal{C}_y$. We claim that F_y must have an inner edge. To see this, assume to the contrary that F_y does not have an inner edge. Consider the set of all curves in $\mathcal{C}$ which contain an edge of F_y and let U be the union of all the discs bounded by these curves. By our assumption, U is a simply connected region. Observe that $y \notin U$, and any arc from y to infinity must cross U. Thus $\mathbb{R}^2 \setminus U$ is not connected, hence U is not simply connected, which yields a contradiction. We conclude that F_y must have an inner edge. Let us choose an inner edge of F_y and draw an arc γ, starting at y, which crosses the inner edge once and does not cross any other curve. Denote by x the endpoint of γ. Observe that every curve in $\mathcal{C}$ that surrounds x must surround y as well, i.e. $\mathcal{C}_x \subseteq \mathcal{C}_y$. Moreover, $|\mathcal{C}_x| = |\mathcal{C}_y| - 1$. By applying the induction hypothesis on x we get an arc γ_x, connecting x to a point at infinity, that intersects every curve in $\mathcal{C}_x$ exactly once and avoids any other curve. By adjoining γ to γ_x, we obtain the desired arc connecting y to a point at infinity. ∎

Lemma 2.2.3 and Lemma 2.2.4 can be combined to the following Lemma:

Lemma 2.2.5. *Let $\mathcal{C}$ be a family of closed curves that has the connected intersection property. Assume that all the curves in $\mathcal{C}$ surround a common point O. Let y be an arbitrary point in $\mathbb{R}^2 \setminus \cup_{C\in\mathcal{C}} C$. Consider the family $\mathcal{C}_y \subseteq \mathcal{C}$ of all the curves in $\mathcal{C}$ that surround y. Then there exists a Jordan arc, connecting y to a point at infinity, that intersects every curve in $\mathcal{C}_y$ exactly once and avoids all the curves in $\mathcal{C} \setminus \mathcal{C}_y$.*

2.3 Sketch of the Proof of Theorem 2.1.1

Our goal is to show that $\mathcal{F}$ can not shatter any $s+2$ points subset of P. Assume to the contrary that $\mathcal{F}$ shatters a set $S = \{v_1, ..., v_{s+2}\} \subseteq P$ of $s+2$ points, i.e. for any subset $V \subseteq S$, there exists a curve $C \in \mathcal{C}$ with $P_C \cap S = V$. For every pair $v_i, v_j \in S$, consider the set of curves $\mathcal{C}_{ij} \subseteq \mathcal{C}$ consisting of all the curves in $\mathcal{C}$ that surround both v_i and v_j. Consider also the set R_{ij} of all the points in the plane which are surrounded by every curve in $\mathcal{C}_{ij}$. Since $\mathcal{C}$ has the connected intersection property, Lemma 2.2.2 implies that R_{ij} is a connected region. Upon drawing an edge (v_i, v_j) between v_i and v_j inside the region R_{ij}, we obtain a drawing of the complete graph on $s+2$ vertices, K_{s+2} as a topological graph in the plane that we denote by $G = (S, E)$. We shall investigate the special properties of G, that will eventually lead us to a contradiction.

2.3.1 The Convex Case

To understand the properties of G let us consider a special case of the problem in which the points are in convex position and the discs bounded by curves in $\mathcal{C}$ are convex sets. Since the points in P are in convex position, there is a cyclic order on the points $S = \{v_1, v_2, ..., v_{s+2}\}$ starting at a point v_1 and moving on the convex polygon formed by the $s+2$ points, counter-clockwise. Note that the intersection of the disc bounded by any pair of curves in $\mathcal{C}$ is either empty or convex. Therefore, the connected intersection requirement is automatically valid and we only need to require that $\mathcal{C}$ has the s-intersecting property. By Helly's Theorem it follows that for every pair of points $v_i, v_j \in S$, R_{ij} is a convex region. We draw a straight segment between v_i and v_j inside the region R_{ij} for all i and j and obtain a drawing of K_{s+2} as a geometric graph to which we also refer as K_{s+2} (see Figure 2. 1).

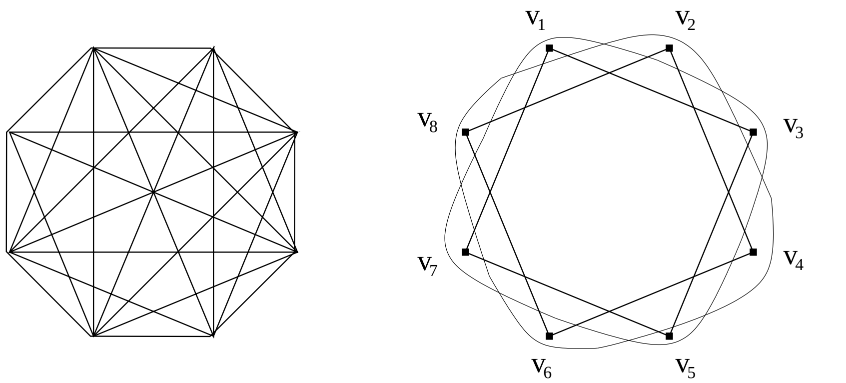
We will show the existence of two curves in $\mathcal{C}$ that intersect $s+2$ times and thus obtain a contradiction.

Consider the following two subsets of S:

$$S_1 = \{v_i | i \text{ is odd}\} \qquad S_2 = \{v_i | i \text{ is even}\}$$

Since $\mathcal{F}$ shatters S, there are curves $C_1, C_2 \in \mathcal{C}$ such that $P_{C_1} \cap S = S_1$ and $P_{C_1} \cap S = S_2$. As figure 2. 1 illustrates, C_1 and C_2 intersects $s+2$ times. We will prove it more formally in the general case.

This observation in the convex case motivated us to search for a similar behavior in the more general case. We will show that there exists a cyclic order on S such that every pair of edges $e_1, e_2 \in E$ cross each other an odd number of times if and only if their corresponding edges in K_{s+2} cross.



Case s=6: we obtain a drawing of. K_8

One curve surrounds all the vertices with odd indices and the other curve surrounds all the veritces with even indices. The curves must intersect 8 times.

Figure 2. 1: The convex case for $s = 6$

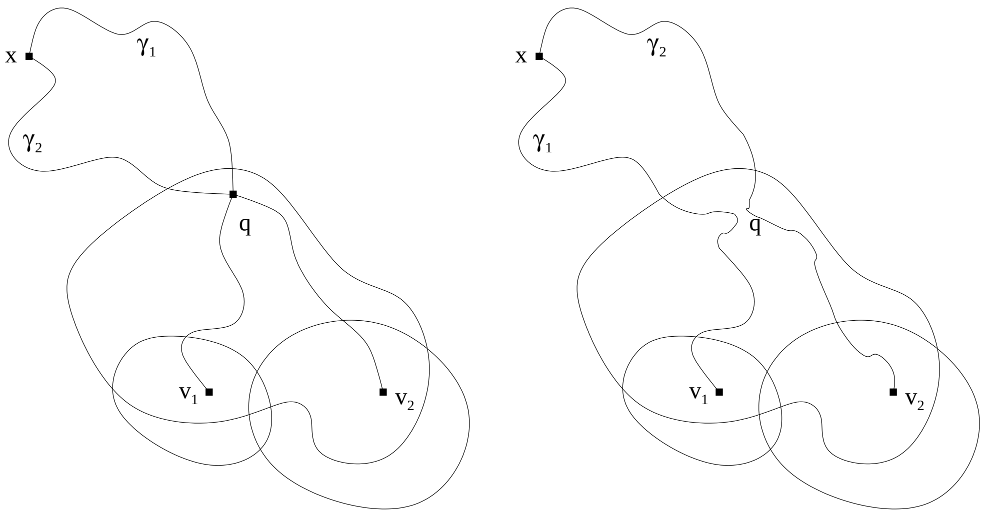
2.4 Proof of Theorem 2.1.1

Using the same notation introduced above, we shall now prove few results that will help us understand the topological graph $G = (S, E)$.

Claim 2.4.1. *Let x be a point in the plane that lies in $\mathbb{R}^2 \setminus \overline{\cup_{C \in \mathcal{C}} disc(C)}$. Then for every vertex $v_i \in S$ one can draw an arc γ_i, connecting v_i and x, that does not intersect any curve $C \in \mathcal{C}$ with $P_C \cap S = S \setminus \{v_i\}$. Moreover, this drawing can be such that no two arcs γ_i and γ_j cross.*

Proof. Let $\mathcal{D}$ be the subset of $\mathcal{C}$ consisting of all the curves $C \in \mathcal{C}$ with $|P_C \cap S| = s+1$. Since $s \geq 2$ it follows that $|\mathcal{D}| \geq 3$ and that any three discs bounded by curves in $\mathcal{D}$ have a non-empty intersection. Furthermore, because $\mathcal{D} \subseteq \mathcal{C}$, any two discs bounded by curves in $\mathcal{D}$ have a connected intersection. By Lemma 2.2.2, there exists a point in the plane that is surrounded by all the curves in $\mathcal{D}$. By Lemma 2.2.3, the union of any set of discs bounded by curves in $\mathcal{D}$ is simply connected. Thus, for every vertex $v_i \in S$ one can apply Lemma 2.2.4 and draw an arc γ_i, connecting v_i with x, such that γ_i avoids any curve $C \in \mathcal{D}$ with $P_C \cap S = S \setminus \{v_i\}$ and crosses any other curve in D exactly once. From all the possible drawings of such arcs, we pick one with minimum number of intersection points among the γ_i's. We shall prove that this minimum is 0. Assume otherwise, then there exists a pair of arcs γ_i and γ_j that cross at a point q. We denote by $\gamma_{i,q}$ and $\gamma_{j,q}$ the portions of γ_i and γ_j, respectively, which connect q with x. Both $\gamma_{i,q}$ and $\gamma_{j,q}$ avoid the curves in $\mathcal{D}$ which do not surround q and intersect once the curves in $\mathcal{D}$ which surround

q. By swapping the portions $\gamma_{i,q}$ with $\gamma_{j,q}$ and by a small modification of the drawing, we can eliminate the crossing point q and obtain a new drawing of arcs that has one less crossing point. See Figure 2. 2. This new drawing still satisfies the property that for every t, each γ_t crosses the curves in $\mathcal{D}$ which surround v_t exactly once and avoids all the other curves in $\mathcal{D}$. This constitutes a contradiction to the minimality of the number of intersection points among the arcs γ_t in the selected drawing. ∎



The arcs γ_1, γ_2 cross in q

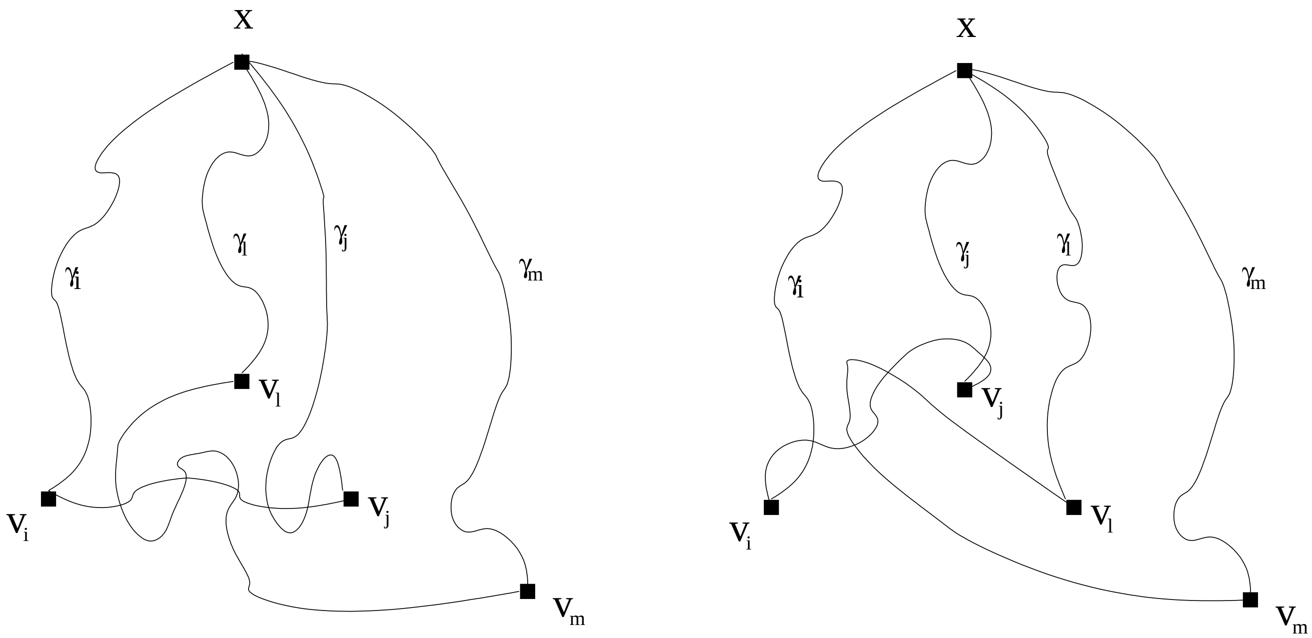
After a small modification of the drawing and swapping the arcs portions, the arcs do not cross

Figure 2. 2: Eliminating intersection point between two arcs

Let us draw an arc γ_i for every $v_i \in S$ according to Claim 2.4.1. Pick an arc, say γ_1, and define a cyclic order on the arcs γ_i, according to the counterclockwise order in which they reach x, starting with γ_1. Assume without loss of generality that this order is $(\gamma_1, \ldots, \gamma_{s+2})$. Then $(v_1, ..., v_{s+2})$ is a cyclic order on S.

Claim 2.4.2. *For every four distinct vertices $v_i, v_j, v_l, v_m \in S$ the edges (v_i, v_j) and (v_l, v_m) of the graph G cross an odd number of times if and only if i and j separate l and m in the natural cyclic order of* $(1, \ldots, s+2)$.

Proof. We denote by $\triangle_{ij}$ the closed curve that is composed by the arcs γ_i, γ_j and the edge (v_i, v_j) in G. We define $\triangle_{lm}$ similarly. The curves $\triangle_{ij}$ and $\triangle_{lm}$ meet at x. Observe that any other intersection point between $\triangle_{ij}$ and $\triangle_{lm}$ must be an intersection point of the edges (v_i, v_j) and (v_l, v_m). To see this, recall that in our drawing no two of the arcs $\gamma_1, \ldots, \gamma_{s+2}$ cross. Moreover, an arc γ_t connecting v_t to x may cross only those edges of G that are incident to v_t. This is because $\mathcal{F}$ shatters S and therefore there exists a curve $C \in \mathcal{C}$ with $P_C \cap S = S \setminus \{v_t\}$. By the construction of γ_t it avoids disc(C). Since any edge in G, not incident to v_t, is contained in disc(C), γ_t cannot cross any edge that is not incident to v_t. We conclude that any intersection point between $\triangle_{ij}$ and $\triangle_{lm}$, other than x, must be an intersection point of the edges (v_i, v_j) and (v_l, v_m). If i and j separate l and m in the natural cyclic order $(1, \ldots, s+2)$, then the curves $\triangle_{ij}$ and $\triangle_{lm}$ properly cross at x. The number of intersection points between two closed curves is even and therefore the edges (v_i, v_j) and (v_l, v_m) must cross an odd number of times. If i and j do not separate l and m in the natural cyclic order, then $\triangle_{ij}$ and $\triangle_{lm}$ *touch* at x. As all other intersection points between $\triangle_{ij}$ and $\triangle_{lm}$ are intersection points of (v_i, v_j) and (v_l, v_m), it follows that (v_i, v_j) and (v_l, v_m) cross an even number of times. See figure 2. 3 ∎



The edges (Vi,Vj) and (Vl,Vm) intersect an odd number of times, since i and j separa te l and m in the cyclic order

The edges (Vi,Vj) and (Vl,Vm) intersect an even number of times, since i and j do not separate l and m in the cyclic order

Figure 2. 3: The oddness of the number of intersections between the edges (v_i, v_j) and (v_l, v_m)

We consider the following two subsets S_1 and S_2 of S:

$$S_1 = \{v_i \in S \mid i \text{ is odd}\} \qquad S_2 = \{v_i \in S \mid i \text{ is even}\}.$$

Since $\mathcal{F}$ shatters S, there exist curves C_1, $C_2 \in \mathcal{C}$ such that $P_{C_1} \cap S = S_1$ and $P_{C_2} \cap S = S_2$. We will show that the curves C_1 and C_2 intersect in at least $s + 2$ points and obtain a contradiction to the assumption that $\mathcal{C}$ has the s-intersection property.

We call each connected component of $\text{disc}(C_1) \setminus \text{disc}(C_2)$ an *ear*. Similarly, each connected component of $\text{disc}(C_2) \setminus \text{disc}(C_1)$ is called an *ear*. We say that C_1 *enters* C_2 at a crossing point u of C_1 and C_2 if a small enough portion of C_1 that starts at u and continues in the counterclockwise orientation along the curve C_1 is contained in $\text{disc}(C_2)$. Otherwise we say that C_1 *leaves* C_2 at u. We use a similar terminology with respect to C_2.

Claim 2.4.3. *Let C_1 and C_2 be two curves with the connected intersection property. Assume that $u_1, u_2, ..., u_m$ is the set of intersection points of C_1 and C_2 arranged in a counterclockwise order along C_1 and $w_1, w_2, ..., w_m$ is the same set of the intersection points of C_1 and C_2 arranged in a counterclockwise order along C_2, and assume without loss of generality that $u_1 = w_1$. Then $u_i = w_i$ for every $i = 1, \ldots, m$.*

Proof. Assume to the contrary that $u_k \neq w_k$ for some $1 \leq k \leq n$, then without loss of generality we can assume that $k = 2$ (otherwise, let i be the maximum index such that $u_i = w_i$ and replace u_1 with u_i). Without loss of generality assume that C_2 enters C_1 at u_1. Then C_1 leaves C_2 at u_1. We will get a contradiction by showing that $w_2 = u_2$. Assume to the contrary that $w_2 = u_j$ for some $2 < j \leq m$. Then $u_2 = w_l$ for some $2 < l \leq m$. The curve C_1 must enter C_2 at the point $u_2 = w_l$ because it leaves C_2 at u_1. Therefore, C_2 leaves C_1 at w_l and consequently must enter C_1 at the point w_{l-1}. It follows that the portion δ of C_2 between w_1 and w_2 in the counterclockwise direction along C_2 is contained in disc(C_1). Similarly, the portion δ' of C_2 between w_{l-1} and w_l in the counterclockwise direction along C_2 is contained in disc(C_1). δ and δ' split disc(C_1) into three regions A_1, A_2, and A_3, where A_1 is the region bounded by δ and a portion of C_1, A_2 is the region bounded by both δ and δ' and two portions of C_1, and A_3 is the region bounded by δ' and a portion of C_1. The portion γ of C_1 between $u_1 = w_1$ and $u_2 = w_l$ in the counterclockwise direction along C_1 is connecting a point on δ, namely, w_1, with a point on δ', namely, w_l. Since u_1 and u_2 are the only intersection points of C_1 and C_2 on γ, it follows that γ is contained in the boundary of A_2. Because C_1 leaves C_2 at u_1 and enters C_2 at u_2, it must be that γ lies entirely outside of disc(C_2). It follows that the interior of A_1 must contain points of disc(C_1) $\cap$ disc(C_2), and similarly, the interior of A_3 must contain points of disc(C_1) $\cap$ disc(C_2). This is a contradiction to the assumption that the interior of disc(C_1) $\cap$ disc(C_2) is a connected set. We conclude that $u_i = w_i$ for every $i = 1, \ldots, m$ ∎

Claim 2.4.4. *If C_1 and C_2 properly cross in exactly m points, then they create precisely m ears.*

Proof. Let $u_1, u_2, ..., u_m$ be the set of intersection points of C_1 and C_2 arranged in a counterclockwise order along C_1. By Claim 2.4.3 this set is the set of intersection points of C_1 and C_2 arranged in a counterclockwise order along C_2. For every $1 \leq i \leq m$ the portion of C_1 and C_2 between u_i and $u_{(i+1) mod(m)}$ forms an ear. Hence, there are at least m ears. We consider $C_1 \cup C_2$ as a planar graph with m vertices and $2m$ edges. By Euler's formula we have $m - 2m + F = 2$, where F is the number of faces created by C_1 and C_2. Hence, $F = m + 2$. This count includes the unbounded face, namely $\mathbb{R}^2 \setminus$ (disc(C_1) $\cup$ disc(C_2)), as well as the intersection disc(C_1) $\cap$ disc(C_2). We deduce that there are exactly m ears. ∎

We now show that the curves C_1 and C_2 cross in at least $s + 2$ points and thus obtain a contradiction to our assumption that $\mathcal{C}$ has the s-intersection property.

Note that each vertex in S_1 is surrounded by C_1 but not by C_2. Therefore, each vertex in S_1 belongs to an ear. Similarly, every vertex in S_2 belongs to an ear. Obviously, a vertex in S_1 and a vertex in S_2 cannot belong to the same ear.

We claim further that even if v_i and v_j are two vertices which belong to S_1, then they cannot belong to the same ear (we argue similarly if the two vertices belong to S_2). Assume to the contrary that $v_i, v_j \in S_1$ belong to the same ear R. R is contained in disc(C_1). Draw an arc γ inside R connecting v_i to v_j (see Figure 2. 4). The edge of G connecting v_i and v_j together with γ form a closed curve $\tilde{C}$ that lies inside disc(C_1). The vertices $v_{i+1}, v_{j+1} \in S_2$ are surrounded by C_2 but not by C_1 and therefore, any arc connecting v_{i+1} and v_{j+1} must cross $\tilde{C}$ an even number of times. By Claim 2.4.2, the edge of G between v_{i+1} and v_{j+1} crosses the edge of G between v_i and v_j an odd number of times but does not cross γ, as γ lies entirely outside disc(C_2). Hence, the edge of G connecting v_{i+1} and v_{j+1} crosses $\tilde{C}$ an odd number of times, a contradiction.

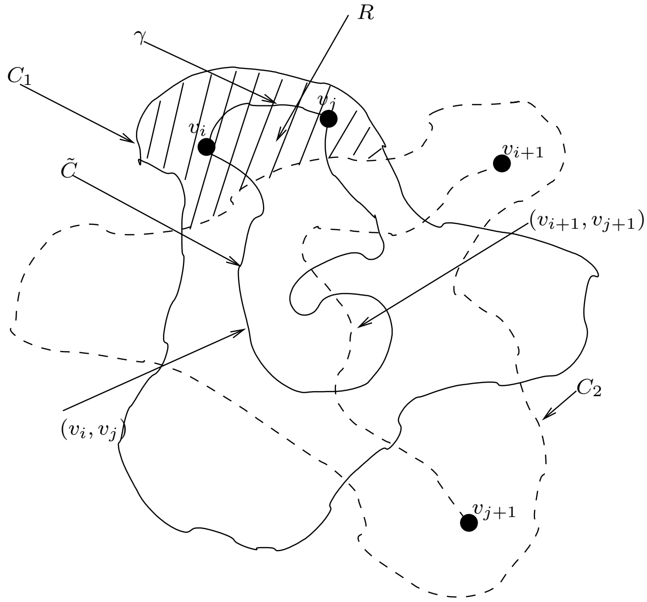


Figure 2. 4: The curves C_1 and C_2

We conclude that each vertex in S belong to a unique ear.

This implies that there are at least $s+2$ ears. It follows from Claim 2.4.4 that C_1 and C_2 intersects in at least $s+2$ points, which is the desired contradiction.

This also concludes the proof of Theorem 2.1.1, as we have shown that $\mathcal{F}$ does not shatter any set of $s+2$ points. ∎

2.5 Proof of Theorem 2.1.3

It is an immediate corollary of Theorem 2.1.1 and the Shatter Function Lemma that if P is a set of n points in the plane and $\mathcal{C}$ is a family of simple closed curves with the s-intersection property and the connected intersection property, then $\mathcal{F} = \{P_C \mid C \in \mathcal{C}\}$ consists of $O(n^{s+1})$ members.

We will show, by a construction, that this bound can indeed be attained. For every fixed even number $s \geq 2$, we will construct a set of n points P and a family $\mathcal{C}$ of bi-infinite x-monotone curves with the s-intersection property such that $\mathcal{F} = \{P_C | C \in \mathcal{C}\}$ consists of $\Omega(n^{s+1})$ members, where P_C is defined as the set of all points in P that lie bellow C. A *bi-infinite x-monotone* curve is a curve that intersects every vertical line exactly once. Note that the intersection of the regions below each pair of curves in $\mathcal{C}$ is a connected region, and therefore any collection of bi-infinite x-monotone curves satisfies the connected intersection property. It is then an easy exercise to modify $\mathcal{C}$ to a family of s-intersecting simple closed curves that has the connected intersection property, closing each curve at infinity.

Let P be the set of integer lattice points $P = \{(a,b) \mid 1 \leq a \leq s+1 \text{ and } 1 \leq b \leq \frac{n}{s+1}\}$. Then for every $(s+1)$-tuple $(b_1, \ldots, b_{s+1}) \in \{1, \ldots, \frac{n}{s+1}\}^{s+1}$, let $C_{b_1,\ldots,b_{s+1}}$ be the graph of the polynomial of degree at most s passing through each of the points $(i, b_i + \frac{1}{2})$ for $i = 1, \ldots, s+1$. Let $\mathcal{C}$ be the collection of all these curves.

Since each of the curves in $\mathcal{C}$ is a graph of a polynomial of degree at most s, it follows immediately that $\mathcal{C}$ has the s-intersection property. Finally, note that the number of curves in $\mathcal{C}$ is $(\frac{n}{s+1})^{s+1} = \Omega(n^{s+1})$. Each curve in $\mathcal{C}$ determines a unique subset of P, P_C, consisting of all the points in P that lie bellow C. Therefore $|\mathcal{F}| = \Omega(n^{s+1})$ and by the Shatter Function Lemma, the VC-dimension of $\mathcal{F}$ is at least $s+1$.

Chapter 3

Sweeping an Arrangement of s-Intersecting Curves By a Ray

3.1 Introduction and Basic Definitions

In this chapter we prove Theorem 3.1.1, which is a generalization of a result of J. Snoeyink and J. Hershberger [15], stated in Theorem 1.0.6. But first we need to define the notion of sweeping an arrangement of curves by a ray. An *arrangement of curves* is a quadruple $(\mathcal{C}, V(\mathcal{C}), E(\mathcal{C}), F(\mathcal{C}))$. $\mathcal{C}$ is a finite set of simple closed curves in the plane. We denote the arrangement by $\mathcal{C}$ as well. We assume that every curve is intersected by any other curve of $\mathcal{C}$ only finitely many times. We assume throughout this work that at any point in which two curves meet they intersect properly and never just touch. A *vertex* of $\mathcal{C}$ is an intersection point of at least two curves in $\mathcal{C}$. An arrangement is *simple* if no three curves share a common point. We assume in this work that the arrangements are simple. $V(\mathcal{C})$ is the set of vertices of $\mathcal{C}$. An *edge* of $\mathcal{C}$ is a connected component of $(\cup_{C\in\mathcal{C}}C)\setminus V(\mathcal{C})$. $E(\mathcal{C})$ is the set of edges of $\mathcal{C}$. A *face* of $\mathcal{C}$ is a connected component of $\mathbb{R}^2 \setminus \cup_{C\in\mathcal{C}}C$. $F(\mathcal{C})$ is the set of faces of $\mathcal{C}$. In this thesis a *ray* is an oriented Jordan arc that starts at a point and goes to infinity. Suppose the curves in the arrangement $\mathcal{C}$ surround a common point O, and that there is a ray γ_0 that starts at O and intersects each curve exactly once. By sweeping the arrangement of the curves by the ray γ_0 we mean that we can move the ray continuously around the point O, say counter-clockwise, such that the ray never intersects any curve of the arrangement more than once. More formally, we denote by Γ the collection of all the rays starting at O and intersect each curve in $\mathcal{C}$ exactly once. Every ray $\gamma \in \Gamma$ can be thought of as a continuous function $\gamma : [0, \infty) \to \mathbb{R}^2$, where $\gamma(0) = O$. A *sweep of $\mathcal{C}$ by the ray γ* is a continuous bijective map $\psi : [0, 1] \to \Gamma$ such that:

- $\psi(0) = \psi(1) = \gamma$.
- For any $s, t \in [0, 1)$, $\psi(s), \psi(t)$ never cross each other.

We sometime refer $\psi(s)$ by its image γ_s.

It follows immediately by its definition that a sweep by a ray is a process that moves the ray around its starting point in either clockwise or counterclockwise direction. From this point on, whenever a sweep by a ray is mentioned, we mean to a sweep that moves the ray in the counterclockwise direction.

We are now able to state the main result of this Chapter:

Theorem 3.1.1. *Let $\mathcal{C}$ be a finite family of simple closed Jordan curves surrounding a common point O. If $\mathcal{C}$ has the connected intersecting property, then $\mathcal{C}$ can be swept by a ray.*

3.2 Proof of Theorem 3.1.1

In order to prove Theorem 3.1.1 we must show the existence of a ray γ, by which we wish to obtain a sweep. Because $\mathcal{C}$ has the connected intersection property, and all the curves in $\mathcal{C}$ surround a common point, it follows immediately from Lemma 2.2.3 that for every subset $\mathcal{D} \subseteq \mathcal{C}$, $\cup_{C \in \mathcal{D}} \text{disc}(C)$ is a simply connected region. We can now apply Lemma 2.2.4 to obtain a ray γ, starting at O and intersecting each of the curves exactly once. Although a sweep by a ray is a continuous process, we can carry it out in discrete steps. Suppose we have already defined the map ψ in the interval $[0, s]$ for some $s \in (0, 1)$. We can carry out the sweep as long as we do not meet a vertex of the arrangement. Every point in the plane that lies on a ray $\psi(t)$ for some $t \in [0, s]$ is said to be *covered* by the sweep and we say that the sweep *covers* this point. Let $V_s \subseteq V(\mathcal{C})$ be the set of all intersection points of curves in $\mathcal{C}$ that the sweep has not covered. Let $\{w_1, w_2, .., w_n\}$, $\{u_1, ..., u_n\}$ be the intersection point of γ_0, γ_s respectively with the curves in $\mathcal{C}$ according to the order in which γ_0, γ_s intersect the curves. Assume that none of the points $\{w_i\}, \{u_i\}$ belongs to $V(\mathcal{C})$. For a face $f \in F(\mathcal{C} \cup \{\gamma, \gamma_s\})$ we say that γ_s *sees the face* f if the sweep hasn't covered the face and if one of the edges of f is a portion of γ_s. We say that γ_s *sees an edge* $e \in E(\mathcal{C})$ if e belongs to a face that γ_s sees. We say that γ_s sees a vertex $v \in V(\mathcal{C})$ if v belongs to a face that γ_s sees. Clearly, the sweep can progress through a vertex of the arrangement if it belongs to a triangle (a face with three edges) that γ_s sees, see Figure 3. 1. So in order to show the existence of a sweep by the ray γ, we need to show that γ_s sees a triangle, for every $s \in [0, 1]$ for which the sweep is well defined in the interval $[0, s]$, except from the stage at which the sweep has covered all the intersection points of $\mathcal{C}$. In that stage γ_s sees $\{w_1, ..., w_n\}$ and the sweep can be completed.

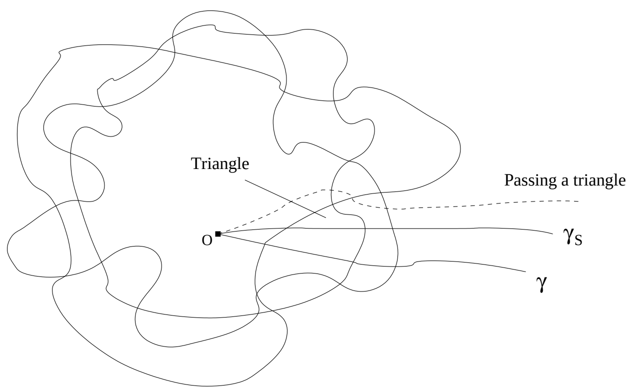


Figure 3. 1: A sweep γ

For every curve $C \in \mathcal{C}$ we define $\eta_C : V_s \cap C \to \{1, 2, ..., |V_s \cap C|\}$ s.t. $\eta_C(v)$ is the position of v on the portion of C that the sweep has not covered yet, starting with the intersection point of C with γ_s and moving on C in the counterclockwise direction, see Figure 3. 2. Since every vertex of the arrangement lies on exactly two curves, we assign to an intersection point v of two curves C, C' two positive integers, namely, $\eta_C(v), \eta_{C'}(v)$.

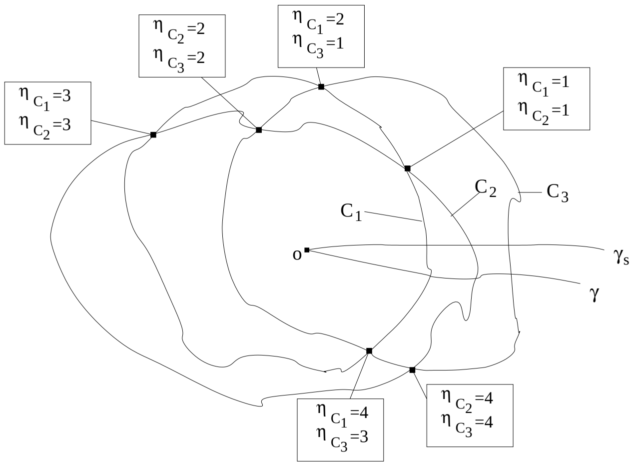


Figure 3. 2: η_C

Recall that our aim is to show that γ_s sees a triangle, and observe that γ_s sees a triangle if and only if there are pair of curves $C, C' \in \mathcal{C}$ and a vertex $v \in V_s \cap C \cap C'$ s.t. $\eta_C(v) = \eta_{C'}(v) = 1$. To this end we would like to define a partial order on V_s:

Definition 3.2.1. Let $v, \tilde{v} \in V_s$ such that $v, \tilde{v}$ lie on the same curve $C \in \mathcal{C}$. We say that $v <_* \tilde{v}$ $\eta_C(v) < \eta_C(\tilde{v})$.

$<_*$ is a relation on V_s. We define the relation $<$ on V_s as the transitive closure of $<_*$:

Definition 3.2.2. Let $v, \tilde{v} \in V_s$. We say that $v < \tilde{v}$ if there exists a sequence $v_1 = v, v_2, ..., v_r = \tilde{v} \in V_s$ such that $v_1 <_* v_2 <_* ... <_* v_r$.

Claim 3.2.3. *$(V_s, <)$ is a poset.*

Proof. The relation $<$ is anti-reflexive: assume that $v < v$, then there are vertices $v = v_1, ..., v_r = v$ such that $v_1 <_* v_2 <_* ... <_* v_r$. We assume without loss of generality that r is minimal. For each $i \in \{1, 2, ..., r-1\}$ the points v_i, v_{i+1} lie on some curve C_i. Denote the portion of C_i connecting v_i with v_{i+1} by e_i, we obtain a drawing of a cycle. Since the edges of this cycle are all oriented at a counterclockwise direction, there are only two possible cases:

Case 1: The direction of the cycle is clockwise. Let y be a point inside the cycle. Since $\mathcal{C}$ has the connected intersection property, and all the curves in $\mathcal{C}$ surround a common point, we can apply Lemma 2.2.3 and Lemma 2.2.4 to conclude that there is an arc from y to infinity that intersects every curve in C at most once. This is clearly a contradiction, since every ray starting at y must enter one of the curves C_i, $i \in \{1, 2, ..., r-1\}$, thus it must intersect C_i at least twice.

Case 2: The direction of the cycle is counterclockwise. In that case O must be inside the cycle. Assume to the contrary that O is outside the cycle and fix a point y inside the cycle. Any arc from O to y must enter one of the curves C_i, $i \in \{1, 2, ..., r-1\}$. Since O is surrounded by all the curves $C_1, C_2, ..., C_{r-1}$, an arc from O to y that enters C_i must leave C_i first. Then, any arc from O to y intersects one of the curves at least twice. This is a contradiction because by Lemma 2.2.3 and Lemma 2.2.4 there exists an arc from O to y that intersects any curve at most once. We conclude that O is inside the cycle and therefore, γ_s must intersect the curve C_i for some $i \in \{1, 2, ..., r-1\}$. Since η_{C_i} numbers the intersection points on C_i according to their position in the counterclockwise direction, starting from the intersection of C_i with γ_s, we conclude that $\eta_{C_i}(v_{i+1} < \eta_{C_i}(v_i)$, which implies that $v_{i+1} <_* v_i$ and so we have reached a contradiction.

$<$ is transitive: If $v < u$ and $u < w$ then there are vertices $v = v_1, v_2 ..., v_r = u$ and $u = \tilde{v}_1, \tilde{v}_2 ..., \tilde{v}_s = w$ such that $v_1 <_* v_2 <_* ... <_* v_r$ and $\tilde{v}_1 <_* \tilde{v}_2 <_* ... <_* \tilde{v}_s$. The sequence $v = v_1, ..., v_r, \tilde{v}_2, ..., \tilde{v}_s = w$ implies that $v < w$.

$<$ is anti-symmetric. If $v < u$ and $u < v$ then by transitivity $v < v$, but we have already proved that $<$ is anti-reflexive.

$<$ is anti-reflexive, anti-symmetric and transitive, and thus $(V_s, <)$ is a poset.∎

Since $(V_s, <)$ is a finite poset, V_s has a minimum point v. Clearly, $v \in V_s \cap C \cap C'$ is a minimum if and only if $\eta_C(v) = \eta_{C'}(v) = 1$. Then γ_s sees a triangle and the sweep can pass v. This shows that the arrangement $\mathcal{C}$ can be swept by a ray and concludes the proof.∎

It follows from Theorem 3.1.1 that a collection of s-intersecting simple closed curves, that are surrounding a common point and satisfy the connected intersection property, can be realized, after one-to-one and continuous transformation of the plane, as a collection of s-intersecting bi-infinite x-monotone curves. Let P be a set of n points in the plane and let $\mathcal{C}$ be a family of s-intersecting bi-infinite x-monotone curves, where $s \geq 0$ is an even integer. Let $\mathcal{F}$ be the set system on the ground set P, consist of all the subset $S \subseteq P$ for which there exists a curve $C \in \mathcal{C}$ such that S is the set of all the points in P that lie below C. For any integer $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$ we define $\mathcal{F}_{\mathcal{C},P,k} = \{S \in \mathcal{F} | |S| = k\}$. In [4] the authors prove Theorem 1.0.7 that states that $|\mathcal{F}_{\mathcal{C}_P,k}| = O((kn)^{\frac{s}{2}})$ and this upper bound on $|\mathcal{F}_{\mathcal{C},P,k}|$ is best possible. Combining Theorem 1.0.7 together with Theorem 3.1.1 we obtain the following Corollary:

Corollary 3.2.4. *Let P be a set of n points in the plane and let $\mathcal{C}$ be a family of simple closed Jordan curves surrounding a common point O. If $\mathcal{C}$ has both the s-intersecting property, for some integer $s \geq 0$, and the connected intersecting property as well, then the number of sets in $\mathcal{F} = \{P \cap disc(C) | C \in \mathcal{C}\}$ of cardinality $k \leq \lfloor \frac{n}{2} \rfloor$ is $O((kn)^{\frac{s}{2}})$. This upper bound is best possible.*

Chapter 4

Paths With No Small Angles on Points in Convex Position

In this Chapter we deal with a special case of a problem of S. Fekete and G.J Woeginger ([8] and [9]). Given a set X of n points in the plane, an ordering of the points of X, $x_1x_2...x_n$ is identified with a polygonal path P on X: Its edges are the straight segments connecting x_i to x_{i+1}. An edge connecting $x, y \in X$ is denoted by (x, y). The angle of P at x_i is $\measuredangle x_{i-1}x_ix_{i+1}$. A path is called *$\alpha$-good* if all of its angles are at least α, where $\alpha > 0$.

Fekete and Woeginger have conjectured that for every finite set X of points in the plane, there exists a $\frac{\pi}{6}$-good path on X. In [3] Bárány, Pór and Valtr proved the existence of $\frac{\pi}{9}$-good path on every finite set of points in the plane. They also generalized this result to higher-dimensional spaces and proved that for every $d \geq 2$ there is a positive α_d, depending only on d, such that for every finite set of points $X \in \mathbb{R}^d$ there exists an α_d-good path on X. In [2] Bárány and Pór have generalized this problem to an infinite set of points in the plane, and proved the existence of α-good path, with $\alpha = \frac{\pi}{20}$. In this chapter we focus on a special case of the problem, where X is a set of points in the plane in convex position. A set of points X in the plane is in *convex position* if the points in X are the vertices of some convex polygon.

The main result of this chapter is the following:

Theorem 4.0.5. *Let X be a set of points in a convex position in the plane. Then there exists a $\frac{\pi}{5}$-good path on X.*

This result cannot be improved:

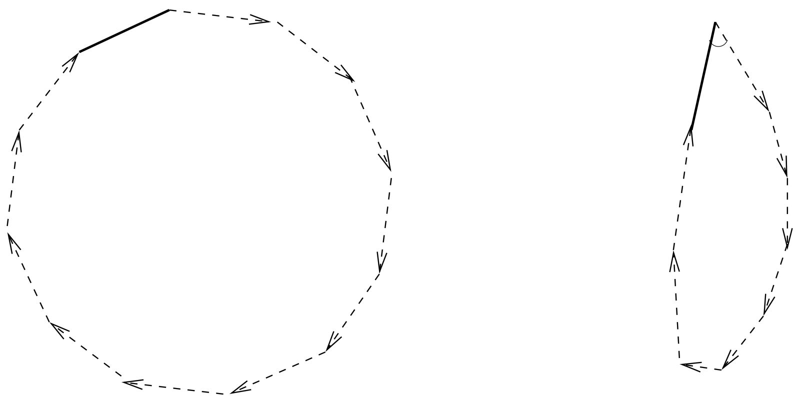
Claim 4.0.6. *For every integer $n \geq 5$, there exists a set X of n points in the plane in convex position with no α-good paths, for any $\alpha > \frac{\pi}{5}$.*

4.1 Proof of Theorem 4.0.5

The points in X are the vertices of some convex polygon which we denote by $conv(X)$. A convex polygon has at most two angles less than $\frac{\pi}{3}$. This is true since the sum of every three angles in a convex polygon is at least π, as every three vertices form a triangle that is contained in the convex polygon, and therefore, has angles that are not greater than the angles of its vertices on the polygon. In particular, a convex polygon has at most two angles less than $\frac{\pi}{5}$. We prove Theorem 4.0.5 by case analysis.

In case $conv(X)$ has no angle less than $\frac{\pi}{5}$, we can obtain a good path by omitting one of the edges of $\partial conv(X)$ (the boundary of $conv(X)$). See the left side of Figure 4. 1.

In case $conv(X)$ has exactly one angle less than $\frac{\pi}{5}$, we can obtain a good path by omitting one of the two edges of $\partial conv(X)$ incident to that angle. See the right side of Figure 4. 1.



Case there is no angle less than $\frac{\pi}{5}$ Case there is one angle less than $\frac{\pi}{5}$

Figure 4. 1: Case there is at most one angle less than $\frac{\pi}{5}$

The only non-trivial case is the case in which $conv(X)$ has exactly two angles that are less than $\frac{\pi}{5}$. In this case any other angle of $conv(X)$ must be at grater than $\frac{3\pi}{5}$. Let x, y be the vertices of the two small angles of $conv(X)$. Note that x, y must be the end vertices of every good path. We define the *distance of x and y on $\partial conv(X)$*, which we denote by $d(x, y)$, as the number of edges in the smallest path on $\partial conv(X)$ connecting x and y. For every pair of points $x_1, x_2 \in X$, we denote the straight segments connecting x_1 and x_2 by (x_1, x_2). For every three points x_1, x_2, x_3 of X we denote the angle between (x_1, x_2) and (x_2, x_3) by $\measuredangle x_1 x_2 x_3$.

We continue the proof by case analysis according to the distance of x and y.

Case $d(x, y) = 1$**:** In that case the straight segment (x, y) is an edge of $\partial conv(X)$. By omitting the edge (x, y) from $\partial conv(X)$ we obtain a $\frac{\pi}{5}$-good path on X. Figure 4. 2 illustrates the $\frac{\pi}{5}$-good path on X.

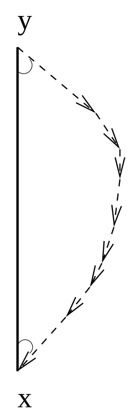


Figure 4. 2: The path P in case $d(x, y) = 1$

The rest of the proof is by induction on $|X| = n$, where $n \geq 7$. Note that the remaining cases are possible only if $n \geq 4$. We will later show the basis of the induction for $n \in \{4, 5, 6\}$.

Case $d(x, y) = 2$**:** there exists a point $z \in X$ such that (x, z) and (z, y) are edges of $\partial conv(X)$. We denote all the other vertices of $conv(X)$ by $u_1, u_2, ..., u_{n-3}$ such that $yu_1u_2...u_{n-3}xz$ is cyclic order in which the points lie on $\partial conv(X)$. The following claim will help us to narrow down the possibilities of the configuration of the points in P.

Claim 4.1.1. *1. We may assume that the angle $\measuredangle zu_1u_2$ is at least $\frac{\pi}{5}$, and similarly, $\measuredangle zu_{n-3}u_{n-4} \geq \frac{\pi}{5}$.*

2. We may assume that the angle $\measuredangle xzu_{n-3}$ is less than $\frac{\pi}{5}$, and similarly, $\measuredangle yzu_1 < \frac{\pi}{5}$.

Proof. Assume that $\measuredangle zu_1u_2 < \frac{\pi}{5}$. Since $\measuredangle yu_1u_2 \geq \frac{3\pi}{5}$ it follows that $\measuredangle yu_1z \geq \frac{\pi}{5}$. By the induction hypothesis we can find good path on the points $X \setminus y$ that connects u_1 and x. By adding to this path the edge (y, u_1) we obtain a good path on X and conclude the proof of 1.

In order to proof the second part of the claim, let us assume that $\measuredangle xzu_{n-3} \geq \frac{\pi}{5}$. It follows by 1 that $xzu_{n-3}u_{n-2}...u_2u_1y$ is a good path.∎

The left side of Figure 4. 3 illustrates the configuration of the points under the assumptions we made above according to Claim 4.1.1. The double arc symbol indicates that the angle is at least $\frac{\pi}{5}$ and the single arc indicates that the angle is less than $\frac{\pi}{5}$.

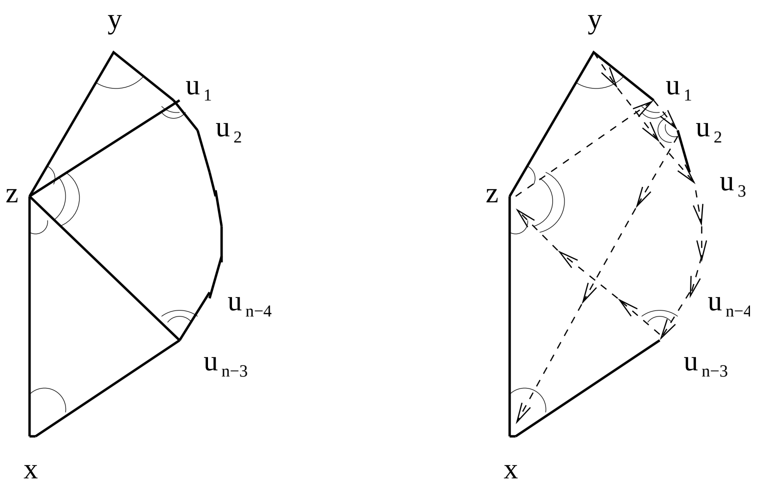


Figure 4. 3: The path P in case $d(x, y) = 2$

Since every other angle in $\partial conv(X)$ is at least $\frac{3\pi}{5}$ we deduce that $\measuredangle u_1zu_{n-3} = \measuredangle xzy - \measuredangle xzu_{n-3} - \measuredangle yzu_1 \geq \frac{\pi}{5}$. Clearly, every angle of a convex polygon which is formed by a subset of X that contains x and y, must be at least $\frac{3\pi}{5}$. Then $\measuredangle yu_3u_4, \measuredangle u_1u_2x \geq \frac{3\pi}{5}$ and the path $yu_3u_4...u_{n-3}zu_1u_2x$ is $\frac{\pi}{5}$-good. See Figure 4. 3.

Case $d(x, y) = 3$**:** There are two points $z, w \in X$ such that $(x, z), (z, w), (w, y)$ are all edges of $\partial conv(X)$. We denote all the other points in X by $u_1, ..., u_{n-4}$ such that $yu_1u_2...u_{n-4}xzw$ is cyclic-order in which the points lie on $\partial conv(X)$.

Claim 4.1.2. *We may assume that the angle $\measuredangle u_1wz$ is at least $\frac{\pi}{5}$, and by symmetry, $\measuredangle wzu_{n-4}, \measuredangle zu_{n-4}u_{n-5}, \measuredangle wu_1u_2$ are at least $\frac{\pi}{5}$.*

Proof. The proof is similar to the proof of Claim 4.1.1. Assume that $\measuredangle zwu_1 < \frac{\pi}{5}$, then $\measuredangle ywu_1 \geq \frac{2\pi}{5}$. By the induction hypothesis there is a $\frac{\pi}{5}$-good path on the points $X \setminus \{y\}$ which connects w and x. Adding to this path the edge (y, w) we obtain a $\frac{\pi}{5}$-good path on X.∎

We denote two of the angles of the triangle formed by y, u_{n-4}, z by $\alpha = \measuredangle yzu_{n-4}, \beta = \measuredangle yu_{n-4}z$. At least one of the angles α, β is grater or equal $\frac{\pi}{5}$, see Figure 4. 4.

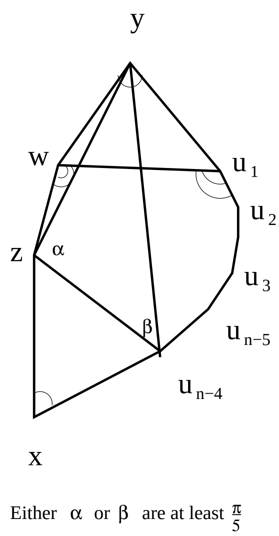


Figure 4. 4: Case $d(x, y) = 3$

If $\beta \geq \frac{\pi}{5}$, then the path $yu_{n-4}zwu_1u_2...u_{n-5}x$ is $\frac{\pi}{5}$-good. This path is illustrated in Figure 4. 5.

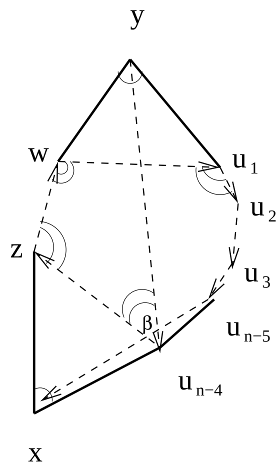


Figure 4. 5: Case $d(x, y) = 3$ and $\beta \geq \frac{\pi}{5}$

We may therefore assume that $\beta < \frac{\pi}{5}$ and by symmetry, we may also assume that the angle $\gamma = \measuredangle wu_1x < \frac{\pi}{5}$. We deduce that $\alpha \geq \frac{\pi}{5}$ (see Figure 4. 6). Since two of the angles of the triangle formed by w, u_1, x are less than $\frac{\pi}{5}$ it follows that the angle $\measuredangle xwu_1$ is at least $\frac{3\pi}{5}$. Then the path $yzu_{n-4}u_{n-3}...u_1wx$ is $\frac{\pi}{5}$-good.

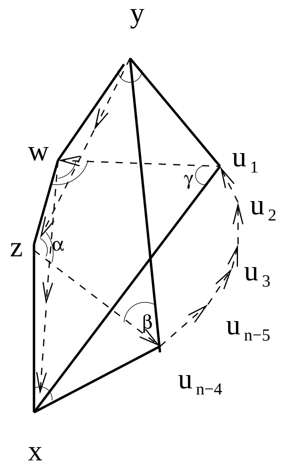


Figure 4. 6: Case $d(x, y) = 3$ and $\beta < \frac{\pi}{5}$

Case $d(x, y) \geq 4$**:** Let $xu_1, u_2, ...u_k, y, v_1, v_2, ..., v_{n-k-2}$ denote the points in X according to the counter-clockwise order in which the points lie in $\partial conv(X)$. We assume that $k, n-2-k \geq 3$. Consider the triangle formed by y, v_{n-k-2} and u_1. At least one of its angles other than $\measuredangle v_{n-k-2}yu_1$ is at least $\frac{\pi}{5}$, say $\alpha = \measuredangle yv_{n-k-2}u_1 \geq \frac{\pi}{5}$. Then $yv_{n-k-2}u_1u2...u_kv1v2..v_{n-k-3}$ is a $\frac{\pi}{5}$-good path on S, as illustrated in Figure 4. 7 below.

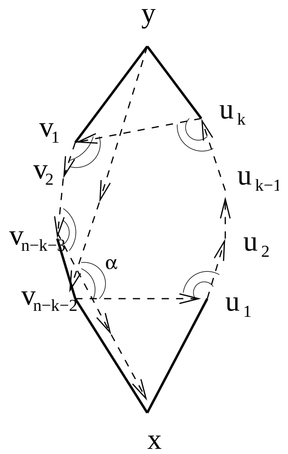


Figure 4. 7: The path P in case $d(x, y) \geq 4$

4.1.1 The Basis of the Induction

In order to prove the basis of the induction we need to show the existence of a $\frac{\pi}{5}$-good path on every set X of $n \in \{4, 5, 6\}$ points in the plane in a convex position for which $conv(X)$ has two angles that are less then $\frac{\pi}{5}$ and the vertices of the small angles are at distance at least 2 on $\partial conv(X)$.

Case $n = 4$**:** Let $X = \{x, y, z, w\}$ and assume that x, y are the vertices of the small angles in $conv(X)$. Then x, z, y, w is a cyclic order in which the points in X lie on

$\partial conv(X)$. We can argue the same as in the first part of Claim 4.1.1 to reduce this case to a case in which the angles $\measuredangle zwy, \measuredangle zwx, \measuredangle wzy, \measuredangle wzx$ are at least $\frac{\pi}{5}$. Then $xzwy$ is a $\frac{\pi}{5}$-good path.

Case $n = 5$**:** Denote the points in X by x, u_1, u_2, y, z according to the cyclic counter-clockwise order in which the points lie on $\partial conv(X)$. By applying the same arguments as in the proof of Claim 4.1.1, we can assume that the angles $\measuredangle zu_1u_2, \measuredangle zu_2u_1$ are at least $\frac{\pi}{5}$ and that the angles $\measuredangle xzu_1, \measuredangle yzu_2$ are less than $\frac{\pi}{5}$. Then the path xu_1zu_2y is $\frac{\pi}{5}$-good (see Figure 4. 8).

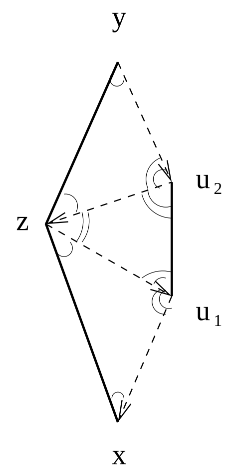


Figure 4. 8: Case $n = 5$

Case $n = 6$**:** In that case $d(x, y) \in \{2, 3\}$. If $d(x, y) = 2$, we denote the points in X by x, u_1, u_2, u_3, y, z according to cyclic order in which they lie in $\partial conv(X)$. Arguments similar to those argued at the proof of Claim 4.1.1 can be applied to reduce to a case in which $\measuredangle xzu_1, \measuredangle yzu_3 < \frac{\pi}{5}$ and $\measuredangle zu_1u_2, \measuredangle zu_3u_2 \geq \frac{\pi}{5}$. It follows that the angle $\measuredangle u_1zu_3 \geq \frac{\pi}{5}$. If both angles $\measuredangle u_1zu_2, \measuredangle u_2zu_3$ are at less than $\frac{\pi}{5}$, then $\measuredangle yu_1z \geq \frac{\pi}{5}$ because the sum of the angles $\measuredangle yu_1z, \measuredangle u_1zu_2, \measuredangle u_2zu_3, \measuredangle u_3zy$ and $\measuredangle zyu_1$ is π. In that case, the path $xu_2u_3zu_1y$ is $\frac{\pi}{5}$-good. We may therefore assume without loss of generality that $\measuredangle u_1zu_2 \geq \frac{\pi}{5}$. One of the angles $\measuredangle u_1u_2z, \measuredangle u_3u_2z$ is at least $\frac{\pi}{5}$. If $\measuredangle u_1u_2z \geq \frac{\pi}{5}$, then the path $xu_1u_2zu_3y$ is $\frac{\pi}{5}$-good. Then we may assume that $\measuredangle u_1u_2z < \frac{\pi}{5}$ and $\measuredangle u_3u_2z \geq \frac{\pi}{5}$. It follows that $\measuredangle u_1zu_2 \geq \frac{\pi}{5}$, because the angles of the triangle formed by x, u_2 and z must sum to π. Figure 4. 9 illustrates the assumptions we made so fur.

The path $xu_1zu_2u_3y$ is $\frac{\pi}{5}$-good (see Figure 4. 10). This covers all possible configurations of six points in the case $d(x, y) = 2$.

If $d(x, y) = 3$, we denote the points by x, u_1, u_2, y, v_1, v_2 according to cyclic order in which the points lie in $\partial conv(X)$. We can argue as in Claim 4.1.2 to conclude that we may assume that $\measuredangle u_1v_2v_1, \measuredangle v_2v_1u_2, \measuredangle v_2u_1v_1, \measuredangle v_1u_2u_1$ are at least $\frac{\pi}{5}$. If $\measuredangle xv_1v2 \geq \frac{\pi}{5}$, then $xv_1v2u_1u_2y$ is a $\frac{\pi}{5}$- good path. Then we may assume that $\measuredangle xv_1v_2 < \frac{\pi}{5}$ and similarly, the angles $\measuredangle xu_2u_1, \measuredangle yu_1u_2, \measuredangle yv_2v_1$ are less than $\frac{\pi}{5}$. Consider the triangle formed by x, u_2, and v_1. One of the angles $\measuredangle xv_1u_2, \measuredangle xu_2v_1$ must be at least $\frac{\pi}{5}$. Similarly, one of the angles $\measuredangle yv_2u_1, \measuredangle yu_1v_2$, of the triangle formed by y, u_1, v_2 is at least $\frac{\pi}{5}$. If both $\measuredangle xv_1u_2, \measuredangle yv_2u_1 \geq \frac{\pi}{5}$ then the path $xv_1u_2u_1v_2y$ is $\frac{\pi}{5}$-good. If $\measuredangle xv_1u_2, \measuredangle yv_2u_1 < \frac{\pi}{5}$, then $xu_2v_1v_2u_1y$ is a $\frac{\pi}{5}$-good path. Therefore, we may assume that $\measuredangle yv_2u_1 < \frac{\pi}{5}$ and $\measuredangle xv_1u_2 \geq \frac{\pi}{5}$ and similarly, we may assume that $\measuredangle yu_1v_2 \geq \frac{\pi}{5}$ and $\measuredangle xu_2u_1 < \frac{\pi}{5}$. We conclude that the angles $\measuredangle xv_2u_1, \measuredangle yu_2v_1$ are at least $\frac{\pi}{5}$. If $\measuredangle xu_1v_2 \geq \frac{\pi}{5}$, then $xu_1v_2v_1u_2y$ is a $\frac{\pi}{5}$-good path.

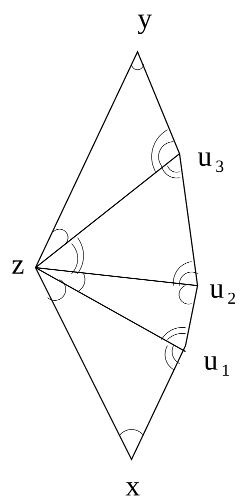


Figure 4. 9: Case $n = 6$ and $d(x, y) = 2$

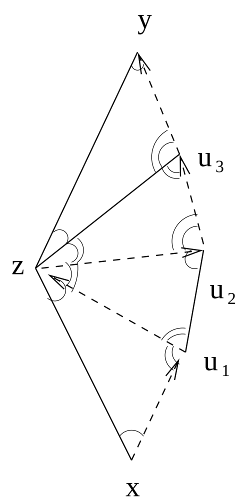


Figure 4. 10: A $\frac{\pi}{5}$-good path in the case $n = 6$ and $d(x, y) = 2$

Then we may assume that $\measuredangle xu_1v_2 < \frac{\pi}{5}$ and similarly, $\measuredangle yv_1v_2 < \frac{\pi}{5}$. Figure 4. 11 illustrates the assumptions we made above.

Consider the triangle yu_1v_1. If $\measuredangle yu_1v_1 < \frac{\pi}{5}$, then $\measuredangle v_1u_1v_2 \geq \frac{\pi}{5}$, because the sum of the angles $\measuredangle yu_1v_1, \measuredangle yv_2u_1, \measuredangle v_2yu_1$ and $\measuredangle v_1u_1v_2$ is π. Moreover, the angle $\measuredangle u_1v_1u_2$ is at least $\frac{\pi}{5}$ because the sum of the angles $\measuredangle u_1v_1u_2, \measuredangle v_1u_1y, \measuredangle yu_1u_2, \measuredangle xu_2u_1$ and $\measuredangle xu_2v_1$ is exactly the sum of the angles of the triangle formed by u_1, u_2 and v_1 and therefore, equals to π. Then the path $xv_2u_1v_1u_2y$ is $\frac{\pi}{5}$-good. A symmetric case is when $\measuredangle xv_1u_1 < \frac{\pi}{5}$. Then we may assume that $\measuredangle yu_1v_1$ and $\measuredangle xv_1u_1$ are at least $\frac{\pi}{5}$ and the path $xv_2v_1u_1u_2y$ is $\frac{\pi}{5}$-good (see Figure 4. 12).

This concludes the proof of the basis of the induction.

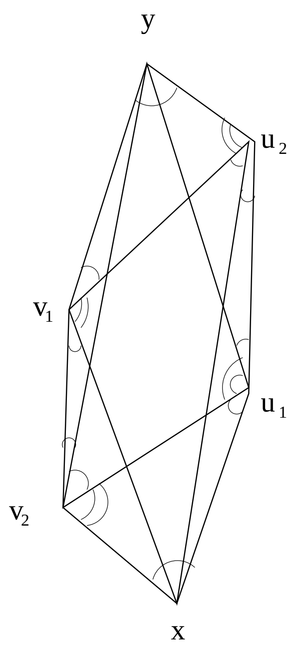


Figure 4. 11: Case $n = 6$ and $d(x, y) = 3$

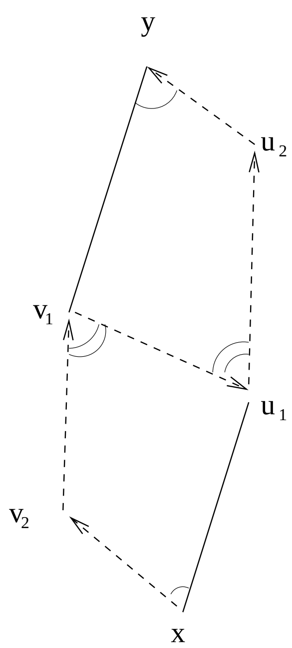


Figure 4. 12: The path p in case $n = 6$ and $d(x, y) = 3$

4.2 Proof of Claim 4.0.6

Let $\alpha > \frac{\pi}{5}$ be a fixed number, and let X be a set of $n \geq 5$ points in the plane, arranged on a triangle as illustrates in Figure 4. 13.

Clearly, there are no α-good paths on X. Although the points in X are not in convex position, they can be perturbed to a set of points in convex position with no α-good path.

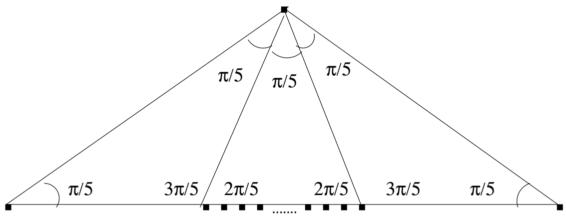


Figure 4. 13: An example of n points with no α-good path, $\alpha > \frac{\pi}{5}$

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בנוסף אנו מראים שלא ניתן לשפר תוצאה זו, כלומר, לכל $\alpha < \frac{\pi}{5}$ ולכל $n \geq 5$, קיימת קבוצה של n נקודות במישור במצב קמור, כך שלכל מסלול עליהן יש זווית בין שתי צלעות עוקבות הקטנה מ־α.

העקומים באוסף מקיפים נקודה משותפת.

המשפט המרכזי בפרק 3 הוא הכללה של התוצאה של Snoeyink ו־Hershberger ואומר שניתן לסרוק באמצעות קרן כל אוסף סופי של עקומים סגורים ופשוטים המקיים את הדרישות הבאות:

1. לכל שני עקומים באוסף מתקיים שהחיתוך של התחומים החסומים על ידם הוא תחום קשיר.

2. קבוצת כל נקודות החיתוך בין עקומים באוסף היא סופית.

3. כל העקומים באוסף מקיפים נקודה משותפת.

אנו מניחים שכל נקודה המשותפת לזוג עקומים באוסף היא נקודת חיתוך ממש, ושאין נקודות השקה.

תוצאה זו למעשה גורסת שכל אוסף של עקומים המקיים את התכונות לעיל, ניתן לממש כאוסף של גרפים של פונקציות.
מכאן שתוצאות המתקיימות על גרפים של פונציות, חלות גם על העקומים באוסף.

בפרק 4 אנו מציגים מקרה פרטי של בעיה שהועלתה לראשונה על ידי Fekete ו־Woeginger.
בהנתן קבוצה סופית X של n נקודות במישור, סידור של הנקודות $x_1x_2...x_n$ קובע מסלול פוליגוני p העובר דרך כל הנקודות.
זווית המסלול בנקודה x_i היא הזווית הקטנה מ־π, הנוצרת על ידי שתי הצלעות העוקבות (x_{i-1}, x_i) ו־ (x_i, x_{i+1}).
נאמר שהמסלול p הנקבע על ידי הסידור $x_1x_2...x_n$ הוא $\alpha - good$ אם לכל $2 \leq i \leq n-1$, זווית המסלול ב־x_i היא גדולה או שווה α.

Fekete ו־ Woeginger שיערו שלכל קבוצה סופית של נקודות במישור קיים מסלול על הנקודות שהוא $\frac{\pi}{6} - good$.
Bárány, Pór ו־Valtr הוכיחו שתמיד קיים מסלול $\frac{\pi}{9} - good$ על קבוצת נקודות סופית במישור.
בנוסף הם הכלילו את הבעיה למרחבים ממימד גבוה והראו שלכל $d \geq 2$ קיים קבוע α_d התלוי רק ב־ d, כך שלכל קבוצה סופית של נקודות ב־ $\mathbb{R}^d$ קיים מסלול על הנקודות שהוא $\alpha_d - good$.
הכללה נוספת של הבעיה, שהוצגה על ידי Bárány ו־ Pór היא לקבוצה בת מניה, לא בהכרח סופית של נקודות במישור.
Bárány ו־ Pór הראו שעל קבוצה כזו, תמיד ניתן למצוא מסלול שהוא $\frac{\pi}{20} - good$.

אנו מראים שלכל קבוצת נקודות במישור הנמצאת במצב קמור קיים מסלול על הנקודות שהוא $\frac{\pi}{5} - good$.

הם תתי הקבוצות של P, המופרדות מהשמים שלהן על ידי עקומים השייכים לאוסף נתון של עקומים סגורים $\mathcal{C}$.
אנו מחינים שאוסף העקומים $\mathcal{C}$ מקיים את הדרישות הבאות:

1. כל זוג עקומים ב־ $\mathcal{C}$ נחתך לכל היותר s פעמים.

2. לכל זוג עקומים C, C' ב־ $\mathcal{C}$ מתקיים שאוסף הנקודות במישור המוקפות על ידי שני העקומים הוא תחום קשיר או קבוצה ריקה.

המערכת $\mathcal{F}$ מוגדרת באופן פורמלי באופן הבא: $\mathcal{F} = \{\mathcal{P} \cap \mathrm{disc}(\mathcal{C}) | \mathcal{C} \in \mathcal{C}\}$, כאשר $\mathrm{disc}(C)$ מסמן את התחום במישור החסום על ידי העקום הסגור C.

אנו מראים שמימד VC של מערכת זו הוא לכל היותר $s+1$.
מהלמה של Perles, Sauer ו־ Shelah נובע שהמערכת מכילה לכל היותר $O(n^{s+1})$ איברים, כאשר n הוא מספר הנקודות ב־ P.

בנוסף, אנו מראים על ידי דוגמה שלכל n ו־s, כאשר s מספר זוגי, קיימים קבוצת n נקודות במישור ואוסף של פולינומים ממעלה s, כך שלעמרכת הקבוצות על P, שאיבריה הם כל תתי הקבוצות של P המופרדות מהשמלים שלהן מלמעלה על ידי פולינום כלשהו באוסף, יש מימד VC השווה ל־$s+1$.
מאחר ו־s זוגי, ניתן לסגור את הפולינומים באוסף באינסוף ולקבל אוסף של עקומים סגורים המקיים שכל שני עקומים באוסף נחתכים לכל היותר s פעמים והחיתוך של התוחימים החסומים על ידם הוא ריק או קשיר, כך שהמערכת המוגדרת על הקבוצה באמצעות העקומים באוסף מכילה לפחות $\Omega(n^{s+1})$ איברים.
מכאן שהחסם העליון שאנו מציגים על גודל המערכת הינו הדוק עד כדי מכפלה בקבוע.

פרק 3 עוסק בסריקה של אוסף של עקומים סגורים במישור באמצעות קרן.
קרן היא עקום פשוט המחבר נקודה במישור לאינסוף.
בהינתן אוסף $\mathcal{C}$ של עקומים סגורים במישור המקיפים נקודה משותפת O, וקרן γ היוצאת מהנקודה O וחותכת כל עקום ב־$\mathcal{C}$ בדיוק פעם אחת, סריקה של $\mathcal{C}$ באמצעות הקרן γ היא הזזה רציפה בזמן של γ סביב הנקודה O, למשל נגד כוון השעון, כך שכל נקודה במישור פוגשת את γ בדיוק פעם אחת ובכל זמן כל עקום ב־$\mathcal{C}$ נחתך על ידי γ בדיוק פעם אחת.

סקירה של עצמים היא כלי שימושי בהוכחות מתמטיות.
היא מאפשרת לקבוע תכונות של אוסף של עצמים ממימד d על ידי התבוננות בסדרה של "פרוסות" עוקבות של העצמים ממימד $d-1$, ובכך היא הופכת בעיה סטטית ממימד d לבעיה דינמית ממימד אחד פחות.

Snoeyink ו־Hershberger הראו שניתן לסרוק על ידי קרן כל אוסף סופי של עקומים סגורים ופשוטים המקיים שכל שני עקומים באוסף נחתכים לכל היותר פעמיים ושכל

תקציר

בעבודה זו נציג תוצאות בגיאומטריה קומבינטורית המחולקות לשלושה פרקים.
האחד עוסק במימד VC של עקומים במישור הנחתכים s פעמים, השני עוסק בסריקה (sweep) של עקומים במישור באמצעות קרן והאחרון עוסק במקרה פרטי של בעיה שהוצעה לראשונה על ידי Fekete ו־ Woeginger ועוסקת במסלולים פוליגונים ללא זוויות קטנות על קבוצת נקודות במישור.

פרק 2 דן במערכת קבוצות על קבוצה X.
בהנתן קבוצה X, מערכת קבוצות על X היא אוסף של תתי קבוצות של X.
מערכת הקבוצות בה נתעסק מוגדרת על קבוצה סופית P של נקודות במישור, כאשר האיברים במערכת הם תתי הקבוצות של P המופרדות על ידי אוסף של אובייקטים גיאומטרים.

דוגמה חשובה למערכת קבוצות היא מערכת ה־k קבוצות (k-sets) של קבוצת נקודות במישור P.
k קבוצה היא תת קבוצה של P בגודל k אשר ניתן להפרידה מיתר הנקודות ב־ P על ידי ישר.
מערכת קבוצות זו נחקרה רבות בארבעת העשורים האחרונים בניסיון למצוא חסמים הדוקים על גודל המערכת.
החסם העליון הטוב ביותר על מערכת ה־k קבוצות הוצג על ידי Dey שהראה שגודל המערכת הוא לכל היותר $O(nk^{\frac{1}{3}})$ כאשר n הוא מספר הנקודות ב־ P.
החסם התחתון הטוב ביותר הידוע כיום הוצג על ידי Tóth והוא $\Omega(n\exp(\sqrt{\log n}))$.

אחד הכלים למציאת חסם עליון על מערכת קבוצות הוא מימד ה־ VC.
נאמר שמערכת קבוצות $\mathcal{F}$ על קבוצה X מנפצת (shatter) תת קבוצה A של X אם לכל תת קבוצה B של A קיים איבר $F \in \mathcal{F}$ כך ש $B = A \cap F$.
מימד VC של $\mathcal{F}$ הוא גודל תת הקבוצה הגדולה ביותר של X ש־ $\mathcal{F}$ מנפצת.
מימד ה־ VC של מערכת משרה חסם עליון על המערכת על ידי הלמה של Perles, Sauer ו־ Shelah (נקראת גם Shatter Function Lemma) הגורסת שלמערכת על קבוצה בגודל n יש לכל היותר $O(n^d)$ איברים, כאשר d הוא מימד VC של המערכת.

המשפט המרכזי שאנו מציגים בפרק 2 עוסק במציאת חסם עליון הדוק על מימד VC של מערכת קבוצות $\mathcal{F}$ על קבוצה סופית P של נקודות במישור. האיברים ב־$\mathcal{F}$

עבודת מחקר זו מוקדשת לבני האהוב, יהלי.

המחקר נעשה בהנחיית פרופ׳ רום פנחסי בפקולטה למתמטיקה.

ברצוני להודות לפרופ׳ רום פנחסי על סבלנותו הרבה, על ההנחיה ועל התמיכה.

אני מודה לטכניון – מכון טכנולוגי לישראל על התמיכה הכספית הנדיבה בהשתלמותי.

מימד ה־VC של עקומים הנחתכים s פעמים

חיבור על מחקר

לשם מילוי חלקי של הדרישות לקבלת התואר
מגיסטר למדעים במתמטיקה

שרית בוזגלו

הוגש לסנט הטכניון– מכון טכנולוגי לישראל

סיון תש"ע   חיפה   יוני 2010

מימד VC של עקומים הנחתכים s פעמים

שרית בוזגלו